Potential frequency-resolution capabilities of spectral analysis

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ON FEASIBILITY OF FREQUENCY RESOLUTION IN SPECTRAL ANALYSIS

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A model of a short signal is considered. It consists of two complex harmonics with unknown amplitudes and frequencies $\Delta\omega = O(T^{-\gamma})$ apart plus a complex white noise, where T is a period of observation. The upper bound γ^* of exponents γ, such that a spectral method resolves these harmonics, is considered as a new measure of resolution capability for small samples. The knowledge of γ^* allows to rank modern high resolution thechniques originally developed for geophysical data proccesing. As a first step on this way, we find $\gamma^{**} = \sup\gamma^*$, where supremum is taken over all spectral methods. For a generic signal $\gamma^{**} = 5/4$. It reduces to 7/6 for harmonics with equal powers. In other words, $\gamma^* \leq 7/6$ for spectral techniques, which are based on power spectrum and thus lose initial harmonic phases. This results are closely related to the old harmonic analysis problem that goes back to Lord Rayleigh. He defined, on the basis of nonoptimal Fourier method, a frequency resoltion condition in the absence of noise by relation: $|\omega_1 - \omega_2| > 2\pi T^{-1}$. However, the frequency resolution condition, even in the presence of noice, takes the form $|\omega_1 - \omega_2| \gg T^{-\gamma^{**}}$. The likelihood of the signal in hand does not posess the property of local asimptotic normality for all $\gamma < \gamma^{**}$, therefore our results are not based on the general asymptotic theory of Le Cam-Hajek-Ibragimov-Khas'minsky.

1. INTRODUCTION

The 1970s launched a number of novel practical techniques of harmonic analysis for use in geophysics. These include Burg's maximum entropy method, Pisarenko's harmonic decomposition technique, Capon's "maximum likelihood" spectral estimation, and extensions of these, both parametric and nonparametric (see the review in Kay and Marple 1981). The new techniques aimed at enhancing frequency resolution for short samples and, for that reason, have come to be known as techniques of high-frequency resolution. The necessity arose to make quantitative evaluations and comparisons of the solutions obtained by those methods.

The techniques were mostly analyzed and compared using an empirical computer approach. Although real signals vary quite widely, the simple model of hidden periodicities is meaningful enough for the purposes of comparison:

$$x(n) = s(n) + \sigma w(n), \qquad 0 \leq n < N \qquad (1.1)$$

where $s(n) = a_+ \exp(i\omega_+ n) + a_- \exp(i\omega_- n)$ is a signal involving the unknown parameters $a_\pm, \omega_\pm$ and $\sigma; w$ is a complex-valued Gaussian white noise: $Ew(n) = 0$, $Ew(n)\bar{w}(m) = \delta_n^m$.

V.F.Pisarenko suggested formalization of observed short or moderately long time series by setting up model (1.1) involving convergent frequencies, the peak frequencies $\omega_\pm$ varying as $N \to \infty$ according to

$$\omega_\pm = \omega_0 \pm h_0 N^{-\gamma}, \qquad \gamma > 0, \;\; h_0 \neq 0 \qquad (1.2)$$

where $(\omega_0,\ h_0)$ are new frequency parameters and γ is a known parameter controlling the closeness of the frequencies relative to sample size N. The upper bound $\gamma^* = \sup\gamma$ over those γ that make frequency resolution possible (see below) is a quantitative parameter of resolution capability at the fixed parameter point $(\omega_0,\ h_0)$; the quantity $\gamma_G^* = \inf_G \gamma^*(\omega_0,\ h_0)$ can perform the same function for a parameter set G. In particular, the method of harmonic decomposition (MHD) resolves any pair of close frequencies if $N^{1/6}|\omega_+ - \omega_-| \gg 1$, i.e., $\gamma_{\text{MHD}}^* = 1/6$ [Molchan and Newman, 1988].

This work (see a short version [Molchan, 1990]) examines the utmost resolution capability of harmonic analysis for (1.1), (1.2), i.e., it finds the upper bound $\gamma^{**} = \sup\gamma^*(\mu)$ over all methods μ for the entire parametric set of frequencies. We shall show that the value γ^{**} is attained at the method of least squares (MLS) and shall obtain γ_{MLS}^* as a function of restrictions on the amplitudes.

Historical remarks. Walker [Walker, 1971] has studied the properties of MLS estimators for amplitudes and frequencies in the general problem of hidden periodicities, that is, in a situation where the signal s is a sum of a known number m of unknown complex-valued harmonics with frequencies $\{\omega_k\}$, when $N \to \infty$ and $|\omega_i - \omega_j| > C$. Recall that the MLS estimators are then found from

$$\sum_{0 \leq n < N} \left| x(n) - \sum_{1 \leq k \leq m} a_k \exp i\omega_k n \right|^2 = \min_{\{a_k, \omega_k\}} \qquad (1.3)$$

The resulting frequency estimators are consistent, which is equivalent to resolvability, asymptotically normal and independent. Their standard deviations are

$$\sigma_{\text{MLS}}(\omega_k) \simeq c\sigma|a_k|^{-1}N^{-3/2} \tag{1.4}$$

The MLS is optimal when the frequencies are fixed [Ibragimov and Khas'minskiy, 1981]. On the other hand, when the frequencies are converging, the order of decay for (1.4) cannot be made higher, hence the optimality of MLS in this situation (see [Ibragimov and Khas'minskiy, 1981]) yields the simplest necessary resolution condition $|\omega_i - \omega_j| \gg N^{-3/2}$ or the estimate $\gamma^{**} \leq 3/2$.

The unconventional exponent in (1.4) that controls the decay of $\sigma(\omega_k)$ as N tends to infinity seems to insure frequency resolution in a more general case in which $0 < \gamma \leq 1$ and $N^{-\gamma}|\omega_i - \omega_j| > C$ (the relevant result is stated without proof by Pisarenko et al. [1976].

This paper examines the case of two complex-valued harmonics with $\gamma > 1$ (the case of short data samples in the accepted terminology). V. F. Pisarenko [Pisarenko et al., 1976] and V. M. Gertsik (1983, unpublished manuscript, available at the Institute of Physics of the Earth, Acad. Sci. USSR) studied MLS estimators in a similar problem where the continuous-time real-valued part of (1.1) and (1.2) is observed and $\gamma > 1$. Their result can be used to obtain nontrivial estimates of γ^{**} in the real-valued version of (1.1), (1.2):

$$7/6 \leq \gamma^{**} \leq 5/4, \text{ if } \omega \neq 0, \ \text{Im}(a_+\bar{a}_-) \neq 0 \ \text{(Gertsik)} \tag{1.5}$$

$$1 \leq \gamma^{**} \leq 7/6, \text{ if } \omega \neq 0, \ \text{Im}(a_+\bar{a}_-) = 0 \ \text{(Pisarenko)} \tag{1.6}$$

To be more specific, they have found the upper bounds for such γ for which the relevant inverse Fisher matrix converges to zero. The bounds are given in (1.5), (1.6) as upper bound estimates of γ^{**}.

Derivation of a lower bound for γ^{**} in (1.5) relies on MLS estimators of frequency parameters under uniform local asymptotic normality (LAN) [Ibragimov and Khas'minskiy, 1981]. The LAN property was proved by Gertsik for $\gamma < 7/6$ and $\text{Im}(a_+\bar{a}_-) \neq 0$, so he could use the general theory of maximum likelihood [Ibragimov and Khas'minskiy, 1981] to prove MLS resolution and to establish a certain optimality (after Wolfowitz, see that paper for the terminology) for MLS estimation of frequency parameters. It can be shown that the same procedure leads to the estimate $\gamma^{**} \geq 9/8$ for the case $\text{Im}(a_+\bar{a}_-) = 0$, i.e., we have $\gamma^{**} \in [9/8, 7/6]$, provided $\omega \neq 0$ and $\text{Im}(a_+\bar{a}_-) = 0$.

The parameters $\{a_\pm, \sigma, \omega, h\}$ are assumed to be *apriori* bounded in the above studies of MLS estimation. In our problem the parameter $h = |\omega_+ - \omega_-|N^{\gamma/2}$ is unusual. It is related to a "microscale" of interfrequency intervals, so that the use of *apriori* bounds on h that are uniform over N is not very convenient in applications. The above remark implies that determination of γ^{**} should only rely on those spectral analysis methods μ that do not assume any uniform *apriori* bounds on h in a sequence of experiments with increasing sample size.

This work uses direct methods for solving (1.1), (1.2) to get exact equalities, $\gamma^{**} = 5/4$ for harmonics with unequal initial phases, and $\gamma^{**} = 7/6$ for the case of equal phases. We do not assume uniform *apriori* bounds on h; since we consider a complex-valued problem, the restriction $\omega \neq 0$ for the central frequency is also eliminated.

Notation. $\Theta = \Theta' \times \Theta''$ is the parametric space where

$$\Theta' = \{\omega, h: -\pi < \omega \leq \pi, \ 0 < h < \infty\}$$
$$\Theta'' = \{\sigma^2, a_+, a_- : \sigma^2 > 0, \ a_\pm \in C^1\backslash\{0\} \quad a_+ + a_- \neq 0\} \tag{1.7}$$

$P(A \mid \theta)$ is the probability of event A with respect to process (1.1), (1.2) involving parameters θ; $E\xi$ is the mathematical expectation of $\xi, o_p(1)$ and $O_p(1)$ are standard symbols to denote small random variables that are bounded in probability, i.e., $P\{|o_p(1)| > u\} \to 0$, $\forall u > 0$ and $O_p(1) \cdot o(1) = o_p(1)$ for any infinitely small quantity $o(1)$.

An inner product on a complex-valued vector space C^N is given by $(x, y) = N^{-1}\sum_{0 \leq i < N} x_i \bar{y}_j$. The linear operator $A = \{a_{ij}\}$ on C^N is given by $(Ax)_i = N^{-1}\sum_j a_{ij}x_j$, the norm of A in L^2 is $\|A\|_2 = \left(\sum_{i,j}|a_{ij}|^2\right)^{1/2} / N$.

2. FREQUENCY RESOLUTION

A method is said to resolve peak frequencies $\{\omega_k\}$, if it leads to construction of such a sequence of frequency estimates $\hat{\omega}_k$ that $\hat{\omega}_k - \omega_k = o_p(\min_{j \neq k}|\omega_k - \omega_j|)$. In terms of estimates $\hat{\omega}$, $\hat{h}$, for ω_0, h_0, that means that for any $n > 0$ we have

$$P\{|\hat{\omega} - \omega_0|_* > uN^{-\gamma}, \ |\hat{h} - h_0| > u \mid \theta_0\} \to 0, \ N \to \infty \tag{2.1}$$

where $|\cdot|_*$ is a metric on the circle $S^1 = (-\pi, \pi)$, i.e., $|\omega_1 - \omega_2|_* = \min(|\omega_1 - \omega_2|, 2\pi - |\omega_1 - \omega_2|)$.

A resolution is said to be F-uniform over frequency, if (2.1) holds uniformly over θ_0 for $F \in \Theta$.

The condition for frequency resolution is identified in [Pisarenko et al., 1976] with consistent estimation of amplitude and frequency parameters. This is sufficient for frequency location, if $|\omega_+ - \omega_-| > C$ and is not sufficient under (1.2). In this latter case consistent estimators $\hat{\omega}, \hat{h} : \hat{\omega} - \omega = O_p(N^{-\gamma})$ do localize the central frequency and provide a correct conclusion about its splitting, but do not resolve the frequencies (with nonintersecting confidence intervals and the confidence coefficient $\varepsilon_N \to 1$, however exactly the distance between these is known).

Replace (2.1) with a more convenient requirement. Functional (1.3) is easily minimized with respect to amplitudes $\{a_k\}$. The relevant values $\{\hat{a}_k\}$ are uniquely determined from the linear equations

$$(x, e^{(p)}) = \sum_{1\le k\le m} \hat{a}_k(e^{(k)}, e^{(p)}), \quad 1 \le p \le m$$

where

$$x = \{x(n)\},\ e^{(p)} \doteq \{\exp(i\omega_p n)\},\ n = 0, 1, \ldots, N-1$$

Elimination of $\{a_k\}$ from (1.3) yields a variational principle for estimation of the frequencies

$$\{\hat{\omega}_p\} = \arg\max_{\{\omega_p\}} \sum_{1\le i,j\le m} (x,\ e^{(i)})[(e^{(i)},\ e^{(j)})]^{-1}(e^{(j)},\ x) \tag{2.2}$$

This principle can be stated as follows: the set of harmonics $\{e^P\}$ is sought having the maximum projection of the signal x onto the subspace $L_{\theta'}$ spanned by the harmonics with the frequencies $\theta' = \{\omega_p\}$.

Let $P_{\theta'}$ be the projection on $L_{\theta'}$ in C^N and let $P_0 = P_{\theta'_0}$ be the projection on the space S of the true harmonics. Then the MLS estimates for ω_0, h_0 are given by

$$\{\hat{\omega},\ \hat{h}\} = \arg\max_{\theta'=(\omega,h)} ||P_{\theta'}x|| \tag{2.3}$$

where the maximum is taken over all $\omega \in (-\pi,\ \pi)$ and $h \in [0,\ \pi/2 \times N^\gamma]$. We also have $2hN^{-\gamma} \le \pi$, because this quantity is the interfrequency distance on S^1. MLS estimators will be labeled "modified" and the method is called MLS(α) if the extremum in (2.3) is taken over h from the interval $[0,\ N^\alpha]$, $0 < \alpha < \gamma$.

It is easy to see that MLS(α) provided frequency resolution, if for any $u > 0$

$$P\left\{\min_{\Theta_u(\alpha)}(||P_0x||^2 - ||P_{\theta'}x||^2) \le 0 \mid \theta_0\right\} \to 0, \quad N\to\infty \tag{2.4}$$

where

$$\Theta_u(\alpha) = \{\omega\ , h:\ |\omega - \omega_0|_* \ge uN^{-\gamma},\\ |h - h_0| \ge u,\ \omega \in (-\pi, \pi],\ h \in [0, N^\alpha]\}$$

A similar reduction in the proof of consistency is used in the general theory of maximum likelihood estimation [Ibragimov and Khas'minskiy, 1981].

3. SUFFICIENT RESOLUTION CONDITIONS FOR MLS(α)

Let F be a compact set from the parametric set Θ.

Theorem 1: In (1.1), (1.2), MLS(α) provides F-uniform frequency resolution under the following restrictions:

1) $F \subset \Theta \cap \{\theta : \mathrm{Im}(a_+\hat{a}_-) \ne 0\}$, $1 < \gamma < 5/4$ and $0 < \alpha < (\gamma-1)/3$ for harmonics with unequal initial phases;

2) $F \subset \Theta \cap \{\theta : \mathrm{Im}(a_+\hat{a}_-) = 0\}$, $1 < \gamma < 7/6$ and $0 < \alpha < (\gamma-1)/4$ for harmonics with equal initial phases.

Outline of proof: It suffices to show that (2.4) holds uniformly in $\theta = (\omega_0,\ h_0,\ \theta'')$ from F. For this reason we must examine the random function

$$||P_0x||^2 - ||P_{\theta'}x||^2 = ||s_{\theta'}^\perp||^2 + 2\sigma\mathrm{Re}(s_{\theta'}^\perp,\ W)\\ +\sigma^2(W,\ (P_0 - P_{\theta'})W) \tag{3.1}$$

where $s_\theta^\perp = s - P_\theta s$ is the component of s orthogonal to the $L_{\theta'}$ plane spanned by

$$e^\pm = \{\exp(i(\omega\pm\Delta)n),\ n = 0, 1, \ldots, N-1\}\\ -\pi < \omega \le \pi,\ 0 \le \Delta \le \pi/2\\ \Delta = hN^{-\gamma} \tag{3.2}$$

We are to show that

$$||P_0x||^2 - ||P_{\theta'}x||^2 = ||s_{\theta'}^\perp||^2(1 + o_p(1)),\quad N\to\infty \tag{3.3}$$

for $\theta' \in \Theta_u(\alpha)$. The estimate $||s_\theta^\perp|| > C(N,\ u) > 0$ on $\Theta_u(\alpha)$ (see Section 4 below) yields (2.4), hence the statement of the theorem. The proof of (3.3) is independent of any restrictions on α; these emerge as purely technical ones in the analysis of the deterministic function $||s_\theta^\perp||$.

Lemma 3.1: Let $P_{\theta'}$ be the projection operator in C^N onto the plane spanned by (3.2). Then for any $\beta > 1$

$$||P_{\theta'_1} - P_{\theta'_2}|| \le CN^\beta(|\omega_1 - \omega_2|_* + |\Delta_1 - \Delta_2|_*)\\ \Delta_i = h_iN^{-\gamma} < \pi/2$$

Proof: The projection operator onto the subspace spanned by $\{e^{(p)}\}$ has the form $Px = \sum_{1\le p,q\le m}(x, e^{(p)})a^{pq}e^{(q)}$, the matrix being $[a^{pq}] = [(e^{(p)},\ e^{(q)})]^{-1}$. In particular, we have for vector (3.2),

$$P_{\theta'} = [P_{n,m}(\omega,\ h)]\\ = [P_{n,m}(0,\ h)\exp i\omega(n-m)]_{n,m=0,\ N-1}, \tag{3.4}$$

where

$$P_{n,m}(0,\ h) = 2\,[\cos(t_n - t_m)\delta - \varphi_N(\delta)\cos(t_n + t_m)\delta]\\ \times[1 - \varphi_N^2(\delta)]^{-1} \tag{3.5}$$

$$t_n = [n - (N-1)/2]N^{-1},\ \delta = N\Delta = N^{1-\gamma}h$$

$$\varphi_N(\delta) = \sin\delta/[N\sin\delta/N] = \int_{-1/2}^{1/2}\cos 2\delta t d\mu_N(t) \tag{3.6}$$

The measure μ_N is concentrated at the points t_n with weights $1/N$, $\int\mu_N(dt) = 1$.

The operator $P_{\theta'}$ is a smooth function of ω, h. Therefore

$$||P_{\theta'_1} - P_{\theta'_2}|| \le |\omega_1 - \omega_2|_* \sup||\partial P_{\theta'}/\partial\omega||\\ + |\Delta_1 - \Delta_2|_* \sup||\partial P_{\theta'}/\partial\Delta|| \tag{3.7}$$

From (3.4) we get

$$||\partial P_{\theta'}/\partial\omega|| = ||i(n-m)P_{n,m}(\omega, h)|| \le N||P_{\theta'}||_2 = \sqrt{d}N \tag{3.8}$$

where $d = 2$ is the dimension of $L_{\theta'}$.

Let us evaluate

$$|\partial P_{n,m}/\partial\Delta| = 2N| - t^- \sin t^-\delta + t^+ \sin t^+\delta \cdot \varphi_N(\delta) - (d\varphi_N(\delta)/d\delta)\cos t^+\delta + P_{n,m}(0,\ \Delta)\varphi_N(\delta)d\varphi_N(\delta)/d\delta| \times (1-\varphi_N^2(\delta))^{-1} \tag{3.9}$$

where $t^\pm = t_n \pm t_m$, $|t^\pm| < 1$.

Let $\Delta \in I_1 = (N^{-(1+\varepsilon)}, \pi/2)$. From the integral representation of $\varphi_N(\delta)$ in (3.6) we derive $|\varphi_N(\delta)| \le 1$ and $|d\varphi_N(\delta)/d\delta| < 1$. For this reason

$$|\partial P_{n,m}/\partial\Delta| < 2N(3 + |P_{n,m}|)(1-\varphi_N^2(\delta))^{-1} \tag{3.10}$$

To evaluate $1-\varphi_N^2(\delta)$, we note that for any $|x| < \pi/2$

$$|\varphi_N(\delta)| < |\varphi_N(x)|,\quad |\delta| > x,\quad |\delta| < \pi N/2 \tag{3.11}$$

and also (see 3.6)

$$\varphi_N(\delta) = 1 - \delta^2/6 + O(\delta^4),\quad \delta \to 0 \tag{3.12}$$

Hence for $N\Delta = \delta \in (N^{-\varepsilon},\ \pi N/2)$

$$1-\varphi_N^2(\delta) > 1 - \varphi_N^2(N^{-\varepsilon}) > 0.2N^{-2\varepsilon},\quad N > N_0 \tag{3.13}$$

Inequality (3.10) can consequently be continued: $|\partial P_{n,m}/\partial\Delta| < CN^{1+2\varepsilon}(1 + |P_{n,m}|)$, and so

$$\|\partial\theta'/\partial\Delta\|_2 < 2CN^{1+2\varepsilon}(1 + \|P_{\theta'}\|_2) = C_1 N^{1+2\varepsilon}\quad \Delta \in I_1 \tag{3.14}$$

Evaluation of (3.9) for small $\Delta N = \delta \in (0, N^{-\varepsilon})$ will be based on (3.12), as well as on

$$t\sin t\delta = t^2\delta + O(\delta^3),\quad \cos t\delta = 1 - (t\delta)^2/2 + O(\delta^4)$$
$$d\varphi_N(\delta)/d\delta = -\delta/3 + o(\delta^3)$$

and (see below) $P_{n,m}(0,h) = 1 + 12t_n t_m + O(\delta^2)$, $\delta = hN^{1-\gamma}$. This yields

$$\partial P_{n,m}(0,\ h)/\partial\Delta = O(N^{1-\varepsilon}),\quad \|\partial P/\partial\Delta\|_2 < CN^{1-\varepsilon}\quad 0 < \Delta < N^{-(1+\varepsilon)} \tag{3.15}$$

Combining the above estimates (3.7), (3.8), (3.14), (3.15) and recalling that $\|\cdot\| \le \|\cdot\|_2$, we obtain the desired lemma.

Lemma 3.2: Let $\theta_0,\ \theta' \subset (-\pi,\ \pi] \times (0,\ \pi N^\gamma/2]$, then for any $\varepsilon > 0$

$$P\{\sup|(w,\ (P_0 - P_{\theta'})w)| > xN^{\varepsilon-1} \mid \theta_0\} \to 0,\ x \to \infty \tag{3.16}$$

uniformly over θ_0' and N.

Proof: We use the following statement, which follows from theorem in [Ibragimov and Khas'minskiy, 1981, Section 1.19]: Let $\xi(\theta)$, $\theta \in K \subset R^2$ be a real-valued continuous random function in a unit square and let

$$E|\xi(\theta)|^m < A,\quad E|\xi(\theta_1) - \xi(\theta_2)|^m < A|\theta_1 - \theta_2|^3$$

for some $m \ge 3$. In that case

$$E\sup_{\theta\in K}|\xi(\theta)| \le C_m A^{1/m}$$

Apply the theorem to the function

$$\xi_N(\omega,\Delta) = N(w,\ (P_0 - P_{\theta'})w),\ |\omega| \le \pi,\quad \Delta = hN^{-\gamma} \in (0,\pi/2) \tag{3.17}$$

To calculate its moments, we remark that the quadratic form (3.17) is related to the self-adjoint operator $P_0 - P_{\theta'}$ and hence it can be converted to the canonical form $\xi_N(\omega,\ \Delta) = \sum_{1\le i\le 2d}\lambda_i|\varepsilon_i|^2$, $0 \le |\lambda_i| \le 1$ by a unitary transformation. The number of nonzero eigenvalues of $P_0 - P_{\theta'}$ is not greater than the sum of the dimensions of L_0 and $L_{\theta'}$, i.e., $2d \le 4$. White noise is invariant under unitary transformations, hence the ε_i are independent complex-valued Gaussian random variables with parameters (0, 1). It follows that

$$E|\xi_N(\theta')|^m \le \|P_0 - P_{\theta'}\|^m E(|\varepsilon_1|^2 + \ldots + |\varepsilon|^2)^m = \|P_0 - P_{\theta'}\|^m a_m \le a_m$$

where $\|P_0 - P_{\theta''}\| \le 1$ has been used.

In view of the preceding discussion, the representation $\xi_N(\theta_1') - \xi_N(\theta_2') = N(w,\ (P_{\theta_2'} - P_{\theta_1'})w)$ yields for $m > 3$

$$E|\Delta\xi_N|^m \le a_m\|\Delta P_{\theta'}\|^m \le a_m\|\Delta P_\theta\|^3$$

Proceeding further, we get by lemma 1

$$\le Ca_m N^{3\beta}(|\omega_1 - \omega_2|_* + |\Delta_1 - \Delta_2|_*)^3$$

It remains to use the above theorem with $A\cdot = N^{3\beta}a_m(C+1)$. Ultimately we derive

$$E\sup_{\omega,h}|\xi_N(\theta')| \le C_m N^{3\alpha/m} \tag{3.18}$$

Since m is arbitrary, the exponent $3\alpha/m$ can be made indefinitely small. Using (3.18) and Chebyshev's inequality, we obtain lemma 3.2.

Lemma 3.3: Let $0 < \alpha \le \gamma$, then for any $(\varepsilon, \varepsilon', u) > 0$

$$P\left\{\sup_{\Theta_u(\alpha)}|(s_{\theta'}^\perp,\ w)|/\|s_{\theta'}^\perp\| > x\frac{N^{\varepsilon'-1/2}\|s\|^\varepsilon}{\inf\{\|s_{\theta'}^\perp\|^\varepsilon : \theta' \in \Theta_u(\alpha)\}}\,\middle|\,\theta_0'\right\} \to 0,\quad x \to \infty \tag{3.19}$$

uniformly over N and $\theta_0'(-\pi,\ \pi] \times (0,\ \pi N^\gamma/2)$.

Proof: The function

$$\eta_N(\omega,\ \Delta) = (s_{\theta'}^\perp,\ w)/\sqrt{E|(s_{\theta'}^\perp,\ w)|^2} = (s_{\theta'}^\perp,\ w)\sqrt{N}\|(s_{\theta'}^\perp\|^{-1},\quad \Delta = hN^{-\gamma}$$

is Gaussian with mean 0 and variance 1. Evaluate its maximum on $\Theta_u(\alpha)$ by using Fernique's result [Fernique, 1975, Theorem 4.1.1]. It is convenient for our purposes to restate it as follows: Let $\eta(t)$, $t \in R^2$ be a centered Gaussian function in the rectangle $T = \{|t_i| < L_i\}$. If

$$\sup_{|t_1-t_2|<\delta} E|\eta(t_1) - \eta(t_2)|^2 \le (A\delta^{\bar\varepsilon})^2,\quad E|\eta(t)|^2 = 1$$

then

$$P\{\sup_T |\eta(t)| \geq x(1+cAL^{\varepsilon}\bar{\varepsilon}^{-1/2})\} \leq c_1 \int_x^{\infty} \exp(-u^2/2)du, \quad x > x_0 \tag{3.20}$$

where $L = \max L_i$.

Now evaluate $\psi^2 = E|\eta_N(\omega, \Delta) - \eta_N(\omega', \Delta')|^2$ for $\{\omega, \omega'\} \subset (-\pi, \pi]$; $\Delta, \Delta' \in (0, \pi/2)$. For the sake of simplicity we assume the notation θ to be the same for (ω, h) and (ω, Δ), setting $\Delta = hN^{-\gamma}$. Let $n = \|s_\theta^\perp\|$, then

$$\begin{aligned}\psi &= \|s_\theta^\perp/n - s_{\theta'}^\perp/n'\| \\ &\leq \|s_\theta^\perp - s_{\theta'}^\perp\|/n + \|s_{\theta'}^\perp\| \times |1/n - 1/n'| \\ &\leq (\|s_\theta^\perp - s_{\theta'}^\perp\| + |n-n'|)n^{-1} \leq 2\|s_\theta^\perp - s_{\theta'}^\perp\|/\|s_\theta^\perp\|\end{aligned}$$

However, we have

$$\begin{aligned}\|s_\theta^\perp - s_{\theta'}^\perp\| &= \|(P_\theta - P_{\theta'})s\| \leq \|P_\theta - P_{\theta'}\| \\ &\leq \|s\| \cdot \|P_\theta - P_{\theta'}\|^{\mu}, \quad 0 < \mu < 1\end{aligned}$$

That inequality can be continued by using Lemma 3.1:

$$\leq c\|s\|(n^{\beta}\delta)^{\mu}, \quad |\omega - \omega'|_* + |\Delta - \Delta'|_* < \delta, \ \beta > 1$$

Therefore

$$\psi \leq 2c\|s\|(N^{\beta}\delta)^{\mu}/n_{u,\alpha}, \quad n_{u,\alpha} = \inf\{\|s_{\theta}^\perp\|, \ \theta' \in \Theta_u(\alpha)\}$$

On the other hand, $\psi < 2$. We raise both estimates to the powers of ε and $1-\varepsilon$ and multiply them to get

$$\psi \leq 2(c\|s\|)^{\varepsilon}(N^{\beta}\delta)^{\varepsilon\mu}/n_{u,\alpha}^{\varepsilon} = A_N\delta^{\bar{\varepsilon}}, \quad \bar{\varepsilon} = \varepsilon\mu \tag{3.21}$$

The set $\Theta_u(\alpha)$ is divided into four rectangles T_i. For each of these we get, by (3.20) and (3.21),

$$P\{\sup_{T_i} |\eta_N(\theta)| \geq xc(\varepsilon, \mu)\|s\|^{\varepsilon}N^{\varepsilon'}/n_{u,\alpha}^{\varepsilon}\} \leq \varphi(x)$$

where $\varepsilon' = \beta\varepsilon\mu$; $\varphi(x) \to 0$ when $x \to \infty$. Hence Lemma 3, since

$$P\{\sup_T |\eta_N(\theta)| \geq x\} \leq \sum_i P\{\sup_{T_i} |\eta_N(\theta)| \geq x\}$$

Lemma 3.4: Suppose that $1 < \gamma < 1 + 1/k$, F *is a compact set from* Θ, *and*

$$\inf_{\Theta_u(\alpha)} \|s_{\theta'}^\perp\|^2 \geq c_F N^{(1-\gamma)k}, \quad c_F > 0, \ u > 0, \ \theta_0 \in F' \times F'' \tag{3.22}$$

Then (2.4) holds uniformly over $F = F' \times F''$.

Remark: The next section contains a proof of the *apriori* estimate (3.22) with $k = 4$ for harmonics with unequal initial phases and with $k = 6$ for the case of equal phases. This will complete the proof of Theorem 1.

Proof: By

$$\|P_0 x\|^2 - \|P_\theta x\|^2 = \|s_\theta^\perp\|^2 \times \left(1 + 2\sigma \mathrm{Re}\frac{(s_\theta^\perp, w)}{\|s_\theta^\perp\|}\|s_\theta^\perp\|^{-1} + \sigma^2\frac{(w, (P_0 - P_{\theta'})w)}{\|s_\theta^\perp\|^2}\right)$$

Use (3.16), (3.19) with $\varepsilon' = \varepsilon^2/2$ in (3.19) to get

$$\|P_0 x\|^2 - \|P_\theta x\|^2 = \|s_\theta^\perp\|^2 \times (1 + 2\sigma O_p(\|s\|^{\varepsilon}V_n^{-(1+\varepsilon)/2}) + \sigma^2 O_p(V_N^{-1}) \tag{3.23}$$

where $V_N = N^{1-\varepsilon}\inf(\|s_\theta^\perp\|^2, \ \theta \in \Theta_u(\alpha))$. The *apriori* estimate (3.22) yields $V_N \geq c_F N^{(k+1-\gamma k)-\varepsilon}$, $\varepsilon < 0$. Setting $2\varepsilon = k + 1 - \gamma k$, we have $V_N \geq c_F N^{\varepsilon}$.

The quantities σ, $|a_\pm|$, hence $\|s\|$ ($\|s\| \leq |a_+| + |a_-|$) are bounded on the compact set F'', so that

$$\begin{aligned}\|P_0 x\|^2 - \|P_\theta x\|^2 &= \|s_\theta^\perp\|^2(1 + O_p(N^{-\varepsilon(1+\varepsilon)/4})) \\ &\geq c_F N^{(1-\gamma)k}(1 + o_p(1))\end{aligned}$$

Therefore we have, uniformly over F,

$$\begin{aligned}&P\left\{\inf_{\Theta_u(\alpha)}(\|P_0(x)\|^2 - \|P_{\theta'}x\|^2) \leq 0 \mid \theta_0\right\} \\ &< P\left\{\inf_{\Theta_u(\alpha)}(\|P_0(x)\|^2 - \|P_{\theta'}x\|^2) \leq 1/2c_F N^{(1-\gamma)k}\right\} \to 0\end{aligned}$$

4. NORM OF SIGNAL RESIDUAL

Preliminary remarks:

1) It follows from the structure of s (1.1) and of the projection operator $P_{\theta'}$ (3.4) that $\|s_{\theta'}^\perp\|^2$ is a function of $\omega - \omega_0$:

$$\|s_\theta^\perp\|^2 = \|s_0\|^2 - \|P_{(\omega-\omega_0, h)}s_0\|^2, \quad s_0 = s\,|_{\omega_0 = 0} \tag{4.1}$$

Consequently, one can set $\omega_0 = 0$ in what follows.

2) $\|s_{\theta'}^\perp\|$ is not affected by the substitution $s \to z_N s$ where $|z_N| = 1$. Remembering the last remark, we shall represent the signal in the form

$$s(t) = a_+\exp(i\delta_0 t) + a_-\exp(-i\delta_0 t), \quad \delta_0 = N^{1-\gamma}h_0, \tag{4.2}$$

where t takes on the values $t_n = [n - (N-1)/2]/N$ in the interval $(-1/2, 1/2)$.

3) Expansion of (4.2) in the small parameter δ_0 leads to the basic vectors $t^p = \{t_n^p, \ n = 0, 1, \ldots, N-1\}$, where $p = 0, 1, \ldots$. Owing to the antisymmetry $t_n = -t_{N-n}$, the vectors t^p involving odd and even powers are orthogonal, i.e.,

$$(t^p, t^l) = \langle t^{p+l}\rangle = \begin{cases} 0, & p+l \text{ is odd} \\ \left[2^{p+l}(p+l+1)\right]^{-1} + O(N^{-1}), & p+l \text{ is even}\end{cases} \tag{4.3}$$

where $\langle\varphi\rangle = \int \varphi(t)d\mu_N(t)$, the measure μ_N being defined in (3.6).

4) Everywhere below $1 < \gamma < 5/4$, $\delta = \Delta \cdot N = hN^{1-\gamma}$, $\delta_0 = \Delta_0 \cdot N = h_0 N^{1-\gamma}$.

Statement 4.1: If

$$|\omega_\pm - \omega_0|_* \geq 2\pi/N \ \text{and} \ \max\{|a_+ + a_-|^{-1}, |a_\pm|, h_0\} < f$$

then

$$||s_{\theta}^{\perp}||^2 \geq c_f > 0, \; N > N_f$$

Proof: It follows from (4.2) that $s(n) = a_+ + a_- + O(N^{1-\gamma})$, hence

$$||s_{\theta'}^{\perp}||^2 = |a_+ + a_-|^2 \times ||1_{\theta'}^{\perp}|| + O(N^{1-\gamma}) \qquad (4.4)$$

where the remainder is uniform over $a_\pm, h_0$ if $(|a_\pm|, h_0) < f$. The magnitude of the projection of $\mathbf{1} \equiv (1, \ldots, 1)$ on L_θ is given by

$$||\hat{1}_{\theta'}||^2 = \left[F_N^2\left(\frac{\omega_-}{2}\right) + F_N^2\left(\frac{\omega_+}{2}\right)\right.$$
$$\left. -2F_N\left(\frac{\omega_-}{2}\right) F_N\left(\frac{\omega_+}{2}\right) F_N(\Delta)\right] / \left(1 - F_N^2(\Delta)\right) \qquad (4.5)$$

where $F_N(x) = \Phi_N(Nx) = \sin Nx/(N \sin x)$ is Féjer's kernel. The function $F_N(x)$ is of interest when $|x| < \pi/2$, since $\Delta < \pi/2$ and $|\omega_\pm| \leq 2\pi$.

Now use the estimate

$$|F_N(y)| < F_N(x), \quad y > x \geq 0, \quad |x| < x_0/N \qquad (4.6)$$

where $x_0 > 0$ is the least root of

$$F_N(x_0) = \left[N \sin \frac{3}{2}\pi/N\right]^{-1}, \quad x_0 \in (\pi/2, \pi)$$

It then follows from (4.5) that

$$||\hat{1}_{\theta'}||^2 \leq 2F^2(x_*)(1 - F(x_*))^{-1}$$
$$x_* = \min(x, \, \Delta, \, |\omega_\pm|/2)$$

The right-hand side of this inequality is less than 1 if $F(x_*) < 0.5$. We have $F_N(x_*) < 0.5$ at $x_* = 1.9/N$ for $N > 15$. Consequently,

$$||\hat{1}_\theta||^2 = 1 - ||\hat{1}_\theta||^2 > c$$
$$\text{if } \{|a_\pm|, \, h_0\} < f, \; \{|\omega_\pm|/2, \, \Delta\} \geq 1.9/N$$

Let $\Delta \leq 1.9/N$ and $|\omega_\pm| \leq 2\pi/N$. From (4.5) we have

$$||\hat{1}_\theta||^2 = \left[\frac{|F_N(\omega_+/2) - F_N(\omega_-/2)|^2}{2(1 - F_N(\Delta))}\right.$$
$$\left. + F_N\left(\frac{\omega_-}{2}\right) F_N\left(\frac{\omega_+}{2}\right)\right] \frac{2}{1 + F_N(\Delta)} \qquad (4.7)$$

Using the estimates

$$|F_N(\omega_\pm/2)| < \max_{x \in J} |N \sin x|^{-1} = y$$
$$J = [\omega_-/2, \, \omega_+/2] \subset [-\pi/2, \, \pi/2]$$

and

$$|F_N(\omega_+/2) - F_N(\omega_-/2)| = |F_N'(x)|\Delta$$
$$= N_\Delta |\cos Nx - F_N(x) \cos x| \cdot |N \sin x|^{-1}$$
$$< N\Delta(1 + y)y, \quad x \in J$$

we derive

$$||\hat{1}_\theta||^2 < [Ay^2(1 + y)^2 + y^2]B \qquad (4.8)$$

where

$$A = \max_{|N\Delta|<1.9} \frac{1}{2}(N\Delta)^2(1 - F_N(\Delta))^{-1} = \max \varphi(\Delta)$$
$$B = \max_{|N\Delta|<1.9} 2[1 + F_N(\Delta)]^{-1} = \max \psi(\Delta)$$

The functions φ and ψ are increasing in $(0, \, \pi/N)$.

To see this, use (3.5) to obtain

$$F_N(x/N) = \int_{-0.5}^{0.5} \cos 2tx d\mu_N(t), \quad \int \mu_n(dt) = 1$$

whence

$$F_N'(x/N) = -\int_{-0.5}^{0.5} 2t \sin tx d\mu_N(t) < 0, \quad o < x < \pi$$

so that $\psi(\Delta)$ is increasing in $(0, \, \pi/N)$.

Similarly

$$\frac{d}{dx}\frac{1 - F_N(x/N)}{4x^2} = \frac{d}{dx}\int \frac{1 - \cos 2tx}{(2x)^2} d\mu_N(t)$$
$$= -\frac{\langle t^4 \rangle}{4!} 2(2x) + \frac{\langle t^6 \rangle}{6!} 4(2x)^3 - \frac{\langle t^8 \rangle}{8!} 6(2x)^5 \ldots$$

For $0 < x \leq \sqrt{15}$, the above alternating series consists of monotone decreasing terms, hence involves negative values:

$$\frac{\langle t^{2k} \rangle}{(2k)!}(2k - 2) > \frac{\langle t^{2k+2} \rangle}{(2k + 2)!} 2k(2x)^2$$
$$k = 2, \, 3, \ldots \qquad (4.9)$$

whence

$$(2x^2) < (\langle t^{2k} \rangle / \langle t^{2k+2} \rangle) \cdot 4(k^2 - 1)(1 + 1/2k)$$

However,

$$4\langle t^{2k+2} \rangle = 4\int_{-1/2}^{1/2} t^{2k+2} d\mu_N \leq \int t^{2k} d\mu = \langle t^{2k} \rangle$$

consequently, (4.9) is true, provided

$$(2x)^2 \leq \min_{k \geq 2} 16(k^2 - 1)(1 + 1/2k) = 4 \times 15$$

It follows that $1/\varphi(\Delta)$ is decreasing in $(0, \, \pi/N)$.

If $\Delta < 1.9/N$, then $A = \varphi(1.9/N)$ and $B = \psi(1.9/N)$. When $N > 1000$, we have $A \leq 3.6$, $B \leq 1.336$, and

$$y = \max(|N \sin x|^{-1}, \; \pi/N < x < \pi/2) \leq 0.3183$$

To sum up, estimate (4.8) leads to the desired inequality $||\hat{bf1}_{\theta'}||^2 \leq 0.982 < 1, N > 10^4$.

The lower bound on $||\hat{1}_{\theta'}||^2$ and estimate (4.4) yield Statement 4.1.

Remark: We shall assume $h < N^\alpha$, $\delta = N^{1-\gamma}h = o(1)$ in what follows. Then the analysis of $||s_{\theta'}^{\perp}||$ can be reduced to the asymptotics with respect to the small parameters δ and δ_0.

Lemma 4.2: a) *Let* $\{|a_\pm|, \, h_0\} < f$. *Then, uniformly over* h_0

$$||s||^2 = |a_+ + a_-|^2 + 2\text{Re}(a_+\bar{a}_-)(-\delta_0^2/3!$$

$$+\delta_0^4/5! - \delta_0^6/7! + o(N^{(1-\gamma)6})) \quad (4.10)$$

b) The matrix elements of the projection operator $P_{(o,h)}$ have the asymptotics

$$p_{n,m}(0,\ h) = \sum_{0\le k\le 3} u_k(t_n,t_m)(-\delta^2)^k/(2k)! + \delta^8 r_1 + N^{-2} r_2 \quad (4.11)$$

where $|r_i(n,\ m,\ \delta)| < c$ for $\delta \downarrow 0$ and $N \uparrow \infty$; the $u_k(x,\ y)$ are polynomials of the form

$$u_0 = 1 + 12xy$$

$$u_1 = 2S[4x^3y + x^2 - 0.6xy - 1/12]$$

$$u_2 = S[4.8x^5y + 8x^3y^3 + 2x^4 + 6x^2y^2 - 4.8x^3y - 2x^2 + (66/175)xy + 1/15]$$

and S is a symmetrizing operation: $Sf(x,\ y) = [f(x,\ y) + f(y,\ x)]/2$.

Proof: a) Signal (4.2) is expanded into a uniformly convergent series in δ_0:

$$s(t) = \sum_{k\ge 0}(a_+ + (-1)^k a_-)(i\delta_0 t)^k/k!$$

Vectors t^p of opposite parities are orthogonal, hence

$$||s||^2 = |a_+ + a_-|^2 + \sum_{p\ge 1} d_{2p}\langle t^{2p}\rangle\delta_0^{2p}/(2p)! \quad (4.12)$$

where

$$d_{2p} = \sum_{0\le k\le 2p} C_{2p}^k(a_+ + (-1)^k a_-)(\bar{a}_+ + (-1)^{2p-k}\bar{a}_-)i^k\bar{i}^{2p-k}$$
$$= (-1)^p(|a_+ + a_-|^2 - |a_+ - a_-|^2)2^{2p-1}$$
$$= 2\mathrm{Re}(a_+\bar{a}_-)(2i)^{2p}$$

Using (4.3) we get (4.10) with the remainder $O(\delta_0^8) + O(\delta_0^2 N^{-1})$. The first term is associated with truncation of (4.12), the second with the asymptotics of $\langle t^{2p}\rangle$. If $1 < \gamma < 5/4$, then $N^{-1} = O(\delta_0^4)$, which completes the proof of (4.10). c) The expansion $p_{n,m}(0,\ h)$ in $\delta = hN^{1-\gamma}$ can be derived directly from (3.5). To do this, the numerator and the denominator of (3.5) are expanded into series up to the order $O(\delta^6)$ inclusive. The functions $\cos(t_n \pm t_m)\delta$ and $\Phi_N(\delta)$ are even, which simplifies the matter. The expansions for $\cos t\delta$ are easily derived from the relation

$$\Phi_N(\delta) = \sin\delta\cdot[N\sin\delta/N]^{-1} = \sin\delta/\delta\cdot(1 + O(\delta^2/N^2))$$

where the remainder term is uniform over $h < f$. Hence

$$1 - \Phi_N^2(\delta) = \frac{1}{2}(\cos 2\delta - 1 + 2\delta^2)/\delta^2 + O(\delta^2 N^{-2})$$

$$= 2((2\delta)^2/4! - (2\delta)^4/6! + (2\delta)^6/8! + O(\delta^2 N^{-2})$$

$$\frac{1}{3}\delta^2[1 - \Phi_N^2(\delta)]^{-1} = 1 + (2/15)\delta^2 + (13/(15)^2\times 7)\delta^4 + O(\delta)^6 + O(N^{-2})$$

It remains to multiply and add the expansions. The results of these elementary operations are summarized in (4.11).

Statement 4.3: Let $0 < h < N^{\gamma-\alpha_1}$, $1 < \alpha_1 < \gamma$,

$$2N^{-\mu} < |\omega - \omega_0|N < 2\sqrt{14}$$

where

$$0 < 4\mu < \min(\gamma - 1.2(\alpha_1 - 1))$$
$$and\ \ \{|a_\pm|,\ h_0,\ |a_+ + a_-|^{-1}\} < f$$

Then $||s_{\theta'}^{\perp}||^2 \ge cN^{1-\gamma}$.

Remark: From Statement 4.1 it follows that $||s_\theta^\perp||^2$ is uniformly separated from zero in the region given by $|\omega - \omega_0|N \ge 2\sqrt{14}$ and $0 < h < 1.2N^{\gamma-1}$, if $\{|a_\pm|,\ h_0,\ |a_+ + a_-|^{-1}\} < f$.

Proof: According to Lemma 4.2,

$$||\hat{\mathbf{1}}_{\theta'}||^2 = |(\mathbf{1}, e_\omega)|^2 + 12|(\mathbf{1}, te_\omega)|^2 + O(\delta^2) + O(N^{-2}) \quad (4.13)$$

where $e_\omega = \{\exp(i\omega N t_n),\ n = 0,\dots,N-1\}$. We have

$$|(\mathbf{1}, e_\omega)|^2 = \left|\frac{\sin(N\omega/2)}{N\sin(\omega/2)}\right|^2 = \left|\frac{\sin\lambda}{\lambda}\right|^2(1 + O(N^{-2}))$$
$$\lambda = \frac{\omega N}{2}$$

$$|(\mathbf{1}, te_\omega)|^2 = \left|\frac{d}{d\omega}\frac{\sin(N\omega/2)}{N\sin(\omega/2)}\right|^2 = \frac{1}{4}\left|\frac{d}{d\lambda}\frac{\sin\lambda}{\lambda}\right|^2 + O(N^{-2})$$

Relations (4.4) and (4.13) yield

$$||s_{\theta'}^\perp||^2 = |a_+ + a_-|^2\varphi(\lambda) + O(N^{1-\gamma}) + O(\delta^2) \quad (4.14)$$

where $\varphi(\lambda) = 1 - V^2(\lambda) - 3(V'(\lambda))^2$, $V(\lambda) = (\sin\lambda)/\lambda$, $\lambda = \omega N/2$. The function $\varphi(\lambda)$ is increasing in $(0,\ \sqrt{14})$.

The proof is as follows. We have $\varphi(0) = 0$, and $\varphi(\sqrt{14}) \simeq 0.88$. Stationary points satisfy the equation $V'(V + 3V'') = 0$.

The case $V' = 0$ yields the sequence of λ_n specified by $\tan\lambda_n = \lambda_n$. The first positive root is $\lambda > \sqrt{14} \simeq 1.19\pi$.

The case $V + 3V'' = 0$ results in the equation $u(\lambda) = 0$ where

$$u(\lambda) = \lambda^{-1}\sin\lambda\times(1 - \lambda^2/3) - \cos\lambda = \frac{2}{3}\sum_{k\ge 2}\frac{(-\lambda^2)^k}{(2k-1)!}\frac{k-1}{2k+1}$$

If $\lambda^2 < 2(k-1)(2k+3)$, $k \ge 2$, the alternating series $u(\lambda)$ involves decreasing terms, i.e., $u(\lambda) > 0$ for $0 < \lambda^2 < 14$. Hence $\varphi(\lambda)$ has no stationary points in $(0,\ \sqrt{14})$.

In the vicinity of zero $\varphi(\lambda) = \lambda^4/45 + O(\lambda^6)$, $\lambda \to 0$, so (4.14) yields

$$\|s_\theta^\perp\|^2 \geq |a_+ + a_-|^2 N^{-4\mu}/45 + O(N^{1-\gamma}) + O(N^{2(1-\alpha_1)})$$

If $|a_+ + a_-| > f^{-1}$, $4\mu < \min(\gamma - 1,\ 2(\alpha_1 - 1))$, then $\|s_\theta^\perp\|^2 > cN^{-4\mu}$.

Statement 4.4: Let $0 < h \leq N^{\gamma-\alpha_1}$, $1 < \alpha_1 < \gamma$, $u\varphi(N)N^{-\alpha_1} < |\omega - \omega_0| < o(N^{-1})$, *where* $\varphi(N) \uparrow \infty$. *Then*

$$\|s_\theta^\perp\| \geq cN^{4(1-\alpha_1)},\quad N > N_0(f,\ \varphi)$$

uniformly over $a_\pm$, h_0 : $\{|a_\pm|,\ h_0,\ |a_+ + a_-|^{-1}\} < f$.

Proof: According to Lemma 4.2,

$$\|\hat{s}_{\theta'}\|^2 = U_N^0(\omega) - \delta^2 U_N^1(\omega) + O(\delta^4) + O(N^{-2})$$

Here

$$U_N^0 = |(s,\ e_\omega)|^2 + 12|(s,\ te_\omega)|^2 \tag{4.15}$$

$$U_N^1 = \mathrm{Re}[4(t^5 e_\omega,\ s)(s,\ te_\omega) + (t^2 e_\omega,\ s)(s,\ e_\omega)] - (3/5)\,|(te_\omega,\ s)|^2 - (1/12)\,|(s,\ e_\omega)|^2 \tag{4.16}$$

The quantities $(t^k e_\omega,\ s)$ can be represented in the form of a uniformly convergent series:

$$(t^k e_\omega,\ s) = \sum_{\substack{p\geq 0\\ p+k=\text{even}}} (a_+(\omega N - \delta_0)^p + a_-(\omega N + \delta_0)^p) \times i^p \frac{\langle t^{p+k}\rangle}{p!} \tag{4.17}$$

Therefore it is possible to calculate U_N^0 and evaluate U_N^1 explicitly. The use of (4.10) yields

$$\|s\|^2 - U_N^0(\omega) = \frac{1}{4}(\langle t^4\rangle - \langle t^2\rangle^2) \times |a_+(\omega N - \delta_0)^2 + a_-(\omega N + \delta_0)^2|^2 + O((\omega N)^6) \tag{4.18}$$

In virtue of (4.17),

$$(t^k e_\omega, s) = \begin{cases} (\bar{a}_+ + \bar{a}_- \langle t^k\rangle + O((\omega N)^2), & k \text{ even} \\ O(\omega N), & k \text{ odd}\end{cases}$$

Therefore

$$U_N^1 = |a_+ + a_-|^2(\langle t^2\rangle - 1/12) + O((\omega N)^2) = |a_+ + a_-|^2 N^{-2} + O((\omega N)^2) = O((\omega N)^2)$$

Then we have

$$\|s_\theta^\perp\|^2 = \|s\|^2 - U_N^0 + O((\delta\omega N)^2) + O(\delta^4) + O(N^{-2}) = (1/6! + O(N^{-1}))|a_+(\omega N - \delta_0)^2 + a_-(\omega N + \delta_0)^2| + R_N = |a_+ + a_-|^2(\omega N)^4(1 + o(1)) + R^N$$

By assumption, $|\omega N| \gg \delta$, $|\omega N| \gg N^{-1/2}$, so that

$$R_N = O((\delta\omega N)^2) + O(\delta^4) + O(N^{-2}) = o((\omega N)^4)$$

and

$$\|s_\theta^\perp\|^2 > c(\omega N)^4 > cN^{4(1-\alpha_1)},\quad N > N_0$$

Statement 4.5: Let $h < N^{\gamma-\alpha_1}$ *and* $|\omega - \omega_0| < N^{-\alpha_2}$, *where* $1 < c(\gamma) < \alpha_2 \leq \alpha_1 \leq \gamma$. *Then*

$$\|s_{\theta'}^\perp\|^2 = \frac{1}{6!}|L_2 - L_0\delta^2|^2 + \begin{cases} o(N^{4(1-\gamma)}), \quad c(\gamma) = (1+2\gamma)/3 & (4.19a)\\ \varphi(\omega, \delta, \delta_0)/8! + o(N^{(}1-\gamma)6) \\ \quad c(\gamma) = (1 + 3\gamma)/4, & (4.19b)\end{cases}$$

where

$$\varphi = 0.4|L_3 - L_1\delta^2|^2 - \frac{2}{3}\mathrm{Re}[(3L_4 - 7L_2\delta^2 + 4L_0\delta^4)(\bar{L}_2 - \bar{L}_0\delta^2)]$$

and

$$L_p = a_+(\omega N - \delta_0)^p + a_-(\omega N + \delta_0)^p \tag{4.20}$$

The remainder terms are uniform over $a_\pm$, h_0, namely, $\{|a_\pm|, h_0\} < f$. *If, in addition,* $|a_+ + a_-|^{-1} < f$, $h_0 > \varepsilon$, $|h - h_0| > u$, $|\omega - \omega_0| > uN^{-\gamma}$, *then for* $N > N_0$

$$\|s_\theta^\perp\|^2 > \text{const}\begin{cases} N^{4(1-\gamma)}, & |\mathrm{Im}(a_+\bar{a}_-)| > \varepsilon_1 & (4.21a)\\ N^{6(1-\gamma)}, & \mathrm{Im}(a_+\bar{a}_-) = 0, \\ & |a_+ a_-| > \varepsilon_1 & (4.21b)\end{cases}$$

Proof:

1. The derivation of (4.19) is mainly given by (4.10), (4.11), (4.17):

$$\|s_{\theta'}^\perp\|^2 = |a_+ + a_-|^2 + 2\mathrm{Re}(a_+\bar{a}_-)\sum_{1\leq k\leq 3}(-\delta_0^2)^k/(2k+1)! + O(\delta_0^8) - \sum_{1\leq k\leq 3}(s,\ U_k s)(-\delta^2)^k/(2k)! + O(\delta^8) + O(N^{-2})$$

where

$$(s,\ U_k s) = \sum_{0\leq p+l\leq 2k+2} (t^p e_\omega,\ s)(s,\ t^l e_\omega) v_{p,l}^{(k)} \qquad p + l = \text{even} \tag{4.22}$$

and the matrices $[v_{p,l}^{(k)}]$ are real-valued and symmetric, $v_{p,l}^{(k)} = v_{l,p}^{(k)}$ (because the projection operator $P_{(0,h)}$ is self-adjoint and real-valued, see (4.11), (4.15), (4.16));

$$(t^p e_\omega,\ s) = \sum_{\substack{0\leq n\leq 6\\ p+n=\text{even}}} L_n i^n \langle t^{n+p}\rangle/n! + O((\omega N)^{7+\varepsilon})$$

$$\varepsilon = \begin{cases} 1 & p+n \text{ even}\\ 0 & p+n \text{ odd}\end{cases} \tag{4.23}$$

the L_n being defined by (4.20).

Expansions (4.23) are multiplied in (4.22) for exponents p, l of the same parity. Therefore,

$$(s,\ U_k s) = \sum_{0\leq n+m\leq 6} L_n \bar{L}_m c_{n,m}^{(k)} + O((\omega n)^8),\quad n + m \text{ even}$$

The remainder terms $O(\delta^8)$ and $O((\omega N^8)$ are of order $o(N^{(1-\gamma)6})$ when $1+3\gamma < 4\alpha_2$. Similarly, $O(\delta^6)$ and $O((\omega N)^6)$ are of order $o(N^{(1-\gamma)4})$ when $1+2\gamma < 3\alpha_2$. The latter case is necessary to derive (4.19a). Lastly, $O(N^{-2}) = o(N^{(1-\gamma)8})$ when $\gamma < 5/4$.

Thus, some terms in (4.22) can be omitted when deriving (4.19):

$$\delta^{2k}(s,\ U_k s) = \delta^{2k}\sum_A L_n\bar{L}_m c^{(k)}_{n,m} + o(N^{(1-\gamma)6}) \qquad (4.24)$$

where $A = \{n,\ m : 0 < n+m \le 6-2k,\ n+m = \text{even}\}$. Matrices $[v^{(k)}_{p,l}]$ are found from (4.11) in explicit form for $k = 0,\ 1,\ 2$. The case $k = 3$ involves cumbersome calculations, but these can easily be avoided. From (4.24) it follows that the contribution from (4.24) with $k = 3$ is given by the term $\delta^6 \times |L_0|^2 \times c^{(3)}_{0,0}$ where the constant alone is unknown.

The above algorithm yields a polynomial $P(\omega N,\ \delta,\ \delta_0)$ of degree 6 to fit $\|s^{\perp}_{\theta'}\|^2$, the accuracy of this approximation being $O(N^{(1-\gamma)8})$ when $\alpha_1 = \alpha_2 = \gamma$. Since $\|s^{\perp}_{\theta'}\|^2 = 0$ for $\omega = 0$, $\delta = \delta_0$, it follows that $P(0,\ \delta_0,\ \delta_0) = 0$ is an equation in $c^{(3)}_{0,0}$.

Note another simplifying circumstance. The moments $\langle t^k\rangle$ in (4.23) are given by (4.3) to within $O(N^{-1})$ or $o(N^{(1-\gamma)4})$ if $\gamma < 5/4$. Hence approximate values of the moments can be used to calculate (s, U_2, s). In fact, this statement is true for $(s,\ U_i s)$, $i = 0,\ 1$.

The results are summarized in (4.19). The next step is to derive (4.21).

2. The principal term in (4.19a) can be represented in the form

$$\varphi_0 = |L_2 - L_0\delta^2|^2 = |A_+ x - A_- y|^2, \quad A_\pm = a_+ \pm a_-$$

where $x = \omega N + \delta_0^2 - \delta^2$, $y = 2(\omega N)\delta_0$. If $\mathrm{Im}(a_+\bar{a}_-) \neq 0$, then $\varphi_0 = 0$ at the single real-valued point $\omega = 0$, $\delta = \delta_0$. Contours of φ_0 as functions of x, y are ellipses. For this reason, when $|y| > y_0$,

$$\varphi_0 > \min_x |A_+ x - A_- y_0|^2 = y_0^2 |2\mathrm{Im}(a_+\bar{a}_-)|^2/|a_+ + a_-|^2$$

However, $y_0 = 2uh_0 N^{2(1-\gamma)}$. Therefore,

$$|L_2 - L_0\delta^2| > cN^{4(1-\gamma)}$$
$$\text{for } \{h_0,\ |\mathrm{Im}(a_+\bar{a}_-)|,\ |a_+ + a_-|\} > \varepsilon$$

which proves (4.21a).

Let $\mathrm{Im}(a_+\bar{a}_-) = 0$. Rewrite (4.19b) using the notation $B_k = L_k - L_{k-2}\delta^2$:

$$\begin{aligned}15\cdot 8!\ \|s^{\perp}_{\theta'}\|^2 &= 15\cdot 56|B_2|^2 + 6|B_3|^2 - 30\mathrm{Re}B_4\bar{B}_2 \\ &\quad +40|B_2|^2\delta^2 + o(N^{(1-\gamma)6}) \\ &= 15\cdot 41|B_2|^2 + 6|B_3|^2 + |15B_2 - B_4|^2 \\ &\quad +40|b_2|^2\delta^2 + o(N^{(1-\gamma)6})\end{aligned}$$

Hence

$$\|s^{\perp}_{\theta'}\|^2 \ge C(|B_2|^2 + |B_3|^2) + o(N^{(1-\gamma)6})$$

If $|B_2| < c_1 N^{(1-\gamma)3}$, then $\|s^{\perp}_{\theta'}\|^2 \ge c_2 N^{(1-\gamma)6}$, $N > N_0$. Let $|B_2| < c_1 N^{(1-\gamma)3}$. We have

$$\begin{aligned}|B_3| &= |(\omega N - \delta_0(a_+ - a_-)/(a_+ + a_-))B_2 \\ &\quad +8N\omega\delta_0^2 a_- a_+/(a_+ + a_-)| \\ &> |N\omega|\cdot|8h_0^2|a_+^{-1} + a_-^{-1}|^{-1} - c_1 N^{1-\gamma)}|N^{2(1-\gamma)} - c_1 h_0| \\ &\quad \times(a_+ - a_-)/(a_+ + a_-)|N^{4(1-\gamma)}\end{aligned}$$

It follows that, when $|h_0| > \varepsilon$, $|a_+^{-1} + a_-^{-1}| < f$, and $|\omega| > uN^{-\gamma}$, the estimate $|B_3| > cN^{(1-\gamma)3}$, $N > N_0$ and so (4.21b) are true.

The following theorem summarizes the result of this Section.

Theorem 2: Let

$$1 < \gamma < 5/4 \quad \textit{and}$$
$$\max(|a_\pm|, |a_+ + a_-|^{-1},\ h_0,\ h_0^{-1}) < f$$

Then the following estimate holds

$$\|s^{\perp}_{\theta'}\|^2 > cN^{(1-\gamma)k}, \quad N > N_0$$

on the set $\{\omega,\ h :\ |\omega - \omega_0|_* > uN^{\gamma-\alpha_1}\}$ *with*
$k = 4$, *if* $|\mathrm{Im}(a_+\bar{a}_-)| > \varepsilon$ *and* $1 + 2\gamma < 3\alpha_1 < 3\gamma$
$k = 6$, *if* $\mathrm{Im}(a_+\bar{a}_-) = 0$, $|a_+ a_-| > \varepsilon$
and $1 + 3\gamma < 4\alpha_1 < 4\gamma$.

Remark: The result stated above is the last link in the proof of Theorem 1 (see Lemma 3.4). To obtain the formulation of Theorem 1, set $\alpha = \gamma - \alpha_1$.

5. NECESSARY CONDITIONS FOR FREQUENCY RESOLUTION

These conditions in problem (1.1), (1.2) are derived from an analysis of the inverse Fisher matrix. The analysis involves ideas from Gertsik's algorithm. The algorithm yields the principal asymptotic term in the inverse Gram matrix when it is close to being degenerate. The restrictions on γ are deduced from the fact that only one parameter, h, can be estimated consistently.

It is more convenient to use another signal parameterization in this section:

$$\theta = (\sigma^2, A_+, A_-, \omega, h) \in R_+ \times C \times C \times S^1 \times R_+ = \Theta$$

where $A_\pm = a_+ \pm a_-$.

Theorem 3: Let $G_1 \in \Theta \cap \{\theta : \mathrm{Im}(A_+\bar{A}_-) \neq 0\}$ *and* $G_2 \in \Theta \cap \{\theta : \mathrm{Im}(A_+ A_-) = 0\}$ *be some open parameter subsets in (1.1), (1.2). If h can be estimated consistently for any* $\theta \in G_i$ *in (1.1), (1.2) with* $\gamma > 1$ *then* $\gamma < 5/4$ *for* i = 1 *and* $\gamma < 7/6$ *for* i = 2.

Lemma 5.1: Assume that $\hat{h}$ *is an estimate of h that is consistent for* $\theta \in G_1$, $E_\theta|\hat{h}|^2 < \infty$ *and* $m(\theta) = E_\theta\hat{h}$ *are differentiable. Then the Rao-Cramer inequality holds:*

$$E_\theta|\hat{h} - h|^2 \ge |m(\theta) - h|^2 + \nabla^* m(\theta)J_N^{-1}(\theta)\nabla\bar{m}(\theta) \quad (5.1)$$

where $\nabla^* = \{\bar{\Delta}_i\}$ *is the row-operator*

$$\nabla = \{\nabla_i\}$$

$$= (\partial/\partial A_+,\ \partial/\partial A_-,\ \partial/\partial \bar{A}_+,\ \partial/\partial \bar{A}_-,\ \partial/\partial \omega,\ \partial/\partial h)'$$

$J_N(\theta)$ *is the Fisher matrix for the parameters* $(A_+,\ A_-,\ \omega,\ n)$:

$$J_N(\theta) = \sigma^{-2} N[(\nabla_i s,\ \nabla_j s) + (\nabla_i \bar{s},\ \nabla_j \bar{s})]_{i,j=1,\dots,6} \quad (5.2)$$

Proof: Inequality (5.1) states an obvious geometrical fact: the element $\eta = \hat{h} - m(\theta)$ in the Hilbert space of centered random variables with the norm $\|\cdot\| = E|\cdot|^2$ has a greater norm than its projection on the hyperplane spanned by

$$\xi_i = \nabla_i \ln p(x_1, \dots, x_N|,\ \theta), \quad E\xi_i = 0$$

that is,

$$E|\eta|^2 \geq \sum E\eta\bar{\xi}_i[E\xi_i\bar{\xi}_i]^{-1}E\xi_i\bar{\eta}$$

Here

$$P_N(x|\theta) = c_N(\sigma)\exp\left\{\sum_{0\leq n<N} |x(n) - s(n)|^2/\sigma^2\right\}$$

is the likelihood function. Hence

$$\xi_i = N\sigma^{-1}[(\nabla_i s,\ w) + (w,\ \nabla_i s)] \quad (5.3)$$

In the region where $m(\theta)$ is differentiable

$$E\xi_i\bar{\eta} = E_\theta(\nabla_i p_N)\cdot \bar{\eta} p_N^{-1} = \nabla_i E_\theta \bar{\eta} = \nabla_i m(\theta)$$

The elements of $J_N(\theta) = E\xi_i\bar{\xi}_j$ are found from (5.3) and from the fact that

$$E|(a,\ w)|^2 = N^{-1}(a,\ a), \quad E(a,\ w)(b,\ w) = 0$$

for complex-valued white noise w.

Asymptotics of $J_N^{-1}(\theta)$. The case $\mathrm{Im}(a_+\bar{a}_-) \neq 0$. From (5.2) it follows that the analysis of the Fisher matrix requires s to be fixed apart from the factor z_N, $|z_N| = 1$. For this reason we represent s, as in Section 4, in the form

$$s(t) = (A_+ \cos\delta t + iA_- \sin\delta t)\exp(i\omega N t)$$
$$\delta = hN^{1-\gamma}$$

where t takes on the values $t_n = (n - \frac{1}{2}(N-1))/N$ and $A_\pm = a_+ \pm a_-$. Hence ∇_s is a column vector of the form

$$\nabla s = \{e_0 \cos\delta t;\ ie_0\sin\delta t;\ 0;\ 0;\ i(A_+\cos\delta t + iA_-\sin\delta t)Ne_1;\ (-A_+\sin\delta t + iA_-\cos\delta t)N^{-\gamma}e_1\}'$$

where $e_p = (it)^P\exp(i\omega N t)$.

For $\cos\delta t$ and $\sin\delta t$ we use expansions that are uniform over $h < H$:

$$\cos\delta t = 1 - (\delta t)^2/2 + O(\delta^4), \quad \sin\delta t = \delta t + O(\delta^3)$$

In that case the elements ∇s are decomposed into three basic functions $e = \{e_0(t),\ e_1(t),\ e_2(t)\}$, i.e.,

$$\nabla s = D_1\{AD_2 e + O(\delta^3)\}$$

where

$$D_1 = \mathrm{diag}(1,\ 1,\ 1,\ 1,\ N^\gamma h^{-1},\ h^{-1}) \quad (5.4)$$

$$D_2 = \mathrm{diag}(1,\ \delta,\ \delta^2) \quad (5.5)$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & A_+ & A_- \\ -1/2 & 0 & 0 & 0 & A_- & A_+ \end{bmatrix}'$$

Similarly, $\nabla_s = D_1\{BD_2\bar{e} + O(\delta^3)\}$ where

$$B = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \bar{A}_+ & \bar{A}_- \\ 0 & 0 & -1/2 & 0 & \bar{A}_- & \bar{A}_+ \end{bmatrix}'$$

Hence

$$\sigma^2 J(\theta) = ND_1[AD_2G_ND_2A^* + BD_2G_ND_2B^* + R_N]D_1 \quad (5.6)$$

where

$$G_N = [(e_k, e_r)]_{k,r=0,1,2} = [i^{k-r}\langle t^{k+r}\rangle] = \begin{bmatrix} 1 & 0 & -1/12 \\ 0 & 1/12 & 0 \\ -1/12 & 0 & (2^4\times 5)^{-1} \end{bmatrix} + O(N^{-1})$$

$$R_N = \delta^3(AD_2E_1 + E_1^*D_2A^* + BD_2E_2 + E_2^*D_2B^*) + O(\delta^6)$$

Here E_i is a 3×6 matrix with uniformly bounded elements.

Now we define the matrix block

$$D = \mathrm{diag}(D_2,\ D_2), \quad \Gamma_N = \mathrm{diag}(G_N,\ G_N)$$

$$C = [A \vdots B], \quad E = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}$$

In that case (5.6) can be written in a more compact form

$$\sigma^2 J_N(\theta) = ND_1[CD\Gamma_N DC^* + R_N]D_1 \quad (5.7)$$

where $R_N = \delta^3(CDE + E^*DC^*) + O(\delta^6)$.

All matrices here are square, 6×6. Of these, D and Γ_N are evidently invertible. We now show that C is also invertible, provided $\mathrm{Im}A_+\bar{A}_- \neq 0$.

A circular permutation of rows (2.3) and columns (2, 3, 5, 4) in C converts it to a block triangular form:

$$C_1 = \begin{bmatrix} I & 0 & \frac{1}{2}I \\ 0 & I & 0 \\ 0 & \alpha & \beta \end{bmatrix}$$

where

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \alpha = \begin{bmatrix} A_+ & \bar{A}_+ \\ A_- & \bar{A}_- \end{bmatrix}, \quad \beta = \begin{bmatrix} A_- & \bar{A}_- \\ A_+ & \bar{A}_+ \end{bmatrix}$$

and

$$C_1 = (2,3) \times C \times (2,3,5,4)$$

The factors in C_1 correspond to matrices that perform circular permutations of columns according to the cycles shown in parentheses. It is now easy to invert C:

$$C^{-1} = (2,3,5,4) \times C_1^{-1} \times (2,3) \quad (5.8)$$

$$C_1^{-1} = \begin{bmatrix} I & \frac{1}{2}\beta^{-1}\alpha & -\frac{1}{2}\beta^{-1} \\ 0 & I & 0 \\ 0 & -\beta^{-1}\alpha & \beta^{-1} \end{bmatrix} \quad (5.9)$$

where β^{-1} is defined, if $\det\beta = 2i\mathrm{Im}(A_+\bar{A}_-) \neq 0$. Setting

$$\varphi = \sigma N^{-1/2} D^{-1} C^{-1} D_1^{-1} \quad (5.10)$$

we have

$$\begin{aligned}\varphi J_N(\theta)\varphi^* &= \Gamma_N + (EC^{*-1}D^{-1} + D^{-1}C^{-1}E)\delta^3 + O(\delta^2) \\ &= \Gamma + O(N^{-1}) + O(\delta), \quad \Gamma = \Gamma_\infty\end{aligned}$$

The remainder term is uniform over the parameters, if

$$|\mathrm{Im}(a_+\bar{a}_-)| > \varepsilon, \quad |a\pm| < c_a, \quad 0 < c_h < h < H \quad (5.11)$$

Therefore

$$\varphi^{*-1} J_N^{-1} \varphi^{-1} = \Gamma^{-1}(I + o(1)) > \Gamma_0, \quad N \to \infty$$

where

$$\Gamma_0 = [\delta_i^3\delta_j^3 + \delta_i^6\delta_j^6]_{i,j=1,\dots,6}, \quad \delta_i^j = 0;\ 1$$

and $\mathcal{A} \geq \mathcal{B}$, if the matrix $\mathcal{A} - \mathcal{B}$ is nonnegative definite. Really, since

$$\Gamma^{-1} = \mathrm{diag}(G, G)$$

$$G = \begin{bmatrix} 9 & 0 & 12 \\ 0 & 12 & 0 \\ 12 & 0 & 180 \end{bmatrix} > \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (5.12)$$

we get

$$J_N^{-1} > \varphi^*\Gamma_0\varphi = \sigma^2 h^{-4} D_1^{-1}[(3,2) \times U \times (3,2)] D_1^{-1} N^{4\gamma-5} \quad (5.13)$$

where U is a block matrix involving 2×2 blocks: $U = [0,\ \beta^{-1}\alpha,\ -\beta^{-1}]^* \times [0,\ \beta^{-1}\alpha,\ -\beta^{-1}]$. Hence $J_N^{-1} \to 0$, provided $\gamma < 5/4$.

We now state the above result, which is of interest in the study of limiting theorems for MLS estimators.

Statement 5.2: The inverse Fisher information matrix $J_N^{-1}(\theta)$ for parameters $(A_+,\ A_-,\ \omega,\ h)$ in (1.1), (1.2) with $\gamma > 1$ and $\mathrm{Im}(a_+\bar{a}_-) \neq 0$ can be factorized in the form $J_N^{-1} = \varphi^*(\Gamma^{-1} + o(1))\varphi$ where φ is given by (5.10), (5.8), (5.9), (5.4), (5.5), and $\Gamma^{-1} = \mathrm{diag}(G,\ G)$ is a scalar matrix (for G see (5.12)). The remainder term is uniform over parameters on compact sets from the intersection $\Theta \cap \{\theta : \mathrm{Im}(A_+\bar{A}_-) \neq 0\}$.

We have $J_N^{-1} \to 0$, provided $\gamma < 5/4$.

Proof of Theorem 3 ($\mathrm{Im}(A_+\bar{A}_-) \neq 0$): Let $\gamma \geq 5/4$ and suppose there is a consistent sequence of estimates $\hat{h}_N$ for the parameter h on some open subset $G_1 \subset \Theta \cap \{\theta : \mathrm{Im}(A_+\bar{A}_-) \neq 0\}$. Let G be a region with a compact closure from G_1. In that case $h < H$ on G. Consider a new sequence of estimates $\hat{h}_N^* = \min(\hat{h}_N, H)$. It is uniformly bounded and consistent for $\theta \in G$. Since the likelihood $p_N(x \mid \theta)$ is smooth over θ and $\hat{h}_N^*$ is bounded, it follows that $E_\theta h^* = m(\theta)$ is a smooth function of θ, and $E|\hat{h}_N^*| < H^2$. Hence Lemma 5.1 yields inequality (5.1). In view of the estimate (5.13), the Rao-Cramer inequality (5.1) can be continued:

$$E_\theta = |\hat{h}^* - h|^2 \geq |m(\theta) - h|^2 + c^2[\partial^* m(\theta)]U[\overline{\partial m(\theta)}] \quad (5.14)$$

where $c^2 = \sigma^2 h^{-4} N^{4\gamma-5} > c_0(G)$, $\gamma > 5/4$ and the operator

$$\begin{aligned}\partial &= (3;2) \times D_1^{-1} \times \nabla \\ &= (\partial/\partial A_+,\ \partial/\partial \bar{A}_+,\ \partial/\partial A_-,\ \partial/\partial \bar{A}_-, \\ &\qquad hN^{-\gamma}\partial/\partial\omega, h\partial/\partial h)'\end{aligned}$$

Let f be an eigenvector of the nonnegative definite matrix U having a maximum eigenvalue $\lambda > 0$. Let $f = (f_0,\ f_1,\ f_2)$, where f_i is of dimension 2. The equation $Uf = \lambda f$ can be written out as follows:

$$\begin{aligned} f_0 = 0,\ \alpha^* B^{-1}\alpha f_1 - \alpha^* B^{-1} f_2 &= \lambda f_1 \\ -B^{-1}\alpha f_1 - B^{-1} f_2 &= \lambda f_2 \end{aligned} \quad (5.15)$$

where

$$B = \beta\beta^* = 2\begin{bmatrix} |A_-|^2 & \mathrm{Re}(A_+\bar{A}_-) \\ \mathrm{Re}(A_+\bar{A}_-) & |A_+|^2 \end{bmatrix}$$

From the condition $\lambda \neq 0$ it follows that

$$\alpha^* f_2 + f_1 = 0, \quad \alpha^* = \begin{bmatrix} \bar{A}_+ & \bar{A}_- \\ A_+ & A_- \end{bmatrix} \quad (5.16)$$

In that case, however, the last of (5.15) yields an equation in f_2:

$$(I + A) f_2 = \lambda B f_2, \quad \|f_2\| = 1 \quad (5.17)$$

where

$$A = \alpha\alpha^* = 2\begin{bmatrix} |A_+|^2 & \mathrm{Re}(A_+\bar{A}_-) \\ \mathrm{Re}(A_+\bar{A}_-) & |A_-|^2 \end{bmatrix}$$

A and B are two real-valued symmetric matrices which are positive if $\mathrm{Im}(A_+\bar{A}_-) \neq 0$. Hence (5.17) has two solutions, real-valued vectors with the eigenvalues λ_1, λ_2. It is easily found from (5.17) that

$$\lambda_1\lambda_2 = 1 + \frac{1}{2}(|A_+|^2 + |A_-|^2 + \frac{1}{2})(\mathrm{Im}A_+\bar{A}_-)^{-2} > 1$$

so that $\lambda = \max\lambda_i > 1$.

If vector $f_2 = (u,\ v)$ and $\mathrm{Re}(A_+\bar{A}_-) \neq 0$, then $v = v(A_+,\ A_-) \neq 0$. Otherwise the second scalar equation of (5.17) would have given $\mathrm{Re}(A_+\bar{A}_-)(1-\lambda)u \neq 0$. Actually, however, we have $u \neq 0$, since $f_2 \neq 0$ and $1-\lambda \neq 0$. Hence

$$U \geq \lambda f \times f^* > f \times f^* \tag{5.18}$$

Using (5.16),

$$f = (0;\ 0;\ -(\bar{A}_+ u + \bar{A}_- v);\ -(A_+ u + A_- v);\ u;\ v)$$
$$\|f\| = 1$$

where $A_+ u + A_- v$ and v are nonzero, if $\mathrm{Im}(A_+\bar{A}_-) \neq 0$ and $\mathrm{Re}(A_+\bar{A}_-) \neq 0$.

An eigenvalue λ can be multiple only on the set $(A_+,\ \bar{A}_+,\ A_-,\ \bar{A}_-)$ with the codimension 1. For this reason u and v are smooth functions of $A_\pm$, $\bar{A}_\pm$ outside of this set.

Using (5.18), one can continue (5.14)

$$E_\theta = |\hat{h}^* - h|^2 \geq |m(\theta) - h|^2 + c_0|\mathcal{D}m(\theta)|^2$$

where $\mathcal{D} = f^* \times \partial$ is a differential operator of degree one

$$\mathcal{D} = -2\mathrm{Re}(A_+ u + A_- v)\partial/\partial A_- + uN^{-\gamma} h\partial/\partial\omega + vh\partial/\partial h$$

Let θ_0 be an internal point in a generic parameter set, i.e., $\mathrm{Re}(A_+\bar{A}_-) \neq 0$, $\mathrm{Im}(A_+\bar{A}_-) \neq 0$, λ has the multiplicity 1. In that case through that point passes the only characteristic θ_t of operator $\mathcal{D}$ for which $\theta_t \subset G$ with $0 < t < \tau$. We have $\mathcal{D}m(\theta_t) = dm(\theta_t)/dt$ on the characteristic. The fact that $\hat{h}^*$ is consistent and bounded yields $E_\theta|\hat{h}^* - h|^2 \to 0$, $N \to \infty$. From (5.19) we also get $\mathcal{D}m(\theta_t) \to 0$, $m(\theta_t) \to h(t)$.

Estimates (5.19) and $E|\hat{h}^* - h|^2 < 4H^2$ show that $\mathcal{D}m(\theta_t)$ is uniformly bounded over N, θ. For this reason the Lebesgue theorem yields

$$\lim_{N\to\infty} \int_0^\tau \mathcal{D}m(\theta_t)dt = \int_0^\tau \lim \mathcal{D}m(\theta_t)dt = 0$$

On the other hand, the above limit equals

$$\lim[m(\theta_\tau) - m(\theta_0)] = h(\tau) - h(0)$$

where $h' = vh$ and $v \neq 0$. In other words, the limit is

$$h(0)\left[\exp \int_0^\tau v(A_+(0), A_-(t))dt - 1\right]$$

and is different from zero with a suitable choice of $\tau > 0$.

The assumption $\gamma \geq 5/4$ has thus led to a contradiction.

Case of equal phases, $\mathrm{Im}(A_+\bar{A}_-) = 0$. The analysis is similar, so we shall only dwell on the points that differ and omit details. The signal can be represented in the form

$$s(t) = (\tilde{A}_+ \cos\delta t + i\tilde{A}_- \sin\delta t)\exp(i\omega N t + i\varphi)$$

where $\varphi = \arg A_+$ is the common initial phase of the harmonics, for definiteness specified as $\tilde{A}_+ = |a_+ + a_-| > 0$. We are going to use the Rao-Cramer inequality similar to (5.1) for the real-valued parameters $\theta_2 = (\tilde{A}_+,\ \tilde{A}_-,\ \omega, h)$ (the subscript of θ will be omitted below).

As above, we expand s in the small parameter δ up to the order δ^3 inclusive. We have

$$\partial s/\partial\theta = D_1\{CDe + O(\delta^4)\}$$

in the basis $e = (e_0,\ e_1,\ e_2,\ e_3/2)$ where now $e_p = (it)^p \exp(i\omega N t + i\varphi)$; here

$$D_1 = \mathrm{diag}(1;\ 2/3;\ h^{-1}N^\gamma;\ h^{-1})$$
$$D = \mathrm{diag}(1;\ \delta;\ \delta^2;\ \delta^3)$$

$$C = \begin{bmatrix} 1 & 0 & 1/2 & 0 \\ 0 & 3/2 & 0 & 1/2 \\ 0 & \tilde{A}_+ & \tilde{A}_- & \tilde{A}_+ \\ 0 & \tilde{A}_- & \tilde{A}_+ & \tilde{A}_- \end{bmatrix} = \begin{bmatrix} I & \frac{1}{2}\mathcal{I} \\ 0 & \alpha \end{bmatrix} \begin{bmatrix} I & 0 \\ \mathcal{I} & I \end{bmatrix}$$

We are using triangular block matrices involving 2×2 blocks:

$$\alpha = \begin{bmatrix} \tilde{A}_- & \tilde{A}_+ \\ \tilde{A}_+ & \tilde{A}_- \end{bmatrix},\ \mathcal{I} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This makes C easier to invert:

$$C^{-1} = \begin{bmatrix} I & 0 \\ -\mathcal{I} & I \end{bmatrix} \begin{bmatrix} I & -\alpha^{-1}/2 \\ 0 & \alpha^{-1} \end{bmatrix}$$

The matrix C is invertible when $\det\alpha = |\tilde{A}_-|^2 - |\tilde{A}_+|^2 = -4a_+\bar{a}_- \neq 0$.

The Fisher matrix can be represented similarly to (5.6):

$$\sigma^2 J_N = 2ND_1[CD\Gamma_N DC^{-1} + \delta^4 R_N]D_1$$

where R_N has a structure analogous to (5.6), while $\Gamma_N = [\langle t^{k+l}\rangle c_k \bar{c}_l]_{k,l=0,\ldots,3}$, where $c_k = i^k$ for $k < 3$ and $c_k = i^k/2$ for $k = 3$. Since $\langle t^{k+l}\rangle = 0$ for odd values of $k + l$, the matrix Γ_N is decomposable into a direct sum of 2×2 matrices, hence its determinant can be easily calculated:

$$\det\Gamma_N = \frac{1}{4}(\langle t^4\rangle - \langle t^2\rangle^2)(\langle t^6\rangle\langle t^2\rangle - \langle t^4\rangle^2)$$
$$= 3(20 \times 10!)^{-1} + O(N^{-1})$$

Similarly to the above, we get

$$J_N^{-1} = 2^{-1}N^{-1}\sigma^2 D_1^{-1}C'^{-1}D^{-1} \times[\Gamma^{-1} + o(1)]D^{-1}C^{-1}D_1^{-1}, \quad \Gamma = \Gamma_\infty$$

where the estimate is uniform over the parameters

$$0 < c_h < h < H;\quad 0 < c_a < |a_\pm| < C_a$$

and $\Gamma^{-1} + o(1) > \|\delta_i^4\delta_j^4\|$ when $N \to \infty$. It follows that

$$J_N^{-1} > \frac{1}{2}\sigma^2 D_1^{-1} f \cdot f^* D_1^{-1} \times N^{7-6\gamma}h^{-6}$$

where the vector f corresponds to row 4 in C^{-1}, i.e.,

$$f = (0;\ -1;\ \frac{3}{2}\tilde{A}_+\Delta^{-1};\ -\frac{3}{2}\tilde{A}_-\Delta^{-1})$$
$$\Delta = A_+^2 - A_-^2 = 4a_+\bar{a}_-$$

Similarly to the above, assuming that $\gamma > 7/6$ and that there exists a consistent estimate for $h \in [c_h,\ H]$, the other parameters belonging to the set

$$\{\sigma > \sigma_0,\ c_a < |a_\pm| < C_a, |a_+ - a_-| > \varepsilon\}$$

we arrive at a contradiction.

CONCLUSION

The main Theorems 1 and 3 reveal the utmost capabilities of harmonic analysis to resolve closely spaced frequencies from "short" samples. For a generic signal, two peak frequencies ω_i can be resolved provided $N^{5/4}|\omega_1 - \omega_2| \gg 1$. The highest order of closeness for the frequencies is considerably reduced (from 5/4 to 7/6) for techniques based on power spectra. This is caused by the loss of phase, which is typical of correlation methods in practical use.

We have not dealt with the important problem concerning the asymptotic distribution of the optimal frequency estimators. Peculiar situations can arise here, as can be seen from our analysis of the method of harmonic decomposition [Molchan and Newman, 1988]. That question is treated in the paper which follows.

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