Analysis of seismic wave dynamics by means of integral representations and the method of discontinuities

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The following paper received the Best Student Poster Paper award at the 1999 SEG meeting. Normally, it would be published in THE LEADING EDGE, but it is presented here because of its quantitative nature. The paper has been revised slightly from its original form, but it has not undergone peer review.

Analysis of seismic wave dynamics by means of integral representations and the method of discontinuities

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ABSTRACT

We analyze the dynamics (amplitudes and phase distortions) of seismic waves as they propagate along the ray. Our analysis is performed via a ray series approximation in the time domain. That is, we concentrate on characterizing the sharp changes (discontinuities) of the signal that are localized near the wavefront. After convolution of the terms of such a series with a proper temporally short (high-frequency) wavelet, one obtains a synthetic seismic signal at a given point of interest. We present an outline of the proposed technique that yields integrals describing the wavefield. These integrals are similar to oscillatory integrals in the frequency domain. This description is uniformly valid near caustics, allowing the calculation of higher order terms of the ray series approximation. Practical use of the technique is illustrated by several examples which show two possible uses of the technique: general understanding of what is happening during wave propagation and practical calculations. First, we show how the structure of the ray decomposition changes near the simple caustic, and then we calculate a synthetic signal near the cusp caustic. The advantage of the technique is that the problem of seismic wave calculation is technically reduced to a problem of double integration of a Dirac δ-function; thus, it is computationally effective.

INTRODUCTION

Geometrical ray tracing, combined with asymptotic methods for estimating the wave amplitudes along these rays, is widely used in seismic studies. Such asymptotic methods are used for both the forward modeling of seismic waves as well as for the generation of Green's functions for some implementations of Kirchhoff migration. The ray method has some advantages over numerical techniques in that ray tracing is computationally efficient, and the interpretation of the results is easy and self-evident, owing to the fact that the concepts of rays and wavefronts are intuitively understandable to everyone.

The usual way to compute the various terms of the ray series is to solve the corresponding transport equations. In the standard implementation, only the first transport equation is taken into account. It is more difficult to calculate the other terms (the higher order transport equations). This is especially true in the vicinity of caustics where the standard ray method fails.

There are many techniques that allow the extension of the range of validity of such asymptotic approximations. It is possible to use an alternative series decomposition (a uniform asymptotic expansion) near a particular caustic (Ludwig, 1966). Other methods use more formal methods of describing the wavefield in a uniformly valid way in the vicinity of caustics (Hanyga, 1988). The following methods should be mentioned: the Maslov method (Maslov, 1972.) and the method of Gaussian beam summation.

In most cases, the uniform field representation is given in terms of an oscillatory integral of the form:

$$\mathbf{u}(\mathbf{x}, \omega) \sim f(\omega) \int_{\mathbf{D}} \mathbf{a}(\mathbf{q}, \mathbf{x}) e^{i\omega\Phi(\mathbf{q},\mathbf{x})} \, d\mathbf{q}, \tag{1}$$

where $\mathbf{x} = (x, y, z)$ is a point in three-dimensional space, $\mathbf{u}(\mathbf{x}, \omega)$ denotes the displacement vector, $\mathbf{q}$ are the variables of integration, and $\mathbf{a}(\mathbf{q}, \mathbf{x})$ and $\Phi(\mathbf{q}, \mathbf{x})$ are, respectively, the amplitude and phase functions. In this formula, one can see a global integration over some domain $\mathbf{D}$; however, the main contribution to $\mathbf{u}(\mathbf{x}, \omega)$ comes from stationary points of the phase function $\Phi(\mathbf{q})$. For an isolated stationary point, this integral can be simplified to yield the standard ray method formulas. Near a caustic, however, several stationary points approach each other and finally coalesce at the caustic itself. Thus, for a uniform

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(asymptotic) description, the integration may be confined to the small region containing these coalescing stationary points.

Asymptotic methods are usually applied in the frequency domain, with frequency being assumed to be a large parameter ($\omega \to \infty$). In time-domain ray series approximation, we consider a small time interval containing the wavefront $[t-\tau(\mathbf{x}) \to 0]$, where the wavefront surface satisfies $t = \tau(\mathbf{x})$. The time-domain ray series is a sum of terms consisting of discontinuous functions that are best understood as distributions (Alekseev et al., 1961). The method that we call "method of discontinuities" (Goldin, 1989) is a wavefield description by the truncated ray series in the time domain:

$$\mathbf{u}(\mathbf{x}, t) \overset{q+p}{\sim} \sum_{n=0}^{p} \mathbf{U}^{(n)}(\mathbf{x}) R^{(+)}_{q+n,\nu}(t - \tau(\mathbf{x})), \tag{2}$$

where $\mathbf{U}^{(n)}(\mathbf{x})$ and $\tau(\mathbf{x})$ are, respectively, the amplitudes and eikonal of the wave. The definition of the discontinuity $R^{(+)}_{q,\nu}(t)$ is the following:

$$R^{(+)}_{q}(t) = \begin{cases} t^q_+ / \Gamma(q+1), & q \neq -1, -2, \ldots, \\ \delta^{(-q+1)}(t), & q = -1, -2, \ldots, \end{cases}$$
$$R^{(+)}_{q,\nu}(t) = \mathbf{H}^{\nu}\left[R^{(+)}_{q}(t)\right], \tag{3}$$

where $\mathbf{H}^{\nu}$ is a fractional Hilbert operator, $\mathbf{H}^{\nu} = \mathbf{E}\cos(\pi\nu/2) + \mathbf{H}\sin(\pi\nu/2)$; and $\mathbf{E}$ and $\mathbf{H}$ are the identity and the Hilbert transforms, respectively. The symbol of equivalence $\overset{r}{\sim}$ means that the difference of equivalent functions has a smoothness of order higher than r. In turn, we say that the function $f(t)$ has smoothness of order r if $\mathbf{D}^r f(t) \in C$ is a smooth function. Operator $\mathbf{D}^r$ is a time differentiation (for fractional r one obtains fractional differentiation; negative r correspond to integration).

In fact, we neglect a smooth part of the seismic signal (Figure 1a) and characterize only sharp changes localized in a small vicinity of the wavefront (Figure 1b). Successful use of the ray method in seismics shows that the "small" vicinity is, in reality, not so small. The discontinuous wavefield description (2) can be regarded as an impulse response. Its convolution with an appropriate temporally short (high-frequency) wavelet yields a synthetic signal that is a good approximation for the entire signal shown on Figure 1a.

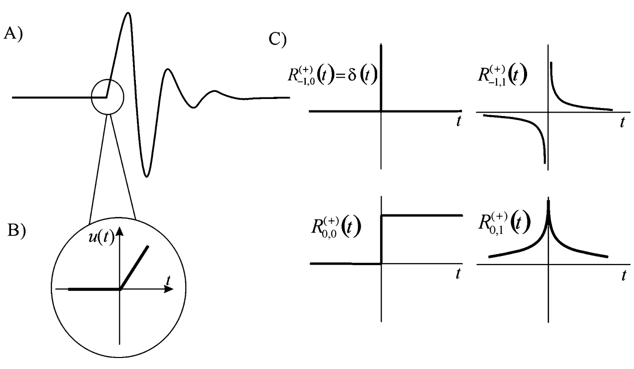

FIG. 1. The method of discontinuities. (a) The model of a seismic signal. (b) Zoom in of the wavefront area: Time-domain ray approximation of this signal. (c) Some examples of discontinuous functions used in the time-domain ray series.

In formula (2), p denotes the order of the ray approximation. The complete ray series ($p = \infty$) gives the full asymptotic description of the wavefield $\mathbf{u}(\mathbf{x}, t)$. In the standard ray method, the leading term only ($p = 0$) is taken into account. In this case, all discontinuities of order higher than q are considered smooth, and are, therefore, neglected.

Discontinuities defined in equations (3) are characterized by the order q and the index ν. The lower the order, the sharper the discontinuity. The value of the index corresponds to the value of the phase shift of the signal. This comes into play if the ray passes near the singularities of the wavefield. For example, when the seismic wave encounters a caustic, it undergoes a phase shift corresponding to the Hilbert transform of the initial signal. Some examples of discontinuities defined in equations (3) are shown in Figure 1c. The use of discontinuities is attractive because these functions are governed by the following simple rules of differentiation and convolution:

$$\mathbf{D}^p R^{(+)}_{q,\nu}(t - t_1(\mathbf{x})) = R^{(+)}_{q-p,\nu}(t - t_1(\mathbf{x})),$$
$$R^{(+)}_{q,\nu}(t - t_1(\mathbf{x})) * R^{(+)}_{p,\mu}(t - t_2(\mathbf{x})) = R^{(+)}_{q+p+1,\nu+\mu}(t - t_1(\mathbf{x}) - t_2(\mathbf{x})). \tag{4}$$

We will not solve differential transport equations. Instead, we propose to use a well-known integral representation of the wave field involving the Green's tensor:

$$u_k(\mathbf{x}_0, t_0) = \int_0^{t_0} dt \iint_S dS(\mathbf{x}) \big\{ G_{\ell k}(\mathbf{x}_0; \mathbf{x}, t_0 - t) \times \left[\mathbf{T_n}\mathbf{u}^{(in)}\right]_\ell(\mathbf{x}, t) - u^{(in)}_\ell(\mathbf{x}, t)[\mathbf{T_n}\mathbf{G}_k]_\ell(\mathbf{x}_0; \mathbf{x}, t_0 - t)\big\}, \tag{5}$$

where $u_k(\mathbf{x}_0, t_0)$ are the components of the displacement vector to be calculated at the point of interest $\mathbf{x}_0$, the Green's tensor $\mathbf{G}(\mathbf{x}_0; \mathbf{x}, t)$, and the initial wave $\mathbf{u}^{(in)}(\mathbf{x}, t)$ (known on the surface of integration S), and $\mathbf{T_n}$ is the differential operator assigning the Cauchy stress at the surface element $dS(\mathbf{x})$, with normal $\mathbf{n}$. We apply the Einstein summation convention when pairs of indices are equal.

The integral representation (5) can be transformed into the time-domain equivalent of the oscillatory integral (1). This can give us the uniform field representation we desire. We mention that the proposed technique is formally identical for both homogeneous and heterogeneous mediums. The only restriction is that we must know the ray-series approximation of the Green's tensor for the medium of interest to apply the technique.

MATHEMATICAL FORMULATION OF THE PROBLEM

Let us consider seismic P-wave propagation in a homogeneous isotropic medium. It is necessary to formulate the mathematical problem to be solved (Figure 2). We are interested in analyzing the wave dynamics (changes in the amplitudes and phases) along a given seismic ray. The coordinate system is oriented in such a way that the ray coincides with the z-axis but is oriented in the direction of decreasing z. The initial wave $\mathbf{u}^{(in)}(\mathbf{x}, t)$ is given on the plane $S: z = h$. In reality, it is enough to know only the Taylor decomposition of the wave amplitude $\mathbf{A}(x, y, h)$ and that of the travetime function $\tau(x, y, h)$

(the eikonal) in the vicinity of the point $\mathbf{x} = (0, 0, h)$. For a homogeneous medium, an exact form of the Green's function $\mathbf{G}(\mathbf{x}_0; \mathbf{x}, t)$ is known. We would like to calculate the displacement vector $\mathbf{u}$ at the point $\mathbf{x}_0 \equiv (0, 0, 0)$ of the same ray. While moving this point along the ray (by varying h) one can compute the seismic signal as it approaches a caustic.

THE TECHNIQUE OF THE SOLUTION

There are three main steps in the simplification of integral (5).

First, we introduce the ray series approximations of $\mathbf{u}^{(in)}(\mathbf{x}, t)$ and $\mathbf{G}(\mathbf{x}_0; \mathbf{x}, t)$ into the integrand of equation (5). It is possible to change the order of the integrations and to perform the time convolution first, taking into account the properties given by equation (4). We then rewrite the integration over S as an integration over the (x, y)-plane. This step is purely technical. After some simplification, we obtain the series of integrals,

$$u_k(0, 0, 0, t) \overset{q+p}{\sim} \sum_{n=0}^{p} I_k^{(n)}(t), \tag{6}$$

$$I_k^{(n)}(t) \equiv \iint_{z=h} L_k^{(n)}(x, y) R_{q+n,\nu}^{(+)}(t - \tilde{\tau}(x, y))\, dx\, dy, \tag{7}$$

where $\tilde{\tau}(x, y) = \tau(x, y) + T(x, y)$, $\tau(x, y) \equiv \tau(x, y, h)$ denotes the eikonal of $\mathbf{u}^{(in)}(\mathbf{x}, t)$ on the plane $z = h$, $T(x, y) \equiv T(x, y, h)$ denotes the eikonal of the Green's tensor $\mathbf{G}(0, 0, 0; \mathbf{x}, t)$ on the same plane, and $L_k^{(n)}(x, y)$ are smooth functions (see the explicit formulas in Goldin and Duchkov, 2000).

Second, let us deal with the integrals given by equation (7). The operators $\mathbf{D}^p$ and $\mathbf{H}^\nu$ were defined earlier. As has already been mentioned, we consider only the discontinuities and neglect smooth functions. In this case, the following equivalence is correct: $\mathbf{E} \sim (\mathbf{H}^{-\nu}\mathbf{D}^p)^{-1}(\mathbf{H}^{-\nu}\mathbf{D}^p)$, where $\mathbf{E}$ is an identity operator. Thus, all integrals from equation (7) may be reduced to a two-fold integral of a δ-function:

$$I_k^{(n)}(t) \sim (\mathbf{H}^{-\nu}\mathbf{D}^p)^{-1} \iint_{z=h} L_k^{(n)}(x, y)\delta(t - \tilde{\tau}(x, y))\, dx\, dy. \tag{8}$$

Third, in equation (8), the δ-function has a complicated argument $\tilde{\tau}(x, y)$. Different types of singularities of the ray system result in different specific properties of the function $\tilde{\tau}(x, y)$ in the vicinity of the coordinate origin (0, 0). Thus, the classification of the singularities may be reformulated in terms of classification of the Taylor series of $\tilde{\tau}(x, y)$. For example, a nondegenerate quadratic form obtained from truncating this series corresponds to regular points of the ray. Finally, therefore, we arrive at a purely mathematical problem: how to find correspondence between the Taylor series structure of the function $\tilde{\tau}(x, y)$ and the type of caustic. Fortunately, this problem has already been solved within the framework of catastrophe theory: Tom's theorem. For typical caustics, it is possible to transform the coordinates $(x, y) \to (\xi, \eta)$, so that the integral in equation (8) can be rewritten as

$$\int_{-\infty}^{\infty} d\eta \int_{-\infty}^{\infty} d\xi \hat{L}_k^{(n)}(\xi, \eta)\delta(t - \hat{\tau}(\xi, \eta)). \tag{9}$$

Here, $\hat{\tau}(\xi, \eta)$ is one of the standard polynomials listed in Tom's theorem (Poston and Steward, 1978). The function $\hat{L}_k^{(n)}(\xi, y)$ is smooth and can be replaced by its Taylor's series to any degree of desired approximation.

The integral in equation (8) is a time-domain equivalent of the oscillatory integral in equation (1). Near a particular caustic, this expression can be transformed into one of the standard integrals (9), yielding a uniform description of the wavefield. An advantage of our technique is that the problem of seismic wave computation is technically reduced to a problem of double integration of a δ-function.

In principal, the solution procedure described by formulas (7)–(9) is valid for an arbitrary medium, provided the Green's tensor is known. We need only know the structure of the function $\tilde{\tau}(x, y)$ along the ray. This is equivalent to knowing the higher order eikonal derivatives for the propagating wave. It is possible to use analytical expressions for such quantities. Continuation of the second eikonal derivative along the ray is a standard technique in seismic modeling. Formulas exist for continuation of the third derivatives (Goldin and Kurdyukova, 1994). In addition algorithms exist for deriving formulas for the higher order derivatives (Klimeš, 1999) as well.

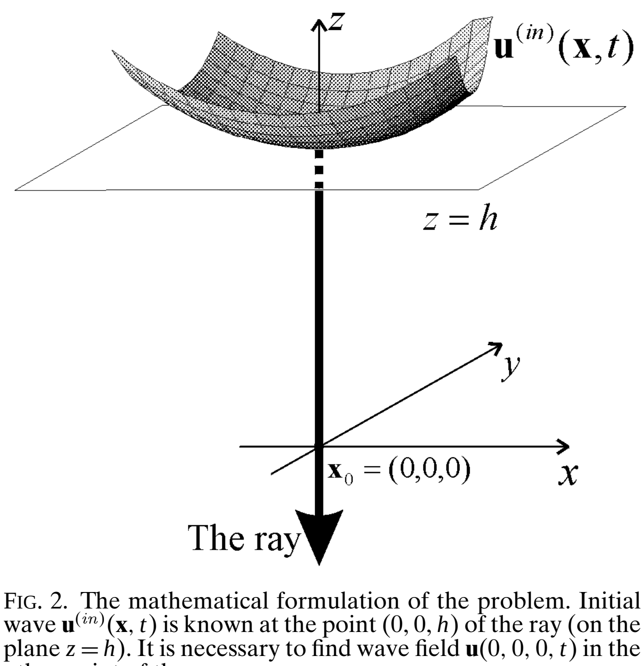

FIG. 2. The mathematical formulation of the problem. Initial wave $\mathbf{u}^{(in)}(\mathbf{x}, t)$ is known at the point $(0, 0, h)$ of the ray (on the plane $z = h$). It is necessary to find wave field $\mathbf{u}(0, 0, 0, t)$ in the other point of the ray.

APPLICATIONS OF THE TECHNIQUE

First-order ray approximation in the regular case

The proposed technique can be directly used for calculation of the higher order ray terms. For the first-order approximation, it is necessary to take into account the two first integrals from the series (6). At regular points of the ray, these integrals are taken explicitly, and the wavefield description takes the form

$$\mathbf{u}(0, 0, 0, t) \overset{q+1}{\sim} \mathbf{V}^{(0)}(\mathbf{b}; h) R_q^{(+)}(t - t_0) + \mathbf{V}^{(1)}(\mathbf{b}; h) R_{q+1}^{(+)}(t - t_0), \tag{10}$$

with analytic formulas for the amplitudes $\mathbf{V}^{(0)}(\mathbf{b}; h)$ and $\mathbf{V}^{(1)}(\mathbf{b}; h)$ (here h denotes the distance along the ray and $\mathbf{b}$ is the vector of initial parameters given on the plane $z = h$). For a complete listing of formulas, see Goldin and Duchkov (2000).

The formulas (10) alone constitute an improvement over traditional ray theory. Popov and Camerlynck (1996) use a ratio of amplitudes of the first and zeroth terms $|\mathbf{V}^{(1)}(\mathbf{x})|/|\mathbf{V}^{(0)}(\mathbf{x})|$ to demonstrate the limitations of the ray method. Their work is conducted in the frequency domain, where the ray series is asymptotically convergent; hence, the ray method is shown to be valid only when the ratio is small. For particular initial data $\mathbf{b}_0$, we have calculated the amplitudes $\mathbf{V}^{(\ell)}(h) \equiv \mathbf{V}^{(\ell)}(\mathbf{b}_0; h)$, where ℓ can take on the values of 1 or 2. Initial data are chosen so that the resulting ray passes a caustic. We have not introduced specific values of amplitudes because such information is not informative for our illustration of the method. The results of performing the computation are shown in Figure 3 (here h is distance from initial point on the ray). As h increases, the wave approaches the caustic. While the zeroth order term $\mathbf{U}^{(0)}(\mathbf{x})$ may still seem being reasonably well behaved (dashed line), the ratio $|\mathbf{V}^{(1)}(\mathbf{x})|/|\mathbf{V}^{(0)}(\mathbf{x})|$ is already growing (dotted line), indicating the region where standard ray formulas fail and must be replaced by uniformly valid formulas.

Structure of the ray series

Another interesting result was obtained for the wave field at a caustic. The result for the example of a simple caustic is

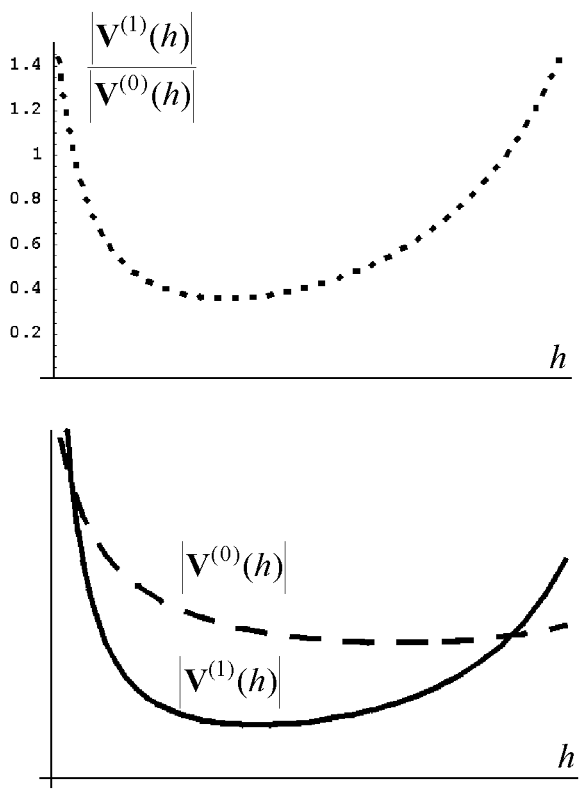

FIG. 3. Variations of the amplitudes of the two first ray-series terms. Horizontal axis shows the distance h from the initial point of the ray. Dashed line = the zeroth term amplitude $\mathbf{V}^{(0)}(h)$ [see series presentation (10)], solid line = the first term amplitude $\mathbf{V}^{(1)}(h)$, dotted line = the ratio $|\mathbf{V}^{(1)}(h)|/|\mathbf{V}^{(0)}(h)|$.

shown in Figure 4. Let us consider different points (the bold dots in Figure 4) of a ray that is tangent to a simple caustic. Far away from the caustic, we can calculate a valid asymptotic approximation of the wavefield using conventional ray series (discontinuities of order q, $q+1$, $q+2, \ldots$). At the caustic itself, such a decomposition does not work. Nevertheless, one can construct an alternative series representation with fractional increase of the discontinuity order: $q-1/6$, $q+1/6$, $q-1/6+1$, $q+1/6+1, \ldots$. Apparently, as we approach the caustic, each term of order q in the conventional ray series splits, producing two terms of respective orders of $q-1/6$ and $q+1/6$. Thus, instead of one conventional ray series, we obtain two series: $q-1/6$, $q-1/6+1, \ldots$ and $q+1/6$, $q\,m+1/6+1, \ldots$. As we move to points on the ray farther away from the caustic, it is again possible to obtain a valid representation via a conventional ray series representation (in the figure, we took $q = -1$). The same analysis has been done for other types of caustics.

Uniform description of a seismic signal near a cusp caustic

In the integral representation (8), the δ-function has a complicated argument containing $\tilde{\tau}(x, y)$. Different orders of singularity of the ray system result in specific properties of $\tilde{\tau}(x, y)$ in the vicinity of the coordinate origin (0, 0). Thus, singularity classification may be reformulated in terms of the polynomials produced by the Taylor representation of $\tilde{\tau}(x, y)$. For example, if the series can be truncated to produce a nondegenerate quadratic form, then this corresponds to regular points of the ray, and so forth. As we mentioned above, the purely mathematical problem becomes one of classifying the type of caustic via examination the structure of the Taylor series representation of $\tilde{\tau}(x, y)$, through the application of Tom's theorem from catastrophe theory.

In previous subsections, we described the wavefield far from any caustics (the regular points of the ray) and have discussed the structure of the series decomposition at a caustic itself. In each case, it was possible to evaluate the integral in equation (8) and to express the result explicitly in terms of the standard discontinuities defined in equations (3). However, in the vicinity of a caustic the standard series representation transforms into another type of series. Therefore, it is impossible to evaluate the integral explicitly. There is, however, the possibility of evaluating it numerically. In this way, we can still obtain the uniform wavefield description. Instead of the series represented in equation (2), the function $\mathbf{u}(0, 0, 0, t)$ is approximated by the series in equation (6). The integrals $I_k^{(n)}(t)$ are the "canonical integrals" of our method. In reality, $I_k^{(n)}(\mathbf{b}; t)$ also depends on the initial data, as was discussed earlier.

As an example, we consider a cusp caustic. We begin with the zeroth order ray-series approximation of the vertical component of the displacement vector $\mathbf{u}$. Far from the caustic, the canonical integral $I_3^{(0)}(\mathbf{b}, t)$ may be reduced to the standard discontinuity $R_q^{(+)}(t)$, while at the caustic itself, the canonical integral reduces to the discontinuity $R_{q-1/4}^{(+)}(t)$. To better visualize the wavefield behavior near the caustic, the discontinuities computed using the standard ray method and those computed using the canonical integral representation $I_3^{(0)}(\mathbf{b}, t)$ were convolved with a function $f(t) = e^{-t^2}\cos(3\pi t)$. The resulting signals are shown in Figure 5 at the distances 0.75, 0.15, and 0 wavelengths away from the caustic. The panels

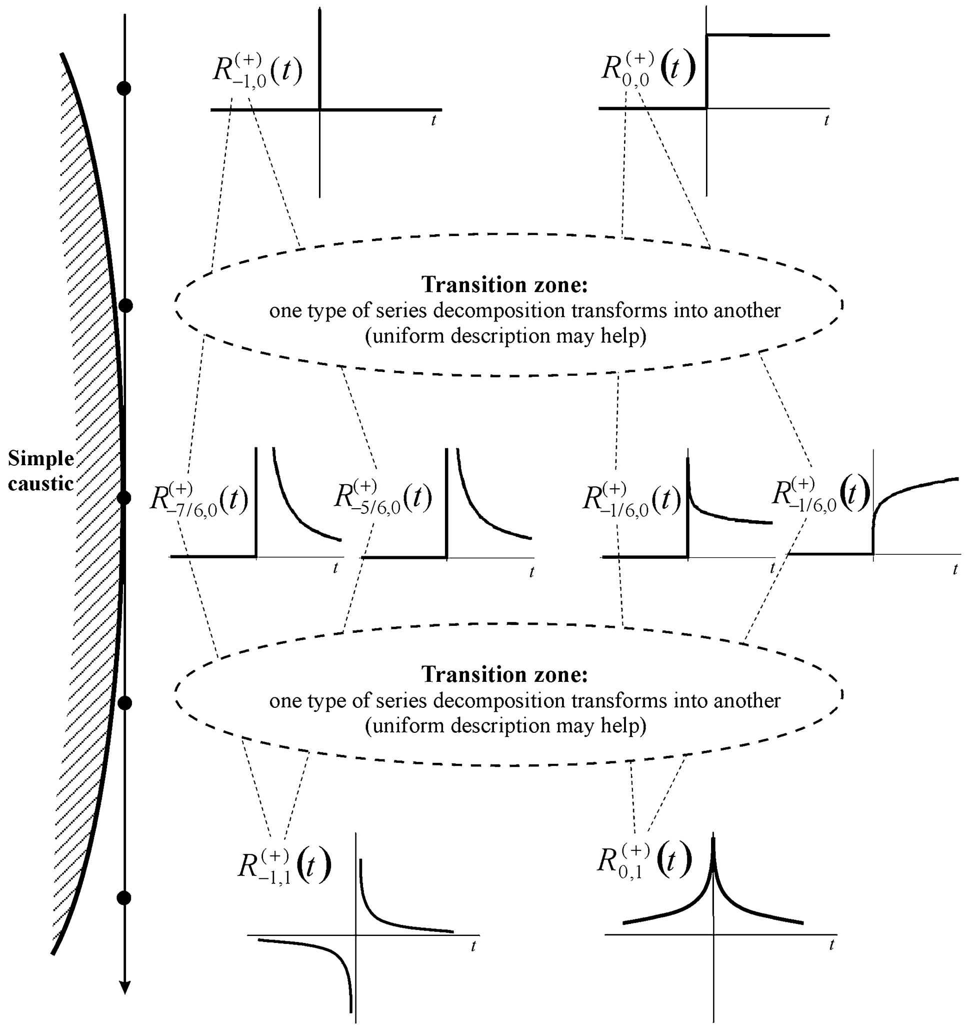

FIG. 4. The structure of the series approximation of the wavefield. The ray is tangent to a simple caustic (hatched zone denotes the caustic shadow). Different types of the series decomposition are possible at different points of the ray (bold dots). First terms of corresponding series are shown together with their sketches (right part of the figure).

on the left (Figure 5a) show the results computed using the standard ray method. The panels on the right (Figure 5b) show the corresponding results computed using the uniform description. The signal form does not change greatly, but it is still possible to detect a small phase shift and amplitude increase that becomes more apparent at positions near the caustic.

CONCLUSIONS

In this paper, we propose a new representation of wavefields that is uniformly valid near a caustic, permitting the computation of higher order terms of the ray series approximation. This technique has the advantage that the computation of wave dynamics (amplitudes and phase distortions) is reduced to a double integration of a δ-function, a less technically difficult operation. The procedure is the same both for homogeneous and heterogeneous media, with the only requirement for heterogeneous media being that we know the ray-series approximation for the Green's tensor. Some practical results include:

1) In the regular case (no caustics), we derived analytic formulas describing the wavefield in the first-order ray approximation (the first two terms of the ray series) for both P- and S-waves.
2) We showed how the structure of the ray series changes at a caustic, as compared with the structure at the regular points of the ray.
3) Our technique allow the computation of seismic signals in a way that is uniformly valid in the vicinity of a caustic. The procedure was demonstrated for the case of a cusp caustic.

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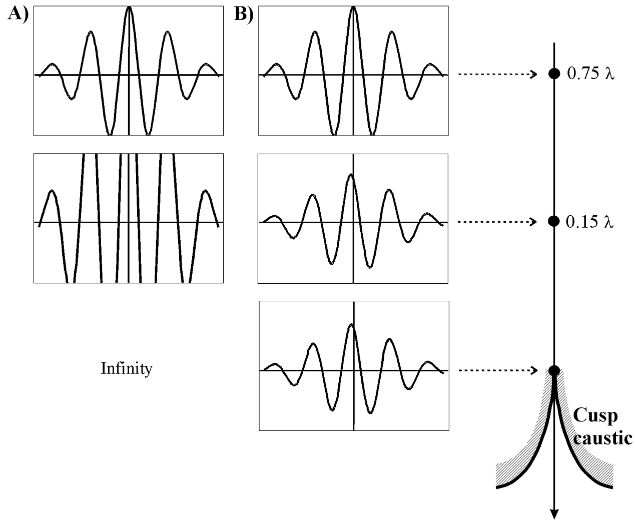

FIG. 5. The synthetic signal within the vicinity of a cusp caustic. The signal shape is given at distances 0.75, 0.15, and 0 wavelengths away from the caustic. (a) The signal shape computed using the standard ray method. (b) The signal shape computed using the uniform representation of the wavefield.

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