Problem on Formation of Parallel Crack System in Brittle Layer

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PROBLEM ON FORMATION OF PARALLEL CRACK SYSTEM IN BRITTLE LAYER

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The process is considered for crushing a brittle layer. The finite-element method and a number of simplified models associated with the averaging are used for calculating the strains and stresses. The chief aim of investigation is to select such value for the parameter of loading under which the failure condition is achieved inside the layer. It is shown that its value depends both on the ratio of layer linear-dimension and the values of material elastic characteristics.

Brittle layer, deformation, failure, fracturing, finite-element method, energy flows

Currently, it is established that rocks are the block medium intersected by cracks of different scale levels; in addition to it, the position, orientation, and other parameters of every single crack are stochastic. However, the distinct and determinate regularities have been traced back to all the averaged characteristics of cracks. Investigation into fracturing of the rock mass is an actual problem for mining, construction of underground structures, hydraulic engineering, etc. In this connection, an intensive search is made for the solutions of different problems on fracturing by means of geological methods and methods of deformable solid mechanics [1 – 3]. In [4], the imitative model is considered for cracking a plane layer subjected to the biaxial nonuniform tension. This model makes it possible to obtain numerically different polygonal structures according to the prescribed parameters. The question concerning the role of actual physical parameters of the rock mass requires additional study.

Let us examine the deformation of thin layer. Assume that its material is ideally brittle and is deformed elastically prior to failure. As a failure criterion, we take the factor when the highest tensile stress reaches the given value. We analyze the problem in two statements: in the first simplified problem, the load is applied to both surfaces (upper and lower) of the layer, and in the second — only to the lower one. The problems are three-dimensional. However, in order to simplify all calculations, we can use the fact that the layer thickness is much less than its cross-dimensions. First, let us consider the problem of uniaxial tension in two-dimensional statement. Then, we reduce it to one-dimensional by means of averaging with respect to layer thickness and make sure that the solution error is small. Let us next carry the results of averaging over to more general three-dimensional case.

Consider the Cartesian coordinate system (x, z) and homogeneous isotropic elastic rectangular domains: $-l \le x \le l$, $-h \le z \le h$ (Fig. 1*a*) and $-l \le x \le l$, $0 \le z \le 2h$ (Fig. 1*b*), where $L = 2l$ is the layer length, and $H = 2h$ is its width.

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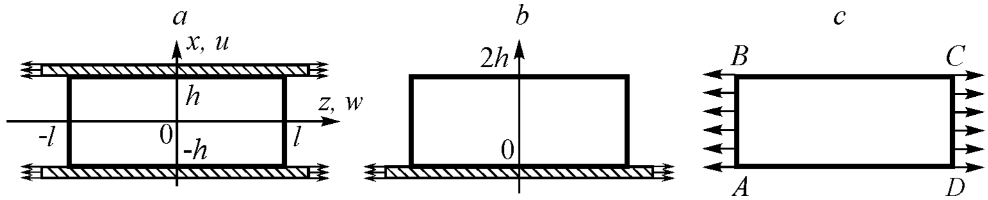

Fig. 1

Assume that the volume forces are absent, the vertical boundaries are free from the stresses, and the prescribed displacements are applied to the bases. We consider two problems with boundary conditions:

$$z=-h \text{ and } z=h \qquad u(x,z)=k_1 x,\ w(x,z)=0 \tag{1}$$

$$\begin{aligned} z&=0 \qquad && u(x,z)=k_2 x,\ w(x,z)=0, \\ z&=2h \qquad && \sigma_{xx}(x,z)=\sigma_{zz}(x,z)=0, \end{aligned} \tag{2}$$

here, $u(x,z)$ and $w(x,z)$ are the displacement vector components; $\sigma_{xx}(x,z)$, $\sigma_{xz}(x,z)$, and $\sigma_{zz}(x,z)$ are the stress tensor components; k_1 and k_2 are the time functions under quasi-static loading.

Let us introduce dimensionless values: $x=H\bar{x}$, $z=H\bar{z}$, $u=H\bar{u}$, and $w=H\bar{w}$, where H is the length scale, and the corresponding dimensionless variables are denoted by the over-bar. In order to choose the scale of stresses, we examine the problem of simple tension when the normal tensile stresses $\sigma_{xx}=\sigma^*=\text{const}$ are assigned on AB and CD (Fig. 1c), and the bases AD and BC are free from stresses. This problem — $\sigma_{xx}=\sigma^*$, $\sigma_{xz}=\sigma_{zz}=0$ is solved over the whole domain. Let the layer undergoes a rupture on the z axis at a certain prescribed value of σ^*. We take the value of σ^* as the scale of stresses: $\sigma_{xx}=\sigma^*\bar{\sigma}_{xx}$, $\sigma_{zz}=\sigma^*\bar{\sigma}_{zz}$, and $\sigma_{xz}=\sigma^*\bar{\sigma}_{xz}$. Hereafter, we shall omit the bar over the variables.

With new variables, the equilibrium equations and Hook's law will have the following form:

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xz}}{\partial z}=0, \qquad \frac{\partial\sigma_{xz}}{\partial x}+\frac{\partial\sigma_{zz}}{\partial z}=0,$$

$$\sigma_{xx}=\frac{\lambda\theta}{\sigma^*}+\frac{2\mu}{\sigma^*}\frac{\partial u}{\partial x}, \qquad \sigma_{zz}=\frac{\lambda\theta}{\sigma^*}+\frac{2\mu}{\sigma^*}\frac{\partial w}{\partial z}, \qquad \sigma_{xz}=\frac{\mu}{\sigma^*}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right),$$

where $\theta=\dfrac{\partial u}{\partial x}+\dfrac{\partial w}{\partial z}$; λ and μ are the Lamé parameters.

Solution by the Finite-Element Method

For problem (1), the layer after deformation is shown in Fig. 2a, and for (2), it is illustrated in Fig. 2b at the ratio $L/H=2$. Triangular elements, a part of which is demonstrated in the figures, were chosen as a grid for the finite-element method. The number of the grid nodes was taken to be proportional to the value of L/H.

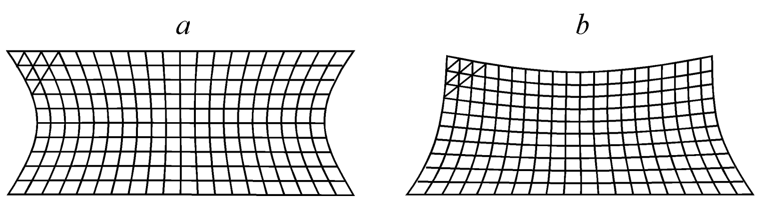

Fig. 2

Let us formulate the brittle failure conditions for problems (1) and (2). By virtue of symmetry of the domain and the boundary conditions relative to the z axis, we suppose that the tensile layer undergoes a rupture at $x = 0$ when the tensile stress $\sigma_{xx}(0, z)$ reaches a certain limit value of σ^*. We find the values of $k_1 = k_1^*$ and $k_2 = k_2^*$ from boundary conditions (1) and (2), under which the brittle failure condition is realized. In problems under consideration, k_1 and k_2, unlike simple tension, are not already constant, and the dependences $k_1 = k_1^*(L/H)$ and $k_2 = k_2^*(L/H)$ take place. The graphs of both functions are depicted in Fig. 3, where the solution of problem on simple tension for $k = 1$ is also given for comparison. It is obvious that as the value of the ratio L/H increases, the loading parameters k_1^* and k_2^* approach more close to the solution of problem on simple tension. Note that for the values of $L/H < 2$, the stresses σ_{xx} become compressive at the point (0, 2h) and in its vicinity on the z axis due to the specific character of the boundary conditions in problem (2), and in problem (1), they become compressive near the point (0, 0) on the z axis when $L/H < 2$.

Two independent parameters — Young's modulus E and Poisson's ratio ν affect the stress state of elastic body. To compare the solutions of problems (1) and (2), the calculation was performed with the same value of $\nu = 0.3$. Now changing the value of ν, we study the dependence of the stress σ_{xx} at the critical points of layer ($(0, -h)$ and $(0, h)$ for problem (1) and at $(0, 0)$ for problem (2)) as the function of ν. For this purpose, for different values of L/H, we formulate boundary conditions (1) and (2) with the previously found values of $k_1^*(L/H)$ and $k_2^*(L/H)$ and solve the problems, varying the value of ν within the range from 0 to 0.5. The graphs of the functions obtained are shown in Fig. 4. As is seen from the graphs, the presence of the upper boundary affects substantially both the values of the tensile stresses σ_{xx} and the stress state at the critical points of the material layer being tested.

Having considered the influence exerted by the loading parameters, we analyze the energy flows [6]. We take the prescribed stress-strain state of the body, where the tensor of stresses σ and the vector of displacements $\vec{u}$ are determined for every point, as the definition of the energy flow. For any area with the normal $\vec{n}$, the stress vector $\vec{\sigma}_n = \sigma\vec{n}$ by the Cauchy formula; and we introduce

$$W_n = -\vec{u}\vec{\sigma}_n. \tag{3}$$

It follows from (3) that $W_n = -\vec{u}\sigma\vec{n} = W\vec{n}$, where $W = -\sigma\vec{u}$, and W_n is the energy transferred through the unit area with the normal $\vec{n}$. If we introduce one more notion — the stress vector in the area with the normal directed along $\vec{u}$, then

$$W = -\vec{\sigma}_u |\vec{u}|, \tag{4}$$

where $\vec{\sigma}_u = \sigma\vec{u}/|\vec{u}|$. From (3) and (4) the mechanical sense of W follows: the direction of W is the direction of the maximal density of the energy flow, and $|\vec{W}|$ is equal to this density. In any other direction, the density of energy flow represents the scalar product of W and $\vec{n}$. Hence it appears that there always exists a certain direction along which the energy is not transferred, it is orthogonal to $\vec{\sigma}_u$. The lines tangent to $\vec{\sigma}_u$ at each point represent the energy flux lines. Figure 5 shows the energy flux lines at $L/H = 2$ for boundary-value problem (1). It is seen from the figure that configurations of these lines differ, depending on the boundary part, from which each of the lines begins to move. Thus, the medium element AB can obtain energy only from the sections A_1B_1 and A_2B_2 which belong to the opposite boundaries and exchange energy with each other. On the other hand, the layer element CD obtains energy only from the sections C_1D_1 and C_2D_2 belonging to the same boundary.

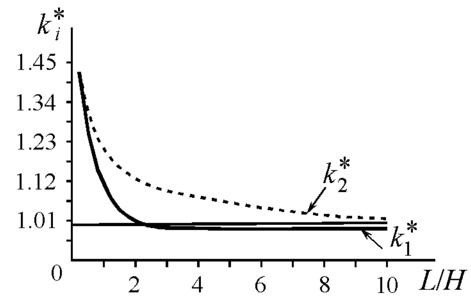

Fig. 3

Thus, the influence exerted by the boundary sections is distinctly observed. As it is known, equations of the elasticity theory belong to the elliptic type, and therefore, the result obtained seems paradoxical at first glance. However, it can be explained in the following way. The equations of the elasticity theory are similar to the Navier – Stokes equations, if we neglect by the inertial terms. These equations describe a slow stationary flow of viscous fluid. Let us now imagine that on a certain boundary part of the region, the entering fluid is colored. In consequence of stationary velocity field, only the region of influence will be colored within it, but this fact does not depend on the ellipticity of the equations. A similar situation takes place in the cases previously discussed for the energy flows.

Thus, the formulation of boundary-value problems (1) and (2) makes it possible to obtain regular structures, i.e., the systems of parallel cracks. In monotonic increasing the loading parameter, the deformation has a discrete character; the principal influence is exerted by the value of L/H. When cracking the material into smaller parts, it is required to assign the greater loading parameter to achieve the brittle failure condition. Poisson's ratio ν also plays an important role in the problem.

Averaging with Respect to Thickness

We solve boundary-value problems (1) and (2) by averaging with respect to layer thickness.

The loading conditions are so that the equations of the shell and plane theories are not acceptable here. Using the specific structure of the deformed domain, we reduce two-dimensional problem to one-dimensional with the help of Kirchhoff's hypothesis.

Averaging of Problem (1). Let the two-dimensional domain $-l \le x \le l$, $-h \le z \le h$ is assigned (Fig. 1*a*). If we superimpose the solution for a uniaxial tension of the band on (1), then we can come to the problem: it is required to find the strain and stress distribution within the band, if the following boundary conditions are fulfilled:

$$\begin{aligned} &\text{at } z=-h \text{ and } z=h \qquad && u(x,z)=0,\ w(x,z)=0, \\ &\text{at } x=\pm l && \sigma_{xx}=k_1E,\ \sigma_{xz}=0, \end{aligned} \tag{5}$$

where k_1 is the loading parameter.

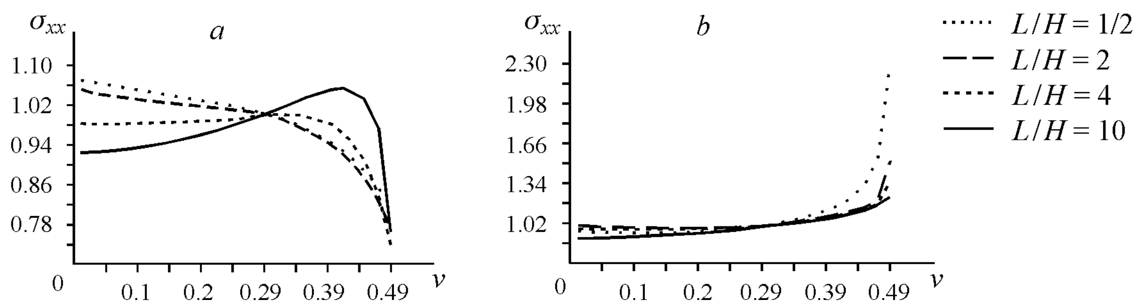

Fig. 4

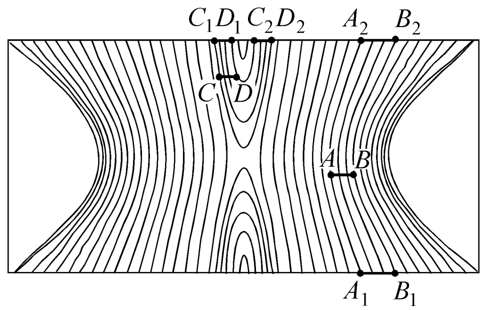

Fig. 5

Let us introduce the averaging operation with respect to layer thickness (variable z)

$$\tilde{f}(x) = \frac{1}{H}\int_0^H f(x,z)\,dz\ ,$$

where $\tilde{f}(x)$ is the averaged value for the arbitrary function $f(x,z)$. If the averaging is applied to the derivative of $f(x,z)$ with respect to z, then we obtain

$$\frac{1}{H}\int_0^H \frac{\partial f(x,z)}{\partial z}\,dz = \frac{f(x,H) - f(x,0)}{H}\ ,$$

and if it is applied to the equilibrium equations, then the number of unknown functions will increase; in lieu of the function $f(x,z)$, a few functions, i.e., $\tilde{f}(z)$, $f(x,0)$, and $f(x,H)$ can arise. Indetermination can be eliminated, if the mean value of the function is calculated using its boundary values. To do this, it is enough to follow the rule: one unknown function of two variables must generate only one function of one variable. It means that for all functions except σ_{xx}, the linear approximation with respect to z must be accepted. Note that in the given statement due to the domain symmetry relative to the axis $z = 0$, a lot of information is being lost in averaging from $z = -h$ to $z = h$, as a result of which the system is inadequate to the problem under consideration. To recover the information, we formulate the conditions, which follow from the symmetry, on the x axis, and then perform the averaging from $z = 0$ to $z = h$. Such approach preserves more information and leads to correct problem.

Loading conditions (5) cause the definite stress and displacement fields within the layer and on its boundaries. In order to find them, we introduce the following designations:

— on the boundary $z = h$

$$\sigma_{xz} = \tau(x)\,, \quad \sigma_{zz} = p(x) + g(x)\,,$$

— at $z = 0$ (6)

$$\sigma_{xz} = 0\,, \quad \sigma_{zz} = p(x)\ .$$

In this statement, four unknown functions, i.e., $\tau(x)$, $p(x)$, $g(x)$, and $u^0(x)$ are to be found. Linear approximation has the form: $u(x,z) = u^0(x)(1 - z/h)$, $w(x,z) = 0$, $\sigma_{xz} = \tau(x)z/h$, and $\sigma_{zz} = p(x) + g(x)z/h$. Hence, for the averaged values

$$\tilde{u} = u^0(x)/2, \quad \tilde{w} \equiv 0, \quad \tilde{\sigma}_{zx} = \tau(x)/2, \quad \tilde{\sigma}_{zz} = p(x) + g(x)/2.$$

Using the equilibrium equations, we obtain

$$\begin{cases} \dfrac{\partial \tilde{\sigma}_{xx}}{\partial x} + \dfrac{\tau}{h} = 0, \quad \dfrac{1}{2}\dfrac{\partial \tau}{\partial x} + \dfrac{g}{h} = 0, \\ \tilde{\sigma}_{xx} = \dfrac{1}{2}(\lambda + 2\mu)\dfrac{\partial u}{\partial x}, \\ p + \dfrac{g}{2} = \dfrac{1}{2}\lambda \dfrac{\partial u^0}{\partial x}, \quad \dfrac{\tau}{2} = -\mu \dfrac{u^0}{h}. \end{cases} \tag{7}$$

The system consists of five equations for five unknown functions: $\tilde{\sigma}_{xx}$, $\tau(x)$, $p(x)$, $g(x)$, and $u^0(x)$.

For $x = \pm l$, the boundary conditions are: $\tilde{\sigma}_{xx} = k_1 E = Q$, and $\tau = 0$. Since at the corner points of the domain the shear stresses are not couple, then the condition $\tau = 0$ is required to remove. Equations (7) can be solved in elementary functions

$$\begin{cases} \sigma_{xx} = Q\dfrac{e^{ax} + e^{-ax}}{e^{aL} + e^{-aL}}, \qquad \sigma_{zx} = \tau(x)\dfrac{z}{h} = -ahQ\dfrac{z}{h}\left(\dfrac{e^{ax} - e^{-ax}}{e^{aL} + e^{-aL}}\right), \\ \sigma_{zz} = p(x) + g(x)\dfrac{z}{h} = \dfrac{1}{4}a^2h^2Q\left(\dfrac{e^{ax} + e^{-ax}}{e^{aL} + e^{-aL}}\right)\left(\dfrac{2z}{h} + \dfrac{\lambda}{\mu} - 1\right), \\ u = u^0\left(1 - \dfrac{z}{h}\right) = \dfrac{ah^2}{2\mu}Q\left(\dfrac{e^{ax} - e^{-ax}}{e^{aL} + e^{-aL}}\right)\left(1 - \dfrac{z}{h}\right), \qquad w = 0. \end{cases} \tag{8}$$

Formulas (8) describe the stress–strain state of the layer at $0 \le z \le h$.

Averaging of Problem (2). We carry out a similar analysis for the boundary conditions in two-dimensional domain: $-l \le x \le l$, $0 \le z \le 2h$ (Fig. 1*b*).

On the boundaries

$z = 0$

$$u = k_2 x, \quad w \equiv 0, \tag{9}$$

$$\sigma_{zz}(x,0) = p(x), \quad \sigma_{xz}(x,0) = \tau(x), \tag{10}$$

$z = 2h$

$$\sigma_{zz}(x, 2h) = 0, \quad \sigma_{zz}(x, 2h) = 0, \tag{11}$$

$$u(x,2h) = kx + u^0(x), \quad w(x,2h) = w^0(x). \tag{12}$$

Functions (9) and (11) are assigned, and functions (10) and (12) are to be found. Linear approximation gives

$$\sigma_{xz}(x,z) = \tau(x)(1 - z/2h), \qquad \sigma_{zz}(x,z) = p(x)(1 - z/2h),$$
$$u(x,z) = kx + u^0(x)z/2h, \qquad w(x,z) = w^0(x)z/2h.$$

Hence, for the averaged values

$$\tilde{\sigma}_{zx} = \frac{\tau(x)}{2}, \quad \tilde{\sigma}_{zz} = \frac{p(x)}{2}, \quad \tilde{u} = k_2 x + \frac{u^0(x)}{2}, \quad \tilde{w} = \frac{w^0(x)}{2}. \tag{13}$$

Using (9) – (13) and the averaging operation with respect to the equilibrium equations, we obtain

$$\begin{cases} \dfrac{\partial\tilde{\sigma}_{xx}}{\partial x} - \dfrac{\tau(x)}{h} = 0, \qquad \dfrac{1}{2}\dfrac{\partial\tau}{\partial x} - \dfrac{p(x)}{h} = 0, \\ \tilde{\sigma}_{xx} = (\lambda + 2\mu)\left(k_2 + \dfrac{1}{2}\dfrac{\partial u^0}{\partial x}\right) + \lambda\dfrac{w^0}{h}, \\ \dfrac{p}{2} = \lambda\left(k_2 + \dfrac{1}{2}\dfrac{\partial u^0}{\partial x}\right) + (\lambda + 2\mu)\dfrac{w^0}{h}, \qquad \dfrac{\tau}{2} = \mu\left(\dfrac{u^0}{h} + \dfrac{1}{2}\dfrac{\partial w^0}{\partial x}\right). \end{cases} \tag{14}$$

Expressions (14) form the closed system of five equations for five unknown functions: $\tilde{\sigma}_{xx}$, $\tau(x)$, $p(x)$, $u^0(x)$, and $w^0(x)$. The conditions on the lower $(z = 0)$ and upper $(z = 2h)$ boundaries are already examined. The boundary conditions on the vertical boundaries remained unused

$$\tilde{\sigma}_{xx}(\pm l) = 0, \qquad \tau(\pm l) = 0. \tag{15}$$

Relations (14) are easily reduced to the system of four equations of the first order relative to four functions. Therefore, problems (14) and (15) will be correct.

From (14) we obtain the system of the second order consisting of two equations for two unknown functions $u^0(x)$ and $w^0(x)$:

$$\begin{cases} \dfrac{1}{2}(\lambda + 2\mu)u^{0\,\prime\prime}(x) - \dfrac{2\mu}{h^2}u^0(x) + \dfrac{1}{h}(\lambda - \mu)w^{0\,\prime}(x) = 0, \\ \dfrac{\mu}{2}w^{0\,\prime\prime}(x) - 2\left(\dfrac{\lambda + 2\mu}{h^2}\right)w^0(x) + \dfrac{1}{h}(\mu - \lambda)u^{0\,\prime}(x) = \dfrac{2\lambda k_2}{h}. \end{cases} \tag{16}$$

The fact that system (16) decomposes into two independent equations at $\lambda = \mu$ $(\nu = 1/4)$ is of great interest. Let us consider this case in detail. From Eqs. (16) we have

$$u^0(x) = C_1 e^{ax} + C_2 e^{-ax}, \qquad w^0(x) = C_3 e^{bx} + C_4 e^{-bx} - \frac{k_2 h}{3}, \tag{17}$$

where $a^2 = \dfrac{4}{3h^2}$, $b^2 = \dfrac{12}{h^2}$, and C_1, C_2, C_3, and C_4 are the unknown constants that are required to determine. We find them, using boundary conditions (15)

$$C_1 = C_2 = \frac{\sqrt{3}AN(B^2 - 1)}{2(A^2(B^2 - 2) + 2B^2 - 1)}, \qquad C_3 = C_4 = \frac{BN(A^2 - 1)}{2(A^2(B^2 - 2) + 2B^2 - 1)}, \tag{18}$$

here, $A = e^{aL}$, $B = e^{bL}$, and $N = -8/3k_2h$. Now knowing the expressions for the displacements $u^0(x)$ and $w^0(x)$, i.e., formulas (17) and (18), and using (14), we obtain the final expressions of the functions desired.

Figure 6 shows a relative error of the loading parameters k_i^* (found by means of numerical solution, and then by means of averaging) depending on the ratio L/H (i = 1, 2 — the problem number): $\delta_i = \delta_i(L/H)$. In the graph, the dashed line illustrates the error for boundary-value problem (1), and the solid line — for problem (2). The comparative analysis of the expressions for tensile stresses σ_{xx} shows that the relative error of the solutions found by averaging for both problems is not more than 0.08 of the solutions obtained numerically for the plane problem. It is seen from the graph that with increase in L/H, the relative error decreases.

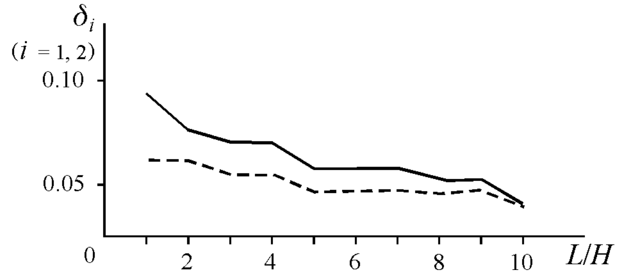

Fig. 6

Thus, proceeding from small deviations of numerical and approximate solutions, we can conclude that the arguments cited above can be used for the problem on layer deformation and crack system formation in three-dimensional case.

Three-Dimensional Statement

We study the three-dimensional formulation of the problem for an arbitrary layer bounded by the contour L (Fig. 7).

Let $u=0$, $v=0$, and $w=0$ on the boundary $z = h$, and $u=u^0(x,y)$, $v=v^0(x,y)$, and $w=0$ on $z=0$.

Similar to (6), we assign $\sigma_{xz}=\tau(x,y)$, $\sigma_{yz}=t(x,y)$, and $\sigma_{zz} = p(x,y)+g(x,y)$ on $z=h$; and $\sigma_{xz}=0$, $\sigma_{yz}=0$, and $\sigma_{zz} = p(x,y)$ on $z=0$.

The operation of averaging gives

$$\widetilde{f}(x,y)=\frac{1}{h}\int_0^h f(x,y,z)dz\,, \qquad \frac{1}{h}\int_0^h \frac{\partial\varphi(x,y,z)}{\partial z}dz=\frac{\varphi(x,y,h)-\varphi(x,y,0)}{h}\,.$$

Applying the average to the equilibrium equations, we obtain

$$\begin{cases} \dfrac{\partial\widetilde{\sigma}_{xx}}{\partial x}+\dfrac{\partial\widetilde{\sigma}_{xy}}{\partial y}+\dfrac{\tau(x,y)}{h}=0\,, \quad \dfrac{\partial\widetilde{\sigma}_{xy}}{\partial x}+\dfrac{\partial\widetilde{\sigma}_{yy}}{\partial y}+\dfrac{t(x,y)}{h}=0\,, \quad \dfrac{1}{2}\dfrac{\partial\tau}{\partial x}+\dfrac{1}{2}\dfrac{\partial t}{\partial y}+\dfrac{g(x,y)}{h}=0\,, \\ \widetilde{\sigma}_{xx}=(\lambda+2\mu)\dfrac{2}{3}\dfrac{\partial u^0}{\partial x}+\lambda\dfrac{2}{3}\dfrac{\partial v^0}{\partial y}, \quad \widetilde{\sigma}_{yy}=\lambda\dfrac{2}{3}\dfrac{\partial u^0}{\partial x}+(\lambda+2\mu)\dfrac{2}{3}\dfrac{\partial v^0}{\partial y}, \quad \widetilde{\sigma}_{xy}=\mu\dfrac{2}{3}\left[\dfrac{\partial u^0}{\partial y}+\dfrac{\partial v^0}{\partial x}\right], \\ p+\dfrac{g}{3}=\lambda\dfrac{2}{3}\dfrac{\partial u^0}{\partial x}+(\lambda+2\mu)\dfrac{2}{3}\dfrac{\partial v^0}{\partial y}, \quad \dfrac{\tau}{2}=-\mu\dfrac{u^0}{h}, \quad \dfrac{t}{2}=-\mu\dfrac{v^0}{h}\,. \end{cases} \tag{19}$$

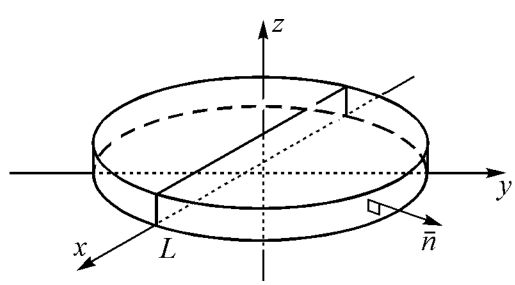

Fig. 7

System (19) consists of nine equations relative to nine unknown functions: $\tilde{\sigma}_{xx}$, $\tilde{\sigma}_{xy}$, $\tilde{\sigma}_{yy}$, τ, t, p, u^0, v^0, and w^0. We can easily exclude a number of functions. As a result, we have

$$\begin{cases} \dfrac{\partial \tilde{\sigma}_{xx}}{\partial x} + \dfrac{\partial \tilde{\sigma}_{xy}}{\partial y} - \dfrac{2\mu}{h^2} u^0 = 0, \quad \dfrac{\partial \tilde{\sigma}_{xy}}{\partial x} + \dfrac{\partial \tilde{\sigma}_{yy}}{\partial y} - \dfrac{2\mu}{h^2} v^0 = 0, \\ \tilde{\sigma}_{xx} = \dfrac{2}{3}\lambda\left(\dfrac{\partial u^0}{\partial x} + \dfrac{\partial v^0}{\partial y}\right) + \dfrac{2}{3} 2\mu \dfrac{\partial u^0}{\partial x}, \\ \tilde{\sigma}_{yy} = \dfrac{2}{3}\lambda\left(\dfrac{\partial u^0}{\partial x} + \dfrac{\partial v^0}{\partial y}\right) + \dfrac{2}{3} 2\mu \dfrac{\partial v^0}{\partial y}, \\ \tilde{\sigma}_{xy} = \dfrac{2}{3}\mu\left[\dfrac{\partial u^0}{\partial y} + \dfrac{\partial v^0}{\partial x}\right]. \end{cases} \tag{20}$$

Expressions (20) coincide with the equations of the plane elasticity theory. The principal difference is in the presence of “body forces” proportional to displacements.

Consider the second problem. We assume that on the boundary $z = 0$

$$u(x,y,0)=k_1 x, \quad v(x,y,0)=k_2 x, \quad w(x,y,0) \equiv 0,$$

$$\sigma_{zz}(x,y,0)=p(x,y), \quad \sigma_{xz}(x,y,0)=\tau(x,y), \quad \sigma_{zy}(x,y,0) = t(x,y);$$

on the boundary $z = 2h$

$$u(x,y,2h) = k_1 x + u^0(x,y), \quad v(x,y,2h) = k_2 x + v^0(x,y), \quad w(x,y,2h) = w^0(x,y),$$

$$\sigma_{zz}(x,y,2h) = 0, \quad \sigma_{xz}(x,y,2h) = 0, \quad \sigma_{zy}(x,y,2h) = 0.$$

Applying the averaging operation to the equilibrium equations, we obtain

$$\begin{cases} \dfrac{\partial \tilde{\sigma}_{xx}}{\partial x} + \dfrac{\partial \tilde{\sigma}_{xy}}{\partial y} - \dfrac{\tau(x,y)}{h} = 0, \quad \dfrac{\partial \tilde{\sigma}_{xy}}{\partial x} + \dfrac{\partial \tilde{\sigma}_{yy}}{\partial y} - \dfrac{t(x,y)}{h} = 0, \quad \dfrac{1}{2}\dfrac{\partial \tau}{\partial x} + \dfrac{1}{2}\dfrac{\partial t}{\partial y} - \dfrac{p(x,y)}{2} = 0, \\ \tilde{\sigma}_{xx} = (\lambda + 2\mu)\left(k_1 + \dfrac{1}{2}\dfrac{\partial u^0}{\partial x}\right) + \lambda\left[k_2 + \dfrac{1}{2}\dfrac{\partial v^0}{\partial y} + \dfrac{w^0}{h}\right], \\ \tilde{\sigma}_{yy} = \lambda\left[k_1 + \dfrac{1}{2}\dfrac{\partial u^0}{\partial x} + \dfrac{w^0}{h}\right] + (\lambda + 2\mu)\left(k_2 + \dfrac{1}{2}\dfrac{\partial v^0}{\partial y}\right), \\ \tilde{\sigma}_{xy} = \mu\left[\dfrac{1}{2}\dfrac{\partial u^0}{\partial y} + \dfrac{1}{2}\dfrac{\partial v^0}{\partial x}\right], \\ \dfrac{p}{2} = \lambda\left(k_1 + \dfrac{1}{2}\dfrac{\partial u^0}{\partial x} + k_2 + \dfrac{1}{2}\dfrac{\partial v^0}{\partial y}\right) + (\lambda + 2\mu)\dfrac{w^0}{h}, \\ \dfrac{\tau}{2} = \mu\left[\dfrac{u^0}{h} + \dfrac{1}{2}\dfrac{\partial w^0}{\partial x}\right], \qquad \dfrac{t}{2} = \mu\left[\dfrac{v^0}{h} + \dfrac{1}{2}\dfrac{\partial w^0}{\partial y}\right]. \end{cases} \tag{21}$$

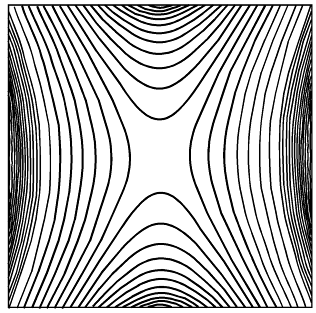

Fig. 8

The system consists of nine equations relative to the following nine unknowns: $\tilde{\sigma}_{xx}$, $\tilde{\sigma}_{xy}$, $\tilde{\sigma}_{yy}$, τ, t, p, u^0, v^0, and w^0.

Further, taking $\lambda = \mu$, we obtain the system of the second order

$$\begin{cases} 3\dfrac{\partial^2 u^0}{\partial x^2} + \dfrac{\partial^2 v^0}{\partial y^2} + 2\dfrac{\partial^2 v^0}{\partial x \partial y} - \dfrac{4}{h^2} u^0 = 0, \\ \dfrac{\partial^2 v^0}{\partial x^2} + 3\dfrac{\partial^2 u^0}{\partial y^2} + 2\dfrac{\partial^2 u^0}{\partial x \partial y} - \dfrac{4}{h^2} v^0 = 0, \\ \dfrac{\partial^2 w^0}{\partial x^2} + \dfrac{\partial^2 w^0}{\partial y^2} + \lambda\left(\dfrac{2}{h} - 1\right)\dfrac{\partial u^0}{\partial x} + \lambda\left(\dfrac{2}{h} - 1\right)\dfrac{\partial v^0}{\partial y} - 6\lambda\dfrac{w^0}{h} = 2\lambda(k_1 + k_2). \end{cases} \tag{22}$$

The first two equations of (22) are the equations of the plane elasticity theory in case of equality of the Lamé parameters ($\lambda = \mu$). The difference is in the presence of “body forces” proportional to displacements.

Consider the boundary conditions. In our statement, for the contour L (Fig. 7), the outward normal $\bar{n} = \{n_1, n_2, 0\}$, consequently, the boundary conditions will be

$$\begin{cases} \tilde{\sigma}_{xx} n_1 + \tilde{\sigma}_{xy} n_2 = 0, \\ \tilde{\sigma}_{xy} n_1 + \tilde{\sigma}_{yy} n_2 = 0, \\ \tau n_1 + t n_2 = 0. \end{cases} \tag{23}$$

For simplicity, let us discuss the case when the layer cross-section with the plane $z = 0$ is the unit square with the center at the point $(0, 0)$. Using (23), we solve systems (20) and (22) numerically by the finite-element method. As an example, Fig. 8 presents the energy flux lines for the loading conditions: $k_1 = 2k_2$. It is seen that the energy entering from the left and right boundaries of the layer “returns” to the same boundary, while the upper and lower boundaries exchange the energy with each other.

Thus, it is obvious from the numerical solution for the problem of uniaxial tension of the layer that both the ratio of linear dimensions of the layer and the values of the material elastic-constants influence the character of formation of the parallel crack system. The energy flux line configuration depends on the dimensions of the region deformed and the values of the loading parameters. Proceeding from small deviations of numerical and approximate solutions of boundary-value problems (1) and (2), we can assert that the assumptions accepted in averaging the two-dimensional problem can be used for the problem on layer deformation in three-dimensional statement.

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