ELSEVIER

Journal of Hydrology 259 (2002) 246–253

Journal of Hydrology

www.elsevier.com/locate/jhydrol

An alternative IUH for the hydrological lumped models

R. Szymkiewicz

Faculty of Hydro- and Environmental Engineering, Technical University of Gdansk, 80-952 Gdansk, Poland

Received 18 June 2001; revised 26 November 2001; accepted 28 November 2001

Abstract

An alternative IUH for the Muskingum model as well as for the linear reservoir is presented. It is shown that the IUH of the Muskingum model can take the form of the IUH for the diffusive wave model. This approach is based on the equivalence of the results given by both models. The instantaneous unit hydrograph obtained in this way has better properties comparing with the classical ones. It is able to reproduce simultaneously a translation of the flood wave along a channel reach and its attenuation. It does not produce any negative discharges at the downstream end of the channel reach as well. © 2002 Elsevier Science B.V. All rights reserved.

Keywords: Instantaneous unit hydrograph; Muskingum model; Numerical diffusion; Diffusive wave

1. Introduction

The simplified flood routing models are derived from the Saint-Venant equations. While neglecting the inertial force in the equation of momentum conservation and introducing the additional assumptions, one obtains well-known diffusive wave model (Miller and Cunge, 1975)

$$\frac{\partial Q}{\partial t} + C\frac{\partial Q}{\partial x} - \nu\frac{\partial^2 Q}{\partial x^2} = 0 \tag{1}$$

in which

$$C = \frac{\mathrm{d}Q}{\mathrm{d}A} = \frac{1}{m}U \tag{2}$$

$$\nu = \frac{Q}{2Bs} \tag{3}$$

where Q is the discharge; C, the kinematic wave celerity; U, the cross-sectional average flow velocity; m, the coefficient which depends on the accepted equation for friction ($m = 3/5$ for Manning formula); ν, the hydraulic diffusivity; B, the channel's width at the water level; s, the bed slope; t, the time; x is the space co-ordinate.

If in the momentum equation apart from the inertial force, the hydrostatic pressure force is also neglected, one obtains the kinematic wave model. It has the form of Eq. (1) with $\nu = 0$. Both models are classified as the simplified models with distributed parameters.

Assuming constant parameters, Eq. (1) can be solved analytically for the initial-boundary conditions which gives the instantaneous unit hydrograph as the solution. Such solution obtained by Hayami has the following form (Eagleson, 1970):

$$h(x,t) = \frac{1}{2\sqrt{\pi\nu}}\frac{x}{t^{3/2}}\exp\left(-\frac{(Ct-x)^2}{4\nu t}\right) \tag{4}$$

for $t > 0$ and $x \geq 0$

It can be considered as the IUH of linear diffusive wave model related to the channel reach of length L,

E-mail address: rszym@pg.gda.pl (R. Szymkiewicz).

0022-1694/02/$ - see front matter © 2002 Elsevier Science B.V. All rights reserved.
PII: S0022-1694(01)00595-9

where $h(t)$ are IUH ordinates. For $x = L$, one obtains:

$$h(t) = \frac{1}{2\sqrt{\pi\nu}} \frac{L}{t^{3/2}} \exp\left(-\frac{(Ct - L)^2}{4\nu t}\right) \tag{5}$$

For the kinematic wave model, when $\nu = 0$, this equation becomes the Dirac δ function which ensures pure translation of the flood wave along the channel reach of length L without any shape deformation.

To flood routing one can apply the hydrologic lumped models as well. They are derived from the storage equation which is obtained by spatial integration of the continuity equation

$$\frac{dS}{dt} = Q_{j-1} - Q_j \tag{6}$$

where S is the storage of the channel reach of length Δx; Q_{j-1}, the inflow; Q_j, the outflow, and j is the index of cross-section.

While introducing an additional formula relating storage, inflow and outflow

$$S = K(XQ_{j-1} + (1 - X)Q_j) \tag{7}$$

one obtains the Muskingum model

$$X\frac{dQ_{j-1}}{dt} + (1 - X)\frac{dQ_j}{dt} = \frac{1}{K}(Q_{j-1} - Q_j) \tag{8}$$

where X is the weighting parameter; K, the time of wave translation between the cross-sections $j - 1$ and j. For $X = 0$, Eq. (8) becomes the linear reservoir model.

The IUH for the Muskingum model was proposed by Venetis (1969). Its equation obtained by the solution of Eq. (8) using the Laplace transformation approach as follows:

$$h(t) = \frac{1}{K(1 - X)^2} \exp\left(-\frac{t}{K(1 - X)}\right) - \frac{X}{1 - X}\delta(t) \tag{9}$$

where $\delta(t)$ is the Dirac δ function.

With $X = 0$, this equation becomes the IUH for the linear reservoir model, commonly used in hydrology:

$$h(t) = \frac{1}{K}\exp\left(-\frac{t}{K}\right) \tag{10}$$

Its application for the linear reservoirs in series gives (Nash, 1957)

$$h(t) = \frac{1}{K\Gamma(N)}\left(\frac{t}{K}\right)^{N-1} \exp\left(-\frac{t}{K}\right) \tag{11}$$

where N is a number of the reservoirs in series, and $\Gamma(N)$ is the gamma function.

The experiences show that the IUHs in the form of Eqs. (9) and (11) have some disadvantages which limit their application. Namely, it is impossible to achieve satisfying agreement between the results of calculation and experimental data when the time lag of output is remarkable. Both IUHs are unable to reproduce the effect of pure translation. For this reason very often the linear reservoir model is used jointly with the model of linear channel (Dooge, 1959; Chow, 1964). Moreover, the IUH for Muskingum model produces the negative ordinates which are caused by the second term of Eq. (9) (Strupczewski et al., 1989). It disagrees with the definition of the instantaneous unit hydrograph which has to be always non-negative: $h(t) \geq 0$ for $t \geq 0$. Consequently, it can produce some unrealistic effects in the form of the initial oscillations in the hydrograph calculated at downstream end. Similar results can be obtained while solving numerically Muskingum model in the form of Eq. (8).

It is well known that the Muskingum equation integrated in time by the method, which does not produce any numerical diffusion, is able to ensure a pure translation of flood wave for $X = 1/2$. For this reason, it seems reasonable to expect the similar effect while using the IUH in the form of Eq. (9). Therefore, this equation should become the Dirac $\delta(t)$ function for $X = 1/2$. Unfortunately, Eq. (9) does not fulfil this condition. The discrepancy is caused by the nature of Eq. (8). In fact this equation is a semi-discrete form of the kinematic wave equation obtained by its spatial discretisation. For this reason, it contains a numerical error introduced during the approximation. Because the IUH in the form of Eq. (9) was derived by an analytical integration of Eq. (8) it contains implicitly this error as well. It has the form of numerical diffusion which disappears for $X = 1/2$, i.e. when the approximation of spatial derivative is made by centred difference.

It seems to be possible to derive a general instantaneous unit hydrograph for all lumped models which is

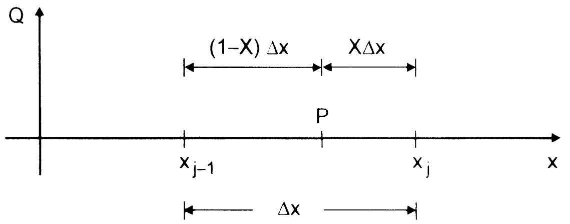

Fig. 1. The discretisation of x-axis for the spatial approximation of the kinematic wave equation.

free from the mentioned disadvantages. In this order, the IUH for the diffusive wave (Eq. (4)) can be applied. It was obtained by an analytical solution of Eq. (1) so that it does not contain any numerical error. The reason for proposing this approach is similarity of the results given by the lumped models and the diffusive wave model.

2. Relation between the simplified distributed models and the lumped systems

The kinematic character of the Muskingum model as well as the numerical nature of the wave attenuation process was noticed by Cunge (1969). While approximating the kinematic wave model by the box scheme and the Muskingum model by the implicit trapezoidal rule, Cunge found out their similarity on condition that

$$K = \frac{\Delta x}{C} \tag{12}$$

The accuracy analysis carried out for the applied approximation showed that it modifies the kinematic wave model to the form:

$$\frac{\partial Q}{\partial t} + C\frac{\partial Q}{\partial x} - \nu_n \frac{\partial^2 Q}{\partial x^2} = 0 \tag{13}$$

where ν_n is the coefficient of numerical diffusion defined as follows:

$$\nu_n = \left(\frac{1}{2} - X\right) C\Delta x \tag{14}$$

Cunge (1969) suggested such value of the parameter X which ensures the numerical diffusivity (Eq. (14)) equal to the hydraulic one given by Eq. (3), i.e. $\nu = \nu_n$. Consequently, the Muskingum model can be used to reproduce the solution of the linear diffusive wave model. This version of the Muskingum model is known as Muskingum-Cunge one (Chow et al., 1988).

In fact the Muskingum model should be regarded as a semi-discrete form of the kinematic wave equation. Moreover, the numerical error generated by this model can be estimated directly from Eq. (8). In order to do this, let us consider the kinematic wave model (Eq. (1) with $\nu = 0$). This equation can be discretised in space. The approximation carried out at the point P located between the nodes $j-1$ and j (Fig. 1) gives

$$\frac{dQ_p}{dt} + C\frac{Q_j - Q_{j-1}}{\Delta x} = 0 \tag{15}$$

where Q_p represents the discharge at the point P. It can be calculated by the linear interpolation between the nodes $j-1$ and j

$$Q_p = XQ_{j-1} + (1-X)Q_j \tag{16}$$

where X is the weighting parameter ranges from 0 to 1. It is defined as follows:

$$X = \frac{x_j - x}{x_j - x_{j-1}} \quad \text{for } x_{j-1} \le x \le x_j \tag{17}$$

Substituting Eq. (16) into Eq. (15) and taking into account Eq. (12), one obtains the Muskingum model in the form of Eq. (8).

The spatial discretisation of the kinematic wave equation introduces a numerical error caused by the truncation of the Taylor series. In order to estimate this error, one can carry out an analysis of consistency. The nodal values of Q in Eq. (8) are replaced by the Taylor series expansion around the point P. While including the terms of second-order, one obtains the modified equation in the form:

$$\frac{\partial Q}{\partial t} + \frac{\Delta x}{K}\frac{\partial Q}{\partial x} - \left(\frac{1}{2} - X\right)\frac{\Delta x^2}{K}\frac{\partial^2 Q}{\partial x^2} = 0 \tag{18}$$

According to the condition of consistency, the modified equation should tend to the governing one while $\Delta x \to 0$ (Fletcher, 1991). For $\Delta x \to 0$, the time of wave translation along the channel reach of the length Δx simultaneously tends to zero ($K \to 0$). Therefore

$$\lim_{\substack{\Delta x \to 0 \\ K \to 0}} \frac{\Delta x}{K} = C \tag{19}$$

and consequently for $\Delta x \to 0$, Eq. (18) tends to the kinematic wave equation. It proves that the Muskingum model is an approximation of the kinematic wave. This approximation introduces a numerical error which is observed in the solution as a numerical diffusion. Its coefficient as follows:

$$\nu_n = \left(\frac{1}{2} - X\right)\frac{\Delta x^2}{K} \quad (20)$$

This expression coincides with Eq. (14) proposed by Cunge (1969). One can add that the mentioned numerical diffusion is caused by the spatial approximation only. An additional diffusion can be generated while integrating Eq. (8) over time by a method of the order lower than two. Usually, the implicit trapezoidal rule is applied. It ensures an accuracy of second-order with regard to t and consequently it is dissipation free.

Summarising, one can say that the numerical solution of the Muskingum model in fact is the numerical solution of the kinematic wave model by the method of lines. In this approach, a solution of the partial differential equation is made in two stages. At first it is discretised in space leading to the system of ordinary differential equations over time. Next, this system is integrated using any well-known method of the numerical solution of an initial problem for the ordinary differential equations.

The classical derivation of the Muskingum model based on the storage Eq. (6) completed by the relationship (7). Therefore, it is interesting how a numerical diffusion is introduced into this model. To explain this problem, let us consider the Muskingum model (Eq. (8)) rewritten in more general form:

$$K\frac{dQ_p}{dt} = Q_{j-1} - Q_j \quad (21)$$

with Q_p defined by Eq. (16).

One can show that this equation can be derived directly from the storage equation without any additional formula relating storage, inflow and outflow. To do this, we have to assume the following:

the storage S is calculated numerically,
the equation for uniform steady flow is applied.

Using both hypothesis one can transform Eq. (6) to Eq. (21).

The storage S can be calculated as follows:

$$S = \int_0^{\Delta x} A\,dx \approx A_p\Delta x \quad (22)$$

where A_p being a cross-sectional area at the point P (Fig. 1) can be expressed as a function of Q_p using the Manning (or Chézy) formula:

$$A_p = \alpha(Q_p)^m \quad (23)$$

with

$$\alpha = \left(\frac{np^m}{s^{1/2}}\right)^m \quad (24)$$

$$m = \frac{3}{5} \quad (25)$$

where n is the Manning coefficient; p, the wetted perimeter; s is the bottom slope.

Therefore, the left side of Eq. (6) can be rewritten as follows:

$$\frac{dS}{dt} = \frac{d}{dt}(\Delta x A_p) = \Delta x\frac{dA_p}{dt} = \Delta x\frac{d}{dt}(\alpha(Q_p)^m) \quad (26)$$

After differentiating with $\alpha = \text{const}$, one obtains:

$$\Delta x\frac{d}{dt}(\alpha(Q_p)^m) = \Delta x\frac{dA_p}{dQ_p}\frac{dQ_p}{dt} = \Delta x\alpha m(Q_p)^{m-1}\frac{dQ_p}{dt} \quad (27)$$

Because the kinematic wave celerity at the point P is defined as:

$$\frac{dQ_p}{dA_p} = C_p = \frac{1}{\alpha m(Q_p)^{m-1}} \quad (28)$$

the right side of Eq. (27) takes the form:

$$\Delta x\alpha m(Q_p)^{m-1}\frac{dQ_p}{dt} = \frac{\Delta x}{C_p}\frac{dQ_p}{dt} = K\frac{dQ_p}{dt} \quad (29)$$

Finally, the left side of Eq. (21) was obtained from the left side of Eq. (6). Therefore, it seems reasonable to accept that an additional formula (7) used to derive the lumped models from the storage equation has double meaning. It can be interpreted as a result of the numerical integration of a storage and of the application of the steady uniform flow equation. A numerical diffusion is introduced by the numerical calculation of the storage S. Note that the kinematic wave model is

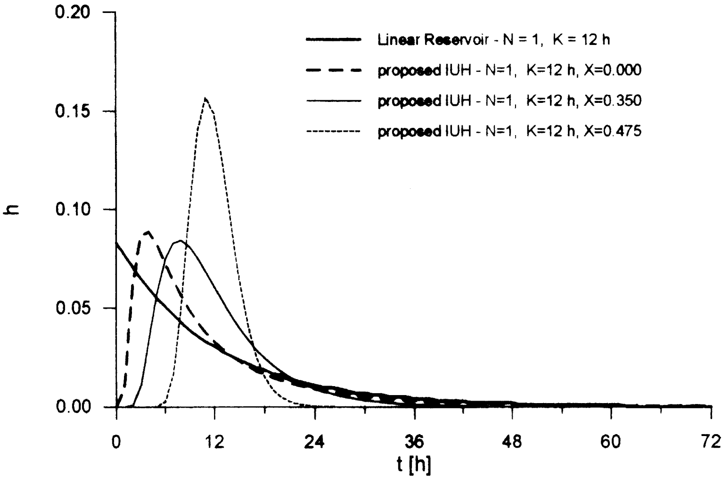

Fig. 2. Instantaneous unit hydrographs of the Muskingum model for $N=1$, $K=12$ h and for various values of the parameter X.

based on the same equations, i.e. the equation of continuity and the steady flow one.

3. Application of the diffusive wave IUH for the lumped systems

If the lumped models being a semi-discrete form of the kinematic wave model are able to reproduce the solution of the diffusive wave, it seems reasonable to expect the reproduction of this solution using the instantaneous unit hydrographs as well. Therefore, one can attempt to apply the IUH of the diffusive wave model for the lumped systems. In order to do this the parameters typical for this kind of model should be introduced into Eq. (4). Namely the hydraulic diffusivity, the kinematic speed and the length of a channel can be replaced by the following expressions:

$$\nu_n = \left(\frac{1}{2} - X\right)\frac{\Delta x^2}{K} \tag{30}$$

$$C = \frac{\Delta x}{K} \tag{31}$$

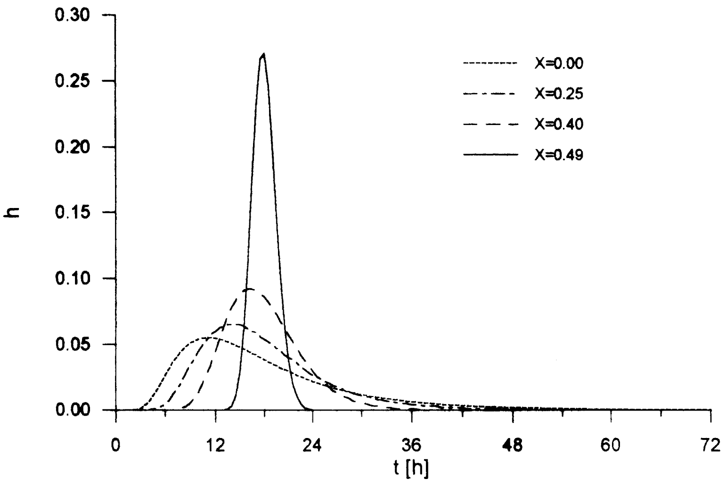

Fig. 3. Instantaneous unit hydrographs of the Muskingum model for $N=3$, $K=6$ h and for various values of the parameter X.

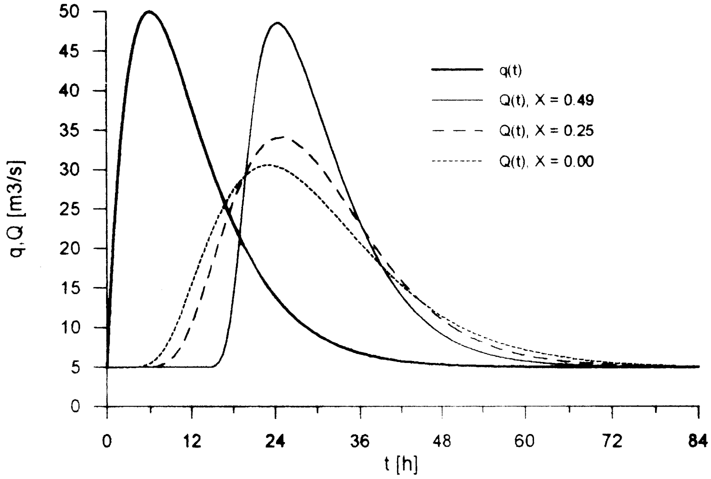

Fig. 4. An example of a food routing by the Muskingum model using the convolution integral and the proposed IUH for $N = 3$, $K = 6$ h and for various values of the parameter X.

$$L = N\Delta x \tag{32}$$

where N corresponds to the number of reservoirs.

Consequently, after a rearrangement, Eq. (4) can be rewritten as follows:

$$h(t) = \frac{1}{(2\pi(1-2X))^{1/2}} \frac{N}{K}\left(\frac{K}{t}\right)^{3/2} \times \exp\left(-\frac{(t-NK)^2}{2(1-2X)Kt}\right) \tag{33}$$

This equation can be considered as an instantaneous unit hydrograph of the Muskingum model for a channel reach of length L.

Eq. (33) has the following properties:

- it holds for $X \leq 1/2$ only, including the negative values as well;
- for $X \to 1/2$ $h(t) \to \delta(t - NK)$;
- $h(t) \geq 0$ for $t > 0$;
- the parameter N can be any positive number, not necessarily an integer.

Some properties listed earlier are illustrated in the figures. In Fig. 2, the proposed IUHs for $N = 1$, $K = 12$ h and for various values of X are plotted. In the same figure, the IUH for classical linear reservoir is presented for comparison. In Fig. 3, the graphs of the IUHs for $N = 3$, $K = 6$ h and for various values of parameter X are plotted. One can notice that the shape of $h(t)$ is mainly determined by parameter X, whereas its location along the axis t depends on the parameters N and K. With increasing in X, the IUH becomes more and more sharper. For extreme case, when $X = 1/2$ its value tends to infinity at the point $t = NK = 18$. It becomes the Dirac δ function. The proposed IUH is able to reproduce the lag time even for $N = 1$.

In Fig. 4, the results of flood routing are presented. The wave at the upstream end was taken in the following form:

$$q(t) = q_b + (q_m - q_b)\left(\frac{t}{t_m}\right)\exp\left(1 - \frac{t}{t_m}\right) \tag{34}$$

with $q_b = 5$ m^3/s, $q_m = 50$ m^3/s, $t_m = 6$ h.

The calculations were carried out for $K = 6$ h and $N = 3$. The wave attenuation depends on the values of the parameter X. It increases with a decreasing in X. On the other hand for increasing value of X, the attenuation of the calculated output at the downstream end is reduced. For example, for $X = 0.49$ it is so small that the flood routing can be regarded as a pure translation of the wave. In this case, the lag time is equal $NK = 18$ h.

The values of the parameters K, N and X can be computed by the method of moments from the hydrographs given at the upstream and downstream ends of a channel. One can show that the integral of the function $h(t)$ over t from zero to infinity is equal to 1.

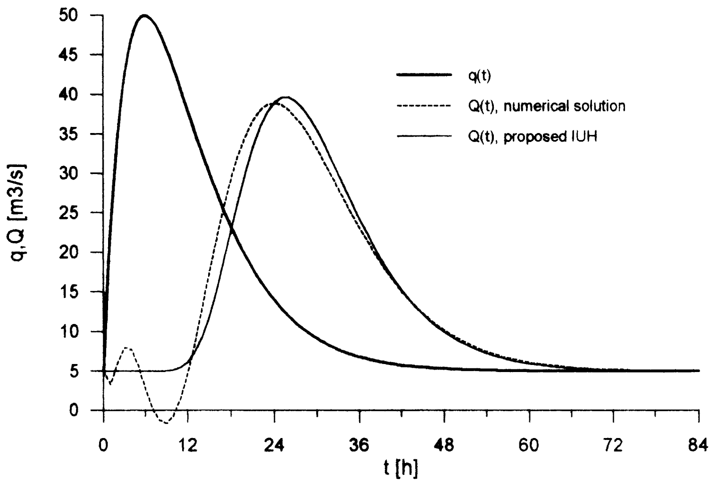

Fig. 5. A comparison of the solutions of the Muskingum model by the finite difference method and by the convolution integral with the proposed IUH for $N=3$, $K=6$ h and $X=0.40$.

Whereas the succeeding moments are as follows:

$$M_1 = NK \tag{35}$$

$$M_2 = N^2K^2 + (1-2X)NK^2 \tag{36}$$

$$M_3 = N^3K^3 + 3(1-2X)N^2K^3 + 3(1-2X)^2NK^3 \tag{37}$$

To calculate N, K and X a method of optimisation can be applied as well.

The examples presented earlier show the great flexibility of the IUH in the form of Eq. (33). It seems that the proper set of the parameters K, N and X is able to ensure both effects, i.e. the wave translation in time and its attenuation. Moreover, one can add that the parameter X can be a real number not greater than 1/2. It can take a negative value as well. This corresponds to the conclusions presented by Szel and Gaspar (2000) on the negative values of X. Whereas N does not have to be an integer. Consequently, conversely to the classical IUH, the proposed one does not need any special treatment for real value of N. Regardless of the assumed values of the parameters the obtained results are always smooth without any oscillations because always $h(t) \geq 0$ for $t > 0$. Whereas while solving numerically, the Muskingum model (Eq. (8)) physically unrealistic results can be very often obtained. Such situation is presented in Fig. 5. For the wave at the upstream end given by Eq. (34), the Muskingum model with $N=3$, $K=6$ h and $X=0.40$ solved numerically using the classical scheme of the finite difference method (Chow et al., 1988) produces the discharge lower than the one imposed as a boundary condition. It takes even the negative values. For the same data, the Muskingum model solved using the convolution integral with the proposed IUH produced the smooth hydrograph at the downstream end.

For $X=0$ one obtains an alternative instantaneous unit hydrograph for a cascade of the linear reservoirs:

$$h(t) = \frac{\sqrt{K}}{\sqrt{2\pi}}\frac{N}{t^{3/2}}\exp\left(-\frac{(t-NK)^2}{2Kt}\right) \tag{38}$$

For a single reservoir, it becomes as follows:

$$h(t) = \frac{\sqrt{K}}{\sqrt{2\pi}}\frac{1}{t^{3/2}}\exp\left(-\frac{(t-K)^2}{2Kt}\right) \tag{39}$$

Both hydrographs presented earlier differ from the classical ones in the forms of Eqs. (10) and (11). This difference can be explained while analysing the way of derivation of Eq. (10). Classical IUH for the linear reservoir was obtained by integration of Eq. (8) resulting from a spatial discretisation of the kinematic wave model. Consequently, an implicit mechanism of a numerical diffusion was included. Whereas Eq. (39) represents an analytical solution of the diffusive wave model with regard to both variables x and t. Note that although the classical IUH for the linear reservoirs in series and the proposed one are different, they have the same first and second moments: $M_1 = NK$, $M_2 = K^2N(N+1)$.

4. Conclusions

It is shown that the classical instantaneous unit hydrographs have not a general character because they are derived from the equations being in fact the kinematic wave model spatially approximated. Consequently, they have imprinted implicitly the mechanism of numerical diffusion. Making use of this fact, the equations of hydrological lumped models were derived directly from the continuity equation and the equation of steady uniform flow.

An analysis of usually applied additional formula relating storage, inflow and outflow show that it can be interpreted as a result of numerical integration of a storage and of the application of the steady uniform flow equation.

Making use of the similarity of the numerical solutions of the diffusive wave model, the kinematic wave model and the Muskingum model, an alternative IUH for the hydrologic lumped systems is proposed. The presented IUH is based on the IUH proposed by Hayami for the linear diffusive wave. It has remarkable advantages. First of all it has a more general character because it holds for a single linear reservoir, a cascade of linear reservoirs, the Muskingum model and the kinematic wave model. Moreover, it is able to ensure simultaneously both important effects: wave translation in time and its attenuation. The proposed IUH has three parameters K, N and X, typical for the lumped models, and is very flexible. It ensures a smooth hydrograph at the downstream end of a channel for any set of the parameters. Consequently, it seems to be interesting alternative for numerical integration of the Muskingum equation, which very often causes remarkable trouble. Additionally, the proposed IUH makes it possible to use this equation to model the rainfall–runoff process as well.

References

Chow, V.T., 1964. Handbook of Applied Hydrology. McGraw-Hill, New York.

Chow, V.T., Maidment, D.R., Mays, L.W., 1988. Applied Hydrology. McGraw-Hill, New York.

Cunge, J.A., 1969. On the subject of a flood propagation computation method (Muskingum method). Journal of Hydraulic Research 7 (2), 205–230.

Dooge, J.C.I., 1959. A general theory of the unit hydrograph. Journal of Geophysics Research 64 (1), 241–256.

Eagleson, P.S., 1970. Dynamic Hydrology. McGraw-Hill, New York.

Fletcher, C.A., 1991. Computational Techniques for Fluid Dynamics, vol. I. Springer, Berlin.

Miller, W.A., Cunge, J.A., 1975. Simplified equations of unsteady flow. In: Miller, W.A., Yevjewich, V. (Eds.). Unsteady Flow in Open Channels. Water Resources Publishing, Fort Collins, CO.

Nash, J.E., 1957. The Form of the Instantaneous Unit Hydrograph. IASH Publication No. 45, vol. 3–4, pp. 114–121.

Strupczewski, W.G., Napiórkowski, J.J., Dooge, J.C.I., 1989. The distributed Muskingum model. Journal of Hydrology (111), 235–257.

Szel, S., Gaspar, C., 2000. On the negative weighting factors in the Muskingum-Cunge scheme. Journal of Hydraulic Research 38 (4), 299–306.

Venetis, C., 1969. The IUH of the Muskingum channel reach. Journal of Hydrology (7), 444–447.