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Downward field continuation in combining satellite and ground-based internal magnetic field data

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Abstract

When ground-based and satellite internal magnetic field data are available they are referred to different concentric spheres outside a sphere containing all the sources. Suppose that ground-based and satellite simultaneous data sets are treated independently. For both independent and simultaneous Spherical Harmonic Analysis (SHA) field models are calculated by the method of least squares via the set of Gauss coefficients $\{g_n^m ; h_n^m\}$ where a definite truncation index N is used. Furthermore, there is supposed that the Gauss coefficients are correctly determined under these conditions so that no error discussion has to be included. By both of the models—the ground-based and the satellite SHA model—for all points outside the sphere containing all the sources the values of the internal field can be determined, but both models will give different results because for the same truncation index N there is a different approximation quality of the infinite series expansion SHA for the ground in comparison to satellite altitude. The mathematical reason is that the convergence quality of the infinite SHA expansion depends on the distance of the reference sphere from the source region to which the expansion is referred. The differences can be evaluated when the relevant functional systems for both SHA models are investigated for which the different set of Gauss coefficients are determined. If ground-based and satellite internal magnetic field data are combined it is necessary to pay attention to such differences. From the physical point of view it is clear that the physical content of the data depends on the distance of the reference sphere from the source region. Shorter wavelengths of the field are strongly attenuated at satellite altitudes. Therefore, it proves to be advantageous to include a downward field continuation when ground-based and satellite internal magnetic field data are combined although this is an ill-posed inverse problem from the mathematical point of view (Anger, G., 1990. Inverse Problems in Differential Equations. Akademie, Berlin: Plenum Press, London; Huestis, S.P. Parker, R.L., 1979. Upward and downward continuation as inverse problems. Geophys. J. R. Astr. Soc. 57, 171–188). From the convergence quality of simultaneous SHA field models referred to related concentric reference spheres a regularizing criterion for a downward continuation of the Gauss coefficients had been derived which uses inherent mathematical characteristics of the SHA procedure [Webers, W., 1998. On Combining Ground-based and Satellite Magnetic Field Data (Scientific Technical Report STR 98/21). Geo Forschungs Zentrum Potsdam (pp. 384–401); Webers, W., 1999. On determining reference fields from ground-based and satellite field data. Phys. Chem. Earth (A) 24, 943–947]. Applying this procedure to downward field continuation for the International Geomagnetic Reference

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Field DGRF 1990 demonstrates the different physical content of simultaneous field models at concentric spheres with their different structures. Moreover, mathematical upward and downward continuation in connection with simultaneous SHA field models based on simultaneous independent data sets at satellite altitude and at the ground will open new insights how internal and external constituents contribute to magnetic field records at different altitudes.

1. Introduction

Modern research in geomagnetism preferably uses ground-based and satellite field data to study and interpret the internal and external magnetic field. Among other advantages recording the magnetic field also using satellites adds a lot of valuable data to the ground-based world-wide measurements and provides a better global distribution. In particular this is very valuable for those areas of the globe with only few magnetic observatories or measurement points, especially oceanic regions. When global internal magnetic field models (e.g. SHA models) are determined they are referred only to a definite reference sphere. Combining ground-based and satellite field data for this purpose requires, therefore, an upward or downward field continuation, where both are ill-posed problems.

Field data close to the source region have more detailed physical content because shorter wavelengths of the field do not reach larger distances. So mostly it would be more convenient to continue the satellite data down to the Earth's surface and to use them together with the ground data at this reference sphere (or ellipsoid etc.). On the other hand the downward field continuation is a more difficult mathematical problem than an upward one. In particular, for the downward continuation of a potential field considerable errors occur that affect the SHA terms increasingly with increasing SHA indices n and m (Rösler, 1981).

2. Mathematical background

SHA field models for the internal magnetic field mathematically are finite partial sums of infinite series expansions.

An infinite spherical harmonic expansion can be achieved by a special arrangement of the expansion terms of an infinite three-dimensional Taylor expansion where the SHA is particularly suitable for representing the magnetic potential and the magnetic field at the surface of a sphere as the Earth's surface (e.g. Kellog, 1929; Kautzleben, 1965). It follows that the mathematical characteristics of the SHA expansion are described by the theory of infinite series expansion, in particular by the theory of infinite power series expansions (e.g. Knopp, 1922; Hobson, 1931; Lense, 1954). Here, the most interesting points are the convergence behaviour of the infinite SHA expansions and their approximation by partial sums when the expansions are referred to different reference spheres.

In geomagnetism we customary use finite approximations of the infinite SHA expansion for the internal magnetic field and determine numerically the SHA coefficients (Gauss coefficients) $\{g_n^m; h_n^m\}$ by the method of least squares from records on a specific sphere of radius r, e.g. of the mean Earth's surface of radius $r = a = 6371.2$ km. Here the method of least squares is applied to the derivatives of the potential V [Eq. (1)], i.e. to the recorded field components where the functional system of the least squares method essentially contains this radius r. Therefore, these coefficients thus determined are referred to that sphere of radius r. Consequently, the coefficients $\{g_n^m ; h_n^m\}$ are based on this relevant convergence behaviour and the approximation quality by the finite approximative partial sum of truncation index N. Despite this, with these specific characteristics referred to the sphere of radius r the field values can be calculated by the SHA for all points external to the source region, i.e. for $r \geqslant a$, being called the convergence area. This means, that the convergence characteristics of the reference sphere implicitly effect also the calculated field values for all the other points beyond the reference sphere. In dependence on the distances of the reference point/the reference sphere of the series expansion from the sources of the field there is a definite convergence behaviour, characterized by the set of coefficients. The coefficients describe convergence quality, convergence area and through dependence on truncation index the approximation quality of the field by finite partial sums.

These theoretical aspects are not so important as far as the investigations concern the internal magnetic field data at the Earth's surface or nearby (Mauersberger, 1961). The problem has to be considered in more detail when field records at satellite altitude of some hundreds of kilometres are taken into account.

As far as one is treating the infinite series expansion for every point within the convergence area the field value can be described correctly by different series expansions having been referred to their reference point/reference spheres. The situation becomes more difficult because of practical reasons only finite partial sums of the infinite series expansions can be used. Limited by certain error restrictions partial sums of the SHA can be used as finite expressions neglecting the background. In particular, for low truncation index $N=6$, or up to 10 often geophysicists are used to apply the same set of Gauss coefficients for other concentric reference spheres near the Earth's surface, i.e. the field is continued upward and downward by the ratio of the relevant radii. This rough approximation appears as insufficient if data from satellite altitudes of some hundreds of kilometers are taken into account or for a downward field continuation, e.g. down to the core-mantle boundary of the Earth. Here, the user of the SHA has to be aware of the mathematical characteristics of the infinite series expansion where a partial sum truncated at any index N is used for practical purposes. Therefore, e.g. at satellite altitude there is a relative rapid convergence giving a definite approximation quality for a truncation index N. For the Earth's surface i.e. at a concentric reference sphere (or ellipsoid etc.) the convergence of the SHA is slower, being closer to the sources of the internal field. Therefore, one would need a higher truncation index $N+M$ with more SHA terms as in satellite altitude in order to receive the same approximation quality for the field by the SHA partial sums. Practically this means that when the field is downwardly continued from the satellite altitude to the ground beyond the truncation index N the necessary M terms are not available for an adequate field representation. This shows the difficult problem of ill-posedness mentioned.

When the functional system of the SHA expansion is evaluated for the contributions in an abstract mathematical space the different convergence behaviour of the SHA expansions at satellite altitude and at the ground can be compared. From these inherent characteristics of both SHA expansions a mathematical procedure is derived in order to determine downward continued Gauss coefficients for a downward field continuation. Mathematically, this procedure is to be characterized as regularization.

In this way also the different physical content of the field data and the relevant SHA models at both concentric reference spheres i.e. satellite altitude and Earth's surface can be compared and evaluated.

3. Method

The procedure is introduced as follows:

The internal magnetic field $\mathbf{B}_{\text{int}}$ is given by

$$\mathbf{B}_{\text{int}} = -\nabla V \tag{1}$$

with the potential

$$V = \sum_{n=0}^{N} \sum_{m=0}^{n} a\left(\frac{a}{r}\right)^{n+1} \left(g_n^m \cos m\lambda \ + h_n^m \sin m\lambda\right) P_n^m(\cos\vartheta) \tag{2}$$

as SHA model where

(r,ϑ,λ)	are the spherical polar coordinates
a	is the radius of the Earth's surface (nominally 6371.2 km)
P_n^m	is a Schmidt quasi-normalized associated Legendre function of degree n and order m,

and g_n^m ; h_n^m are the Gauss coefficients.

The Gauss-coefficients $\{C_k\} = \{g_n^m \ ; \ h_n^m\}$ are calculated as coefficients of the global SHA model, e.g. published as the coefficients of the International Geomagnetic Reference Field (IGRF or DGRF) authorized by the relevant IAGA working group or they are published as coefficients of similar global internal magnetic field models.

For the Earth's surface $r = a$ the set of Gauss coefficients $\{g_n^m \ ; \ h_n^m\} = \{C_k\}$ is calculated by the well defined method of least squares as coefficients of the functional system $\{f_k\}$ where

$$\begin{aligned}
f_1 &= a \cdot P_1^0 \\
f_2 &= a \cdot \cos\lambda P_1^1 \\
f_3 &= a \cdot \sin\lambda P_1^1 \\
f_4 &= a \cdot P_2^0 \\
f_5 &= a \cdot \cos\lambda P_2^1 \\
f_6 &= a \cdot \sin\lambda P_2^1 \\
f_7 &= a \cdot \cos 2\lambda P_2^2 \\
f_8 &= a \cdot \sin 2\lambda P_2^2 \\
&\ \ \vdots
\end{aligned} \tag{3}$$

etc.

and $\mathrm{k} = 1, 2, \ldots, \mathrm{N} \cdot (\mathrm{N} + 2)$.

The comparison of the SHA field models at different concentric spheres (or ellipsoids etc.) is best given by comparing both functional systems and the set of relevant coefficients in an abstract mathematical space. From this the contribution of every function of the infinite functional system can be evaluated when the stepwise construction of this functional space is characterized by the infinite sequence of stepwise formed parallelepipeds spanned by them. So the convergence behaviour of the SHA models (infinite series expansions) is transposed into the convergence behaviour of the infinite sequence of volumes of parallelepipeds in that mathematical space depending on the index k, the number of functions. Herewith, treating the functional system in an infinite dimensional functional space is made analogously to the geometrical understanding of the vector analysis where the volume of the k dimensional vector space is calculated by the GRAM determinant, the elements of which are formed by the scalar products of the k vectors spanning this space. The GRAM determinant has the form

$$G = \psi(k) = \begin{vmatrix} \langle\varphi_1\rangle, \langle\varphi_1\rangle & \langle\varphi_1\rangle, \langle\varphi_2\rangle & \langle\varphi_1\rangle, \langle\varphi_3\rangle & & \\ \langle\varphi_2\rangle, \langle\varphi_1\rangle & \langle\varphi_2\rangle, \langle\varphi_2\rangle & \langle\varphi_2\rangle, \langle\varphi_3\rangle & & \\ \langle\varphi_3\rangle, \langle\varphi_1\rangle & \langle\varphi_3\rangle, \langle\varphi_2\rangle & \langle\varphi_3\rangle, \langle\varphi_3\rangle & & \\ & & & \ddots & \\ & & & & \langle\varphi_k\rangle, \langle\varphi_k\rangle \end{vmatrix} \quad (4)$$

In functional analysis the system of k vectors is now replaced by the system of k functions, the scalar products are replaced by integration, i.e. by integration over the sphere where

$$\varphi_k = C_k \cdot f_k. \quad (5)$$

Because of orthogonality of the system $\{f_k\}$ [Eq. (3)] for the SHA expansion of the potential V [Eq. (2)] there is

$$\langle\varphi_i, \varphi_k\rangle = \frac{4\pi a^2}{2n+1} \cdot C_k^2 \quad \text{for} \quad i = k$$

$$= 0 \qquad \text{for} \quad i \neq k$$

The GRAM determinant [Eq. (4)] consists only of the products $\langle\varphi_k, \varphi_k\rangle \neq 0$ in the principal diagonal. Therefore, it makes good sense to investigate the stepwise contribution of each φ_k to the volume G in functional dependence on index k.

$$G = \psi(k) \quad (6)$$

being an inherent, invariant mathematical characteristic of the SHA model made by its coefficients. Moreover, the relation (6) characterizes the convergence behaviour of the SHA model, analogously to the well-known spatial spectrum W_n (Mauersberger, 1956; Lucke, 1957; Lowes, 1974; Meyer, 1986)

$$W_n = (n+1)\sum_{m=0}^{n}\left[\left(g_n^m\right)^2+\left(h_n^m\right)^2\right]$$

that gives a combination of the terms of the same degree of V [Eq. (2)] as characteristics of the mean energy density of the related multipole field and results in a different slope for the linear approximation

$$\log W_n = F(n) = a_0 + a_1 n$$

at the ground i.e. for $r=a$ in comparison to the satellite orbit, as for example for $r=a+400$ km that characterizes the different convergence behaviour of the SHA.

To consider the stepwise construction of the GRAM determinant shows the contribution of all the functions φ_k and their coefficients C_k in dependence on index k in the stepwise forming of the volume of the parallelepiped in that mathematical space.

Analogously to the spatial spectrum W_n it proves to be useful that the functional dependence of G on index k is represented in a logarithmic scale

$$\psi(k) = \psi\left(\log\langle\varphi_k, \varphi_k\rangle\right) \quad ; \quad k = 1, 2, \ldots, N(N+2). \tag{7}$$

For a first (and very practical) approximation Eq. (7) can be represented by a linear function

$$\psi(k) = c_0 + c_1 \cdot k \tag{8}$$

Because the convergence behaviour of the SHA expansions at the ground and at satellite altitude is different, necessarily, there is a significant difference for the functional dependence $\psi(k)$ [Eq. (7)] for both. This difference can be shown clearly by the coefficients c_0 and c_1 of Eq. (8) and can be demonstrated by the slope of this linear approximation in a two-dimensional coordinate system with the axes k and log $\langle\varphi_k, \varphi_k\rangle$, respectively.

The goal is to derive a transformation formula for the Gauss coefficients from the comparison of the SHA model for the potential referred to the different reference spheres of radius r_1 and r_2, respectively, in the form of comparing the function $G=\psi(k)$ [Eq. (6)] for both.

As discussed in Section 2, there is a relatively rapid convergence of the SHA expansion at satellite altitude ($r=r_1$). This gives a relatively rapid steep slope of the linear approximation of Eq. (8). For the concentric sphere representing the Earth's surface ($r_2<r_1$) there is a slower convergence, being closer to the sources of the internal field. This gives a not so steep slope in Eq. (8).

A lower convergence quality for the reference sphere of radius r_2 in comparison to the sphere of r_1 with $r_2<r_1$ would need M more SHA terms beyond the truncation index N in order to receive the same approximation quality. These M additional SHA terms are not available in satellite altitude when the calculations have been made there with the truncation index N.

Therefore, the downward continuation necessarily results in a loss of information, mathematically represented by the different convergence quality. Physically, this means: those shorter wavelengths of field sources that do not reach a definite satellite altitude cannot be introduced by any form of downward field continuation. There is no information about them. This loss of information is the physical interpretation of the ill-posedness of the downward field continuation. Constructing a solution under this loss of information is called a regularization procedure from the mathematical point of view. There it is essential to use inherent characteristics or objective criteria if it may be possible.

From the theoretical background presented above it proves to be worthwhile to derive a regularizing procedure from the convergence behaviour of the SHA at concentric reference spheres r_1 and r_2, $r_2 < r_1$. The characteristics of the expansions are used as regularizing criterion when the same convergence quality of the downward continuation is enforced that is given for the original reference sphere in order to guarantee that the continuation procedure preserves the physical content of the original data and no erroneous constituents are added. This means that for the downward continuation the same slope in Eq. (8) is to be enforced as is given for the satellite altitude. In this way the problem of different convergence quality of the SHA expansions for the reference spheres with the radius r_1 and r_2, respectively, has been transformed mathematically into the comparison of the related functions $\psi(k)$ in that two-dimensional space where the mutual relation of the $\psi(k)$ is to be described, i.e. geometrically, what is the mutual behaviour of both curves $\psi(k)$ to each other. Necessarily, this mathematical problem does not depend on the coordinate system or any scale used.

When the linear relation [Eq. (8)] is supposed as sufficient approximation then the same convergence quality for the downward continuation as for the original model of the data set of reference sphere of r_1 (satellite altitude) is enforced when the same slope results for Eq. (8). By the way, already from the physical point of view is obvious that the usual upward and downward continuation only by the ratio of the radii, i.e. by

$$\left(\frac{r_1}{r_2}\right)^{n+1} \quad \text{and its inverse} \tag{9}$$

is insufficient for large differences between r_1 and r_2 because of the fact that those shorter wavelengths that do not reach r_1 cannot be introduced by a downward continuation. Therefore, the regularizing procedure derived here will correct this simple procedure [Eq. (9)].

If $c_0^{(1)}$ and $c_1^{(1)}$ denote the coefficients of the linear approximation [Eq. (8)] for the reference sphere of radius r_1 with $c_1^{(1)} = \tan\alpha_1$ and if $c_0^{(2)}$ and $c_1^{(2)}$ are the related coefficients for the reference sphere of radius r_2, with $c_1^{(2)} = \tan\alpha_1$ then the difference of the slope angles

$$\gamma = \alpha_2 - \alpha_1 \tag{10}$$

describes the different convergence quality.

The mutual relation between the linear approximations

$$\begin{aligned} \log \left\langle \varphi_k^{(1)}, \varphi_k^{(1)} \right\rangle &= c_0^{(1)} + c_1^{(1)} \cdot k \quad \text{for} \quad r_1 \\ \log \left\langle \varphi_k^{(2)}, \varphi_k^{(2)} \right\rangle &= c_0^{(2)} + c_1^{(2)} \cdot k \quad \text{for} \quad r_2, r_2 \leqslant r_1 \end{aligned} \tag{11}$$

in a two dimensional space with the coordinate axes k and $\log \langle \varphi_k, \varphi_k \rangle$ is described by a linear transformation, i.e. by a translation by $c_0^{(2)}$ and a rotation by γ. When the downward continuation of the Gauss coefficients C_k requires to enforce the same slope $c_1^{(1)}$ for r_2 as being given for r_1. Applying the well-known geometrical transformation formula of mathematical text books

$$y'' = -x\ \sin\gamma + y\ \cos\gamma + (1 - \cos\gamma)\cdot c_0$$

if the coordinate system x, y is transformed to x'', y'', therefore, gives:

$$\log\left(C_{k\mathrm{reg}}^2 \bullet \frac{4\pi r_1^2}{2n+1}\right) = -k\sin\gamma + (1-\cos\gamma)c_0^{(2)} + \log\left(C_k^2 \bullet \frac{4\pi r_1^2}{2n+1}\left(\frac{r_1}{r_2}\right)^{2(n+1)}\right) \bullet \cos\gamma$$

so that finally there is a relation for the downward continued Gauss coefficients $C_{k\mathrm{reg}}$ to be calculated from the coefficient C_k at r_1 as

$$C_{k\mathrm{reg}}\left(n, k, C_k, r_1, r_2, \gamma, c_0^{(2)}\right) =$$

$$\mathrm{sign}(C_k) \bullet \left((2n+1) \bullet 10^{(1-\cos\gamma)c_0^{(2)} - k\sin\gamma} \bullet \left(\frac{C_k^2}{2n+1}\left(\frac{r_1}{r_2}\right)^{2(n+1)}\right)^{\cos\gamma} \bullet \frac{\left(4\pi r_1^2\right)}{4\pi r_1^2}\right)^{\frac{1}{2}} \tag{12}$$

for $r_2 < r_1$ and $\gamma = \alpha_2 - \alpha_1$.

As argued above, Eq. (12) gives the downwardly continued Gauss coefficients from the concentric sphere of radius r_1 down to that of radius r_2 according to the criterion of same convergence quality.

Eq. (12) can be proved for $r_2 \rightarrow r_1$, i.e. $\gamma \rightarrow 0$.

For $r_2 = r_1$ necessarily there is $\gamma = 0$ and $C_{k\mathrm{reg}} = C_k$.

According to Eq. (10) the angle γ is given in the difference in the slope angles. Therefore, γ describes the different convergence behaviour at the concentric spheres of radius r_1 and r_2, respectively, and therefore the difference of the characteristics of the relevant SHA expansions. Necessarily, the numerical values are referred to the uniform scales used in the apparatus of formulae.

By $C_{k\mathrm{reg}}$ the field components and their global charts can be calculated and compared.

Mathematically, Eq. (12) means a regularization which gives how the Gauss coefficients have to be corrected in comparison to the insufficient calculation only by the ratio of the radii of the reference spheres. The correction is increasing with increasing index k so that the above mentioned increasing of errors of higher SHA terms is damped increasingly.

The effectiveness of the procedure is proved also when the slopes of log W_n are compared for the satellite altitude and the downwardly continued SHA expansions which are enforced to be the same according to the implemented regularization criterion.

The linear relation Eq. (8) proved to be a relatively good approximation for Eq. (7) so that the relation between both curves $\psi(k)$ is represented by a simple transformation of the coordinate system as given above. Thus the regularizing formula (12) is invariant of scale. Consequently, the downward transformation of the Gauss coefficients C_k according to Eq. (12) mathematically is invariant with respect to any scale used, but it is influenced by the quality, how well $G = \psi(k)$ is approximated, e.g. by Eq. (8) or otherwise. Calculations have been carried out to approximate

$G=\psi(k)$ by a polynomial of higher degree, of second, third or forth degree, but no improved results.

With respect to the much more serious consequences from the inverse problems of theoretical mathematics referred to above, the linear approximation by Eq. (8) really proves to be sufficient. According to its derivation $C_{k\text{reg}}$ has the same dimension as C_k.

The way the numbering index k is introduced [Eq. (3)] is customary in geophysics and proves to be the optimum and very useful, in particular defining the appearance of the curve of Eq. (6) for $G=\psi(k)$. The fact and the principle of the procedure that $\psi(k)$ characterizes the convergence quality of the related SHA model is not lost if the index k is introduced in another way. Then there results another, more complicated form of the function $\psi(k)$ that is to be compared for the SHA models. It is clear that then more and more complicated mathematical tools are necessary to describe the mutual relations of the $\psi(k)$ for the SHA models.

From Eq. (12), it follows, that in dependence on the difference of the radii r_1 and r_2 and on the other parameters γ and $c_0^{(2)}$ the Gauss coefficients C_k are changed by the downwardly continuing procedure. The "damping" effect of the regularization increases with increasing index k and thus affects all the coefficients C_k.

Therefore, in general spatial structures of the field cannot be maintained when a field continuation is calculated. In particular, the higher harmonics, i.e. the small structures are affected. Consequently, also the temporal structure of the magnetic field is changed by a downward field continuation when SHA field models at different epochs are compared. Moreover, spatial and temporal structures of the field at satellite altitude cannot be transferred when the field at the Earth's surface or even at the core-mantle boundary is calculated by a downward field continuation. That makes discussions about the physical conditions within the Earth's interior more difficult.

Because the downward continuation of the field is calculated by the downward continuation of the set of Gauss coefficients of a SHA model of a definite epoch the field transformation is based on the same data quality of the original data set. For each field model of a definite epoch separately an adequate downward continued field is available also for the past when the measurements could not be made with the same quality as today. There is no time smoothing of uncertainties caused by different time intervals and/or different data quality at the used observatories or repeat stations in comparison to treating transformation and time dependence in a joint procedure. Alternatively, the procedure of Bloxham and Jackson (1992) used basis functions for the time dependence that had been derived from the inhomogeneous data sets of world-wide observations and repeat stations with different and inhomogeneous time series and interval lengths as well as of different data quality. Because of time smoothing (see also Harrison, 1994) as a necessity, the calculated charts are to be characterized as averaged results so that interpreting in detail can be difficult when these averages are to undergo a separation procedure.

From the theoretical background it is obvious that necessarily any upward or downward continuation must differ from models received by separate, simultaneous measurements data sets at these different altitude levels because of the different physical content there. Any extrapolation from a two dimensional figure into the three dimensional space can only restrictedly be made. Consequently, the comparison of the models having been upwardly or downwardly continued to other reference spheres/altitudes with field models based on simultaneous field records there can be used for separating the different field constituents and in particular for discussing the contributions of the internal and the external magnetic field.

4. Results and discussion

For demonstration purposes numerical calculations have been carried out for the published International Geomagnetic Reference Field DGRF 1990. Mathematically, a simultaneous field model DGRFs 1990 has been calculated for a satellite altitude of $h = 400$ km by an upward continuation to simplify matters using the ratio of the radii of the relevant reference spheres $r_2 = a$ and $r_1 = a + h$, respectively. According to Eqs. (7) and (8), respectively, the Figs. 1 a–c present the functional dependence

$$\psi(k) = \psi\big(\log\langle\varphi_k, \varphi_k\rangle\big).$$

Because of the orthogonality of the functional system $\{f_k\}$ for the potential V the scalar products $\langle\varphi_k, \varphi_k\rangle$ in Eq. (8) are essentially formed by the set of Gauss coefficients $\{C_k\} = \{g_n^m\ ;\ h_n^m\}$ that are supposed to be determined correctly within the procedure of their calculation.

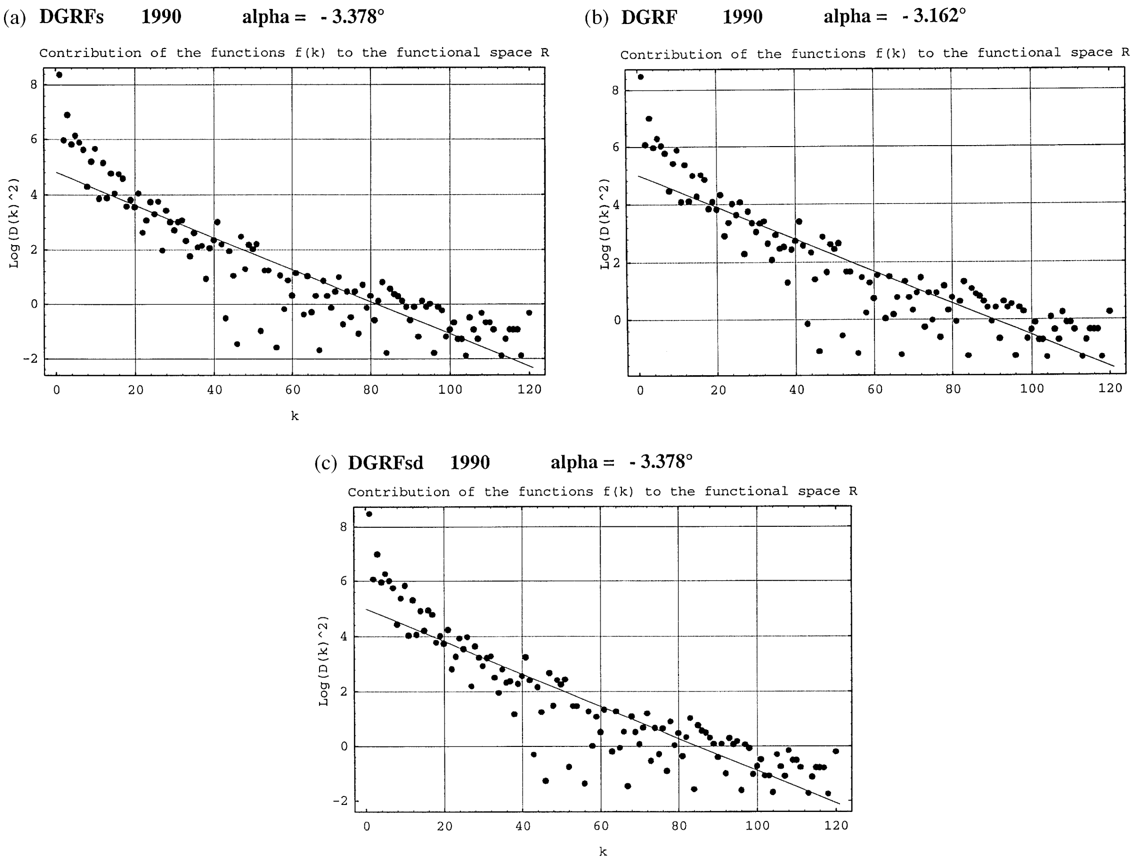


Fig. 1. $\psi(k)$ according to Eq. (7) and its linear approximation by Eq. (8) for DGRF 1990, (a) mathematically upwardly continued to the satellite altitude $h = 400$ km as DGRFs 1990, (b) at the Earth's surface, (c) where the DGRFs 1990 is downwardly continued to the ground by Eq. (12) as DGRFsd 1990.

For the slope of the linear approximation according to Eq. (8) in the logarithmic dependence [Eq. (7)] and in the scale used it follows that

DGRFs 1990: $\alpha = -3.378°$ (Fig. 1a)
DGRF 1990: $\alpha = -3.162°$ (Fig. 1b)

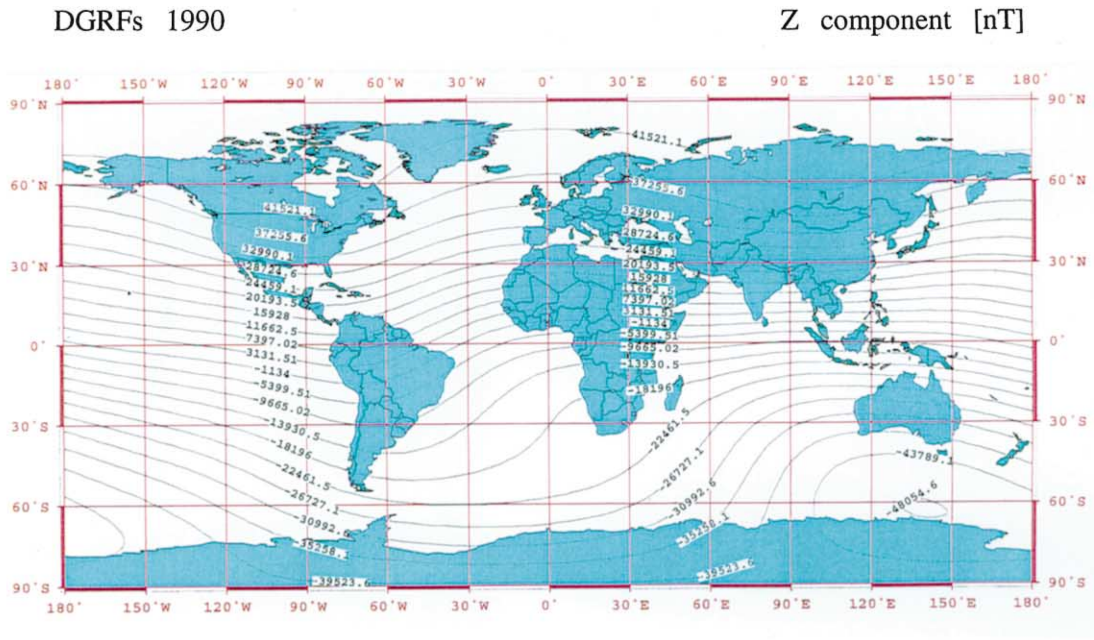
so that the difference $\gamma = 0.216°$ describes the different convergence quality and finally the different physical content of both SHA models with the same truncation index $N = 10$ as theoretically discussed in Section 2. Applying Eq. (12) for determining the downward continued Gauss coefficients $C_{k\text{reg}}$ enables to calculate the downward continued SHA model DGRDsd 1990 (Fig. 1c). Its slope according to Eq. (8) shows from the value $\alpha = -3.378°$ that sufficiently well the convergence quality of the satellite altitude (i.e. of DGRFs 1990) results according to the regularizing criterion that had been derived [Eq. (12)]. The analogous proof is possible for the slope values of the spatial spectrum W_n.

Fig. 2 gives the global charts for the Z component of the geomagnetic field calculated for the satellite altitude $h = 400$ km (Fig. 2a: DGRFs 1990), for the Earth's surface (Fig. 2b: DGRF 1990) and for the downward continuation of the satellite altitude to the Earth's surface (Fig. 2c: DGRFsd 1990). The isolines are given in the usual units nano Tesla (nT). The chart for the satellite altitude (Fig. 2a) shows that more distant from the field sources the general structures are also given similar as at the ground (Fig. 2b). The field values are smaller. There are some differences with respect to small detailed structures. Downward continuation (Fig. 2c) gives the field at the ground in general, but not the correct field values. Some small details where obviously the smaller wavelengths have not reached the satellite altitude, are not contained as theoretically discussed above. The comparison with the ground-based chart, i.e. the difference of the Z-components for DGRF 1990—DGRFsd 1990 (Fig. 2d) demonstrates in detail which physical content cannot be determined when a satellite field model in the altitude $h = 400$ km is downwardly continued to the ground in comparison to the simultaneous ground field model.

This comparison calculated here (Fig. 2a–d), refers as mentioned above, to the mathematical upward continuation for DGRFs 1990 being an approximative calculation and to the truncation index N. Herewith, the principal problem is shown. Simultaneous field models based on recordings at the ground and in satellite altitude and higher values of the truncation index N give relevant modifications of the numerical results but do not change the general as discussed in Section 2.

Despite the aforementioned upward continuation for attaining the simultaneous field model at satellite orbit, the differences of Fig. 2d give an estimation of the problems to consider when ground-based and satellite field data are to be combined. From the physical point of view, Eqs. (1) and (2) obviously show that the wavelength content of the field models at concentric spheres differs (Alldredge, 1982). Moreover, those wavelengths that do not reach a definite reference sphere necessarily cannot be reflected in the related set of Gauss coefficients. Therefore, any downward transformation to a sphere closer to the source region only gives a smoothed model without these wavelengths (regularization). The lack of information cannot be compensated by any mathematical trick. As is obvious from Eq. (12), the downward continuation procedure affects the complete set of Gauss coefficients. Consequently, all elements of the field structure are concerned.

In comparison to other mathematical regularizing procedures that are applied to ill-posed problems there is to emphasize that here the characteristics of the SHA expansions had been used



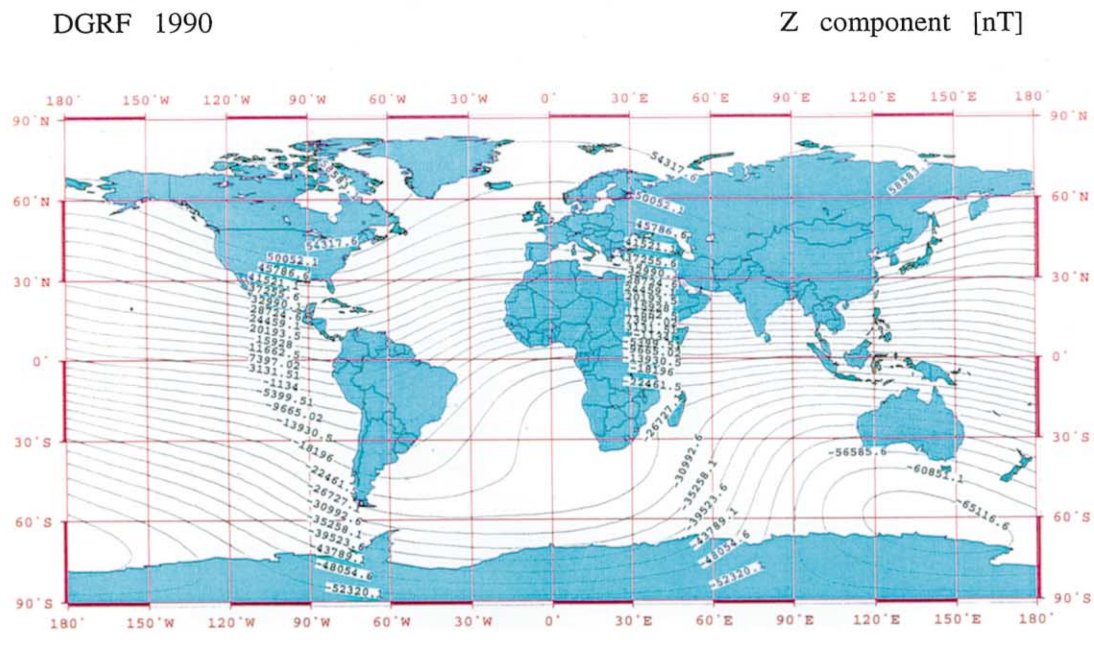
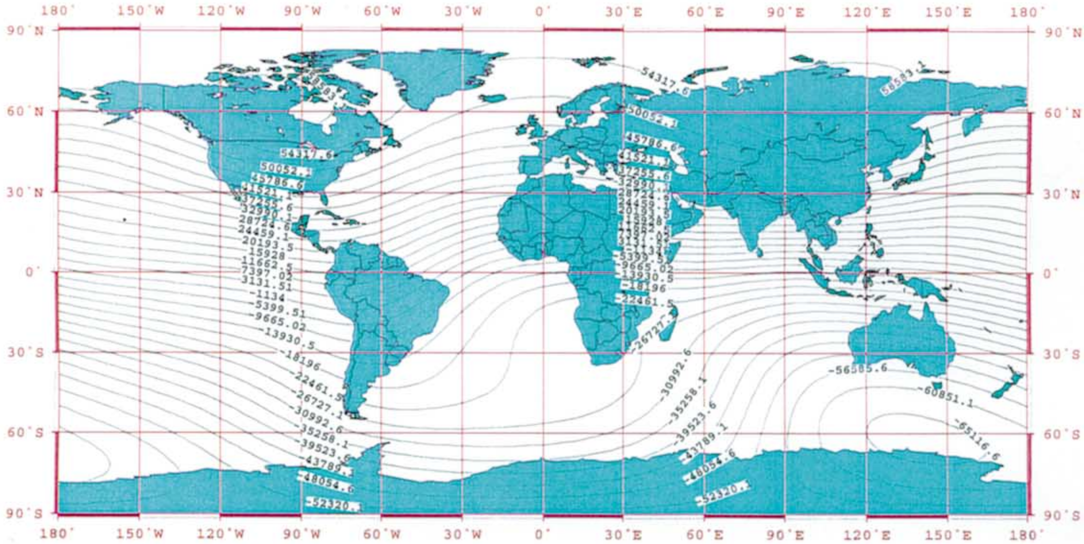


Fig. 2. DGRF 1990 (a) mathematically upward continued to the satellite altitude of $h = 400$ km as DGRFs 1990, Z-component in nT, (b) at the Earth's surface, Z component in nT, (c) where the DGRFs 1990 at satellite altitude (Fig. 2a) is downwardly continued to the ground by Eq. (10) as DGRFsd 1990, Z component in nT. (d) Difference chart DGRF 1990 (of Fig. 2b)—DGRFsd 1990 (Fig. 2c), Z component in nT.

DGRFsd 1990 Z component [nT]



DGRF 1990 – DGRFsd 1990 Z component [nT]

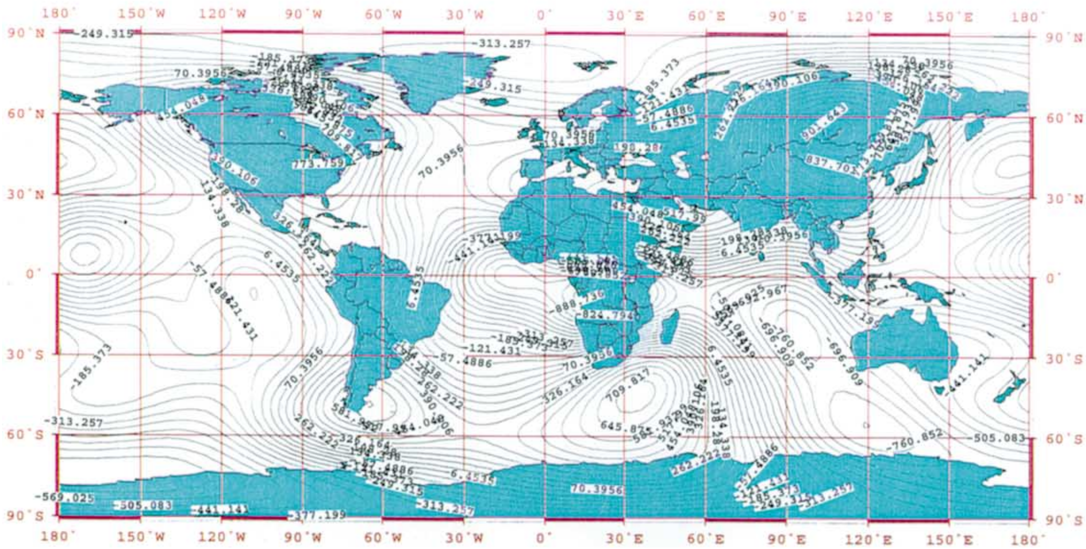


Fig. 2. (*continued*)

to derive the continuation formula [Eq. (12)]. Consequently, no additional assumptions had been introduced that would have to be explained for their physical meaning. Usually, the solution for ill-posed problems is constructed by regularizations where according to mathematical or physical criteria additional conditions are introduced in order to restrict the manifold of mostly infinite number of solutions. Doing so these criteria are most essential for the characteristics of the solution received. Therefore, these criteria must be carefully investigated for their consequences and their physical interpretations.

Deriving the *regularizing formula* [Eq. (12)] makes a downward continuation of the Gauss coefficients and finally that of the internal field much more convenient because *only the own inherent characteristics of the SHA expansion had been used* there without any additional assumptions or e.g. smoothing time functions (Bloxham and Jackson, 1992) or plausible conditions from statistics etc. Calculations for the time interval 1900–1995 for the published global reference fields IGRF/DGRF show impressively that spatial and temporal behaviours of the SHA models are different at the ground in comparison to the satellite altitude of 400 km or 750 km (as the Danish satellite Ørsted) or downwardly continued. For example, the time dependences of the differences of the Gauss coefficients $C_k - C_{k\mathrm{reg}}$ differ significantly and as a consequence the field components too.

These numerical results clearly confirm the mathematical background given above.

5. On aspects of applying field continuation for physical interpretation

From the point of view of physical interpretation the application of mathematical field continuation opens additional possibilities. First of all recording the magnetic field also using satellites gives the advantage of adding a lot of valuable data to the ground-based world-wide measurements. Furthermore, the different physical content of simultaneous data sets at the ground and at satellite altitude in combination with a upward and downward field continuation that principally preserves the original physical content enables interesting aspects.

The physical content of the simultaneous data sets is different for the internal magnetic field because of the distances to the internal magnetic field sources and furthermore for contributions of the external magnetic field to the records. Therefore, it is useful to compare the related SHA models.

Supposing simultaneous field models

$\mathbf{B}_{\mathrm{s}}$ for $r = a + h = r_1$
$\mathbf{B}_{\mathrm{E}}$ for $r = a = r_2$

then the downwardly continued satellite field model $\mathbf{B}_{\mathrm{s}}$, now denoted as $\mathbf{B}_{\mathrm{sd}}$ will be different from the ground-based field

$$\mathbf{B}_{\mathrm{E}} \neq \mathbf{B}_{\mathrm{sd}} \text{ for } r = a \tag{13a}$$

The mathematically upward continued ground-based field model $\mathbf{B}_{\mathrm{E}}$, now denoted as $\mathbf{B}_{\mathrm{ms}}$ will be different from the satellite field model

$$\mathbf{B}_{\mathrm{s}} \neq \mathbf{B}_{\mathrm{ms}} \text{ for } r = a + h \tag{13b}$$

Finally, the ground-based field model $\mathbf{B}_{\mathrm{E}}$, mathematically upward continued to the satellite altitude and then downwardly continued by the procedure [Eq. (12)], now denoted as $\mathbf{B}_{\mathrm{msd}}$ will differ from the satellite field model that is downwardly continued ($\mathbf{B}_{\mathrm{sd}}$)

$$\mathbf{B}_{\mathrm{msd}} \neq \mathbf{B}_{\mathrm{sd}} \quad \text{for} \quad r = a \tag{13c}$$

These comparisons Eqs. (13a–c) necessarily show the differences of the physical content of the field model, in particular also the different contributions of the internal and the external field for r_1 and r_2. The numerical calculations will show which amount of field values can be expected when the simultaneous field models are available and how it can be used practically. For such investigations the quality of the field models as well as the quality of the upward and downward field continuation will be very essential.

As presented above the theoretical background proves that such comparisons will enable to receive valuable information on the different contributions to the magnetic field when the quality of the data, of the models and of the mathematical procedures are sufficiently well.

6. Conclusions

For practically treating SHA models of potential fields, finite partial sums of the SHA with the finite truncation index N are used as reasonable approximations. Consequently, the set of Gauss coefficients characterizes the convergence behaviour and the physical information content of the finite model with reference to the related sphere. Analogue sets of Gauss coefficients of concentric spheres differ significantly.

The regularization procedure [Eq. (12)] for calculating a downwardly continued SHA field model is derived from own inherent and invariant mathematical characteristics of the SHA expansion, i.e. the information content of the original data is used as criterion, practically in the form of the convergence quality of the SHA.

Beyond these own characteristics absolutely no assumptions are introduced.

As differences to the ground-based model the simultaneous satellite field model downwardly continued to the ground provides the different physical information content of both data sets.

As a function of the field points this difference shows the loss of information that has to be taken into account when the wavelength contributions of the sources of the internal magnetic field are discussed.

Consequently, the different spatial and temporal structure of the ground-based field and the simultaneous satellite field which was downwardly continued to ground level open interesting aspects to study the contributions of the field sources—internal and external ones—to the field data at the ground and at satellite altitude so that more detailed conclusions will be possible.

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