Rock Mech. Rock Engng. (2002) 35 (2), 115–126

**Rock Mechanics and Rock Engineering**


# Technical Note

# Confinement Effects and Energy Balance Analyses for Buckling Failure Under Eccentric Loading Conditions

By

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## 1. Introduction

Buckling is a type of failure that has been observed around underground openings in highly stressed as well as in jointed rock masses. Slabs can be formed due to the presence of joint planes parallel to the excavation surfaces in the case of highly jointed rock masses, or under high compressive stress that acts parallel to the excavation boundaries in intact rock masses (Hoek and Brown, 1980). Two possible mechanisms of slab formation are illustrated in Fig. 1. A combination of them can be considered as a mechanism for slab formation and buckling in the case of moderately jointed rock masses, with discontinuous structural features, under high stress. Buckling and slabbing failure under such conditions has been observed in underground mines in the Sudbury Basin in close proximity to mine openings (Swan and Semadeni, 1992) and resulted in hazardous conditions for the mine personnel due to rockburst activity (Fig. 1c). A relatively small confining pressure, compared to in-situ stresses, provided by filling a stope or a pass is often adequate to stop failure at the stope walls.

The mechanism for the formation of a slab that is exposed to buckling load is not examined in this paper. For example, axial splitting has been the focus of extensive research in the past (Fairhurst and Cook, 1966; Dyskin and Germanovish, 1993), while numerical solutions for buckling have been provided by Papamichos and Vardoulakis (1990); Hu (1997) and others. The analytical formulae for slab buckling provided here are based on an eccentric loading approach as described by Timoshenko (1976). It is assumed that buckling failure occurs towards the opening. The focus of the analysis has been the examination of the effect of small confining pressures (in the order of 1 MPa) to control failure and the determination of the amount of energy stored in column under eccentric loading conditions. Eccentricity is defined as the distance between the central axis of the column and the applied load. In the classical Euler approach, it is assumed that no lateral

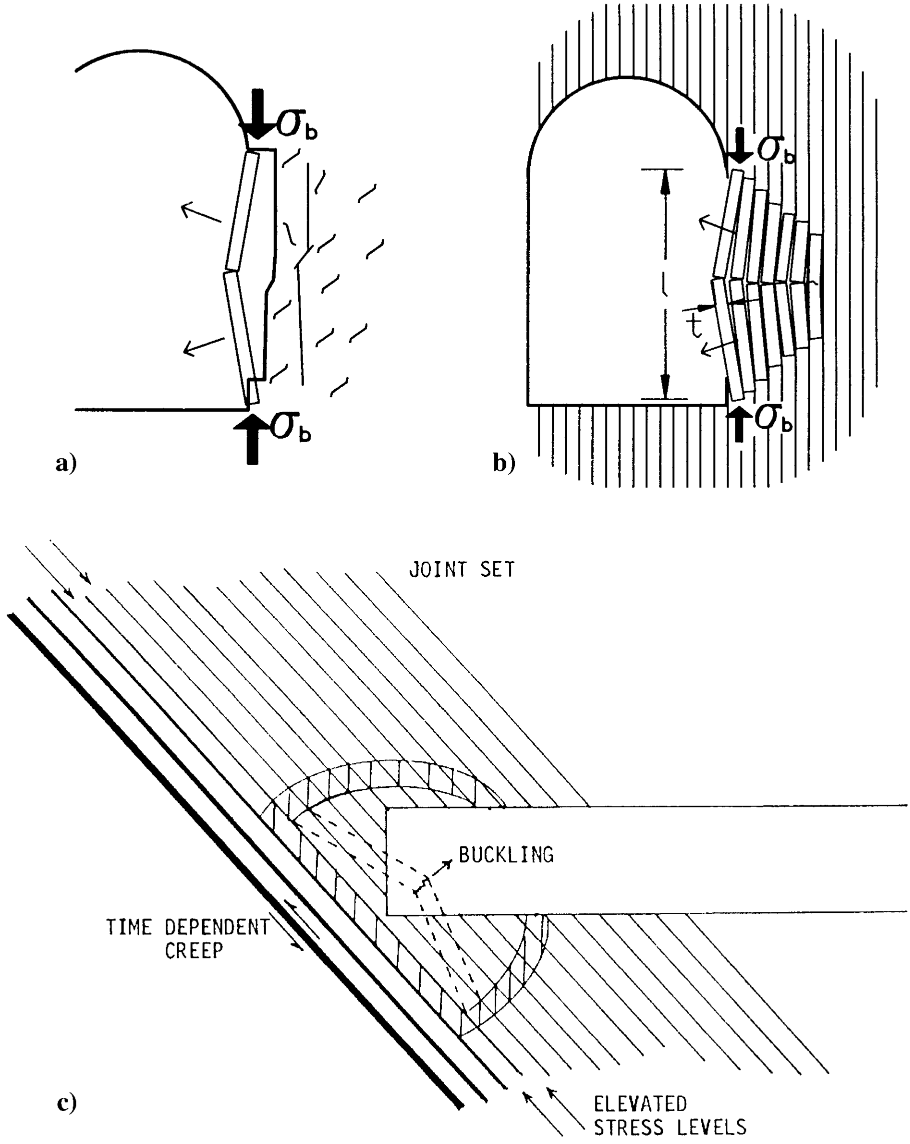


**Fig. 1.** Mechanisms for slab formation and buckling in intact **(a)** and jointed **(b)** rock masses. **(c)** buckling failure in a mine (after Swan and Semadeni, 2001)

movement (i.e., zero deflection) occurs at the column under axial load until the column buckles (Jaeger and Cook, 1976).

## 2. Analysis Using Euler's Formula

Euler's formula is a common analytical approach followed to examine buckling failure. The critical load $P_{cr}$ for a column to buckle under axial load (Fig. 2) can be determined using Euler's formula (Gere and Timoshenko, 1990). For the fundamental case of a pinned-ended column (as opposed to clamped ended), where the ends are free to rotate but are restrained from lateral displacement, the axial stress $\sigma_b$ under which a slab will buckle is given by the equation:

$$\sigma_b = \frac{P_{cr}}{A} = \frac{\pi^2 EI}{Al^2} = \frac{\pi^2 E}{12(l/t)^2}, \tag{1}$$

where:

$P_{cr}$: the critical load;
$E$: Young's Modulus;
$l$: the length of the column;
$t$: the thickness of the column;
$w$: the width of the column;
$A$: the cross section under loading, $(A = wt)$;
$l/t$: the slenderness ratio of the column;
$I$: the moment of inertia, $(I = wt^3/12$, for rectangular cross sections).

The above equation indicates that the buckling stress is a function of the modulus of elasticity and the slenderness ratio. The effect of these parameters on the determination of the buckling stress is shown in Fig. 2. It is evident that the smaller the slenderness ratio, the larger the required stress for buckling. Inherently, it is assumed that buckling load is maintained as the slab deforms.

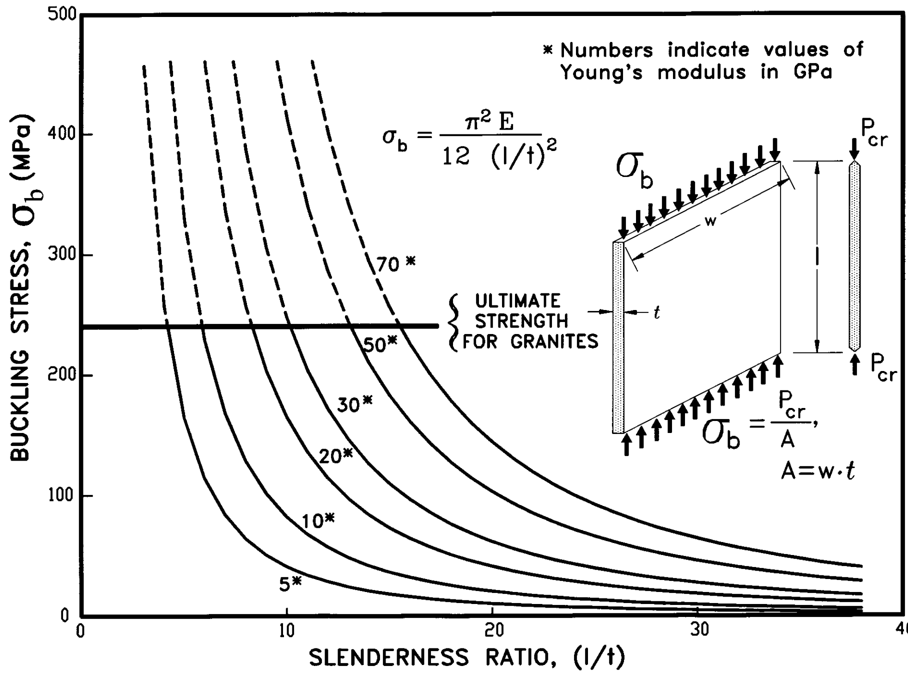


**Fig. 2.** Buckling stress versus slenderness ratio for buckling failure using Euler's formula

It can be seen that the stress required for buckling failure to occur is rather high for the practical range of elastic modulus and slenderness ratio. For a Young's Modulus of 30 GPa and a slenderness ratio of 10, the stress required for buckling to occur is in the range of 250 MPa. It would be rather difficult for such high stress to be found near the boundaries of openings since its value exceeds the strength of the granitic rocks in uniaxial compression. Unless failure initiation in the slab results in Young's Modulus lower than 30 GPa, buckling failure can occur only if the granitic slab has a slenderness ratio greater than 10, as shown in the analysis presented in Fig. 2.

The buckling approach using Euler's formula assumes that the load is applied axially on the column and that failure occurs when the critical stress is reached. That critical stress defines a bifurcation point above which the equilibrium is unstable, and below which it is stable. That stress is not dependent on the strength of the material, and is also independent of the deflection of the column under the load, subject only to end constraints. Thus, the *Euler load* $P_{cr}$ describes the *upper bound* of the value of the critical buckling load.

## 3. The Eccentric Loading Approach

The axial loading analysis in Euler's approach can give a first estimate of the critical buckling load. However, as is pointed out by Timoshenko (1976), the weakness of the approach lies in the fact that as the slenderness ratio increases, various imperfections, such as the initial crookedness of the column, are likely to increase. Then buckling can occur under loads smaller than $P_{cr}$. This introduces a certain arbitrariness into the selection of the proper safety factor against buckling. In addition, the strength of the material is not taken into account, and the column deflections are undeterminable. To make the design procedure more rational, another method has been proposed (Timoshenko, 1976; Gere and Timoshenko, 1990), where the unavoidable inaccuracies in the column could be represented by a small eccentricity $e$, of the applied load. Then the criterion of failure is based on the magnitude of compressive and tensile stresses in the column, rather than the *Euler load*.

The maximum deflection of a column under eccentric load is given by Eq. (2) (secant formula in Timoshenko, 1976):

$$\delta = e\left[\frac{1}{\cos\frac{\kappa l}{2}} - 1\right] \Rightarrow \delta = e\left[\frac{1}{\cos\left(\frac{l}{t}\sqrt{3\frac{\sigma}{E}}\right)} - 1\right], \tag{2}$$

where:

$\kappa^2 = P/EI$; $I = wt^3/12$; $A = wt$
$P$: the eccentric load, $P = \sigma A$;
$e$: the eccentricity.

The corresponding maximum moment will be:

$$M_{\max} = P(e + \delta), \tag{3}$$

and the maximum compressive and tensile stresses are:

$$\sigma_{compr}^{max} = \sigma + \frac{M_{max}}{S}; \quad \sigma_{tens}^{max} = \sigma - \frac{M_{max}}{S}, \tag{4}$$

where: $S$: section modulus ($S = wt^2/6$, for rectangular cross sections).

After linking Eqs. (2), (3) and (4), the formulae for max compressive and tensile stresses can be determined:

$$\sigma_{compr}^{max} = \sigma + 6\frac{\sigma}{t}\left[\frac{e}{\cos\left(\frac{l}{t}\sqrt{3\frac{\sigma}{E}}\right)}\right]; \quad \sigma_{tens}^{max} = \sigma - 6\frac{\sigma}{t}\left[\frac{e}{\cos\left(\frac{l}{t}\sqrt{3\frac{\sigma}{E}}\right)}\right], \tag{5}$$

It can be seen that the stress in the column is a function of Young's modulus, the applied stress, the eccentricity and the slenderness ratio. Then, for a rock slab with a certain strength, there should be a critical slenderness ratio below which failure will initiate in the middle of the column under eccentric loading, and buckling may occur, as shown in the analysis illustrated in Fig. 3. For example, assuming that the tensile strength of the rock is not exceeded, it can be seen that for a column under a stress of 100 MPa, the slenderness ratio for compressive failure to initiate should be less than 13. The selected value of Young's modulus represents the one of the intact rock. For smaller values of Young's modulus or larger eccentricities, the value of the critical slenderness ratio will decrease. Although the approach can give an excellent theoretical description of column behaviour, difficulties can arise because the eccentricity of the load may not be known accurately. This approach was used to analyse the effects of confining pressure on buckling, and to calculate the stored strain energy.

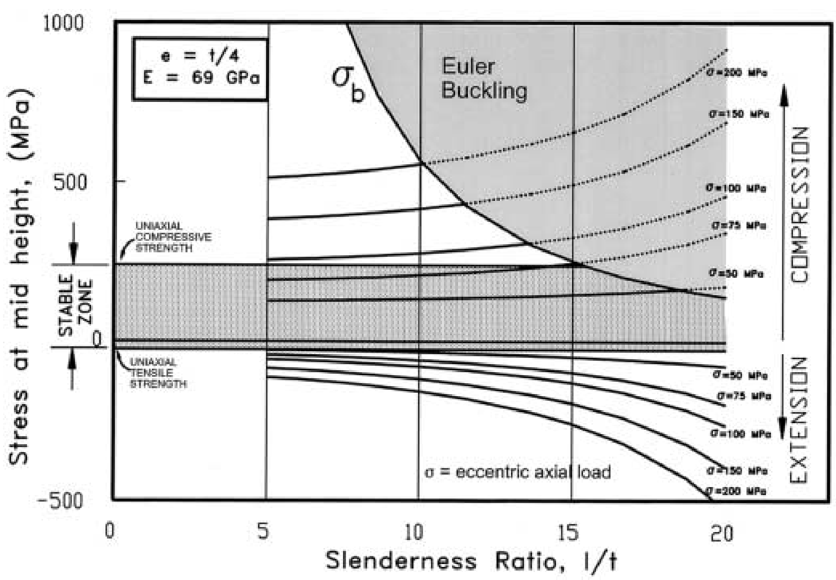


**Fig. 3.** Determination of critical slenderness ratio for buckling analysis under eccentric load

## 4. Influence of Confinement

A comparative study was first carried out in order to examine the maximum deflections generated due to eccentric axial loading and confining pressure. For the case of eccentric axial loading, and for small loads (e.g., $\sigma < 1/10\sigma_b$), the maximum deflection of the column is given by the approximate formula shown in Eq. (6) (Timoshenko, 1976):

$$\delta_{\max} \cong \frac{Pel^2}{8EI} = \frac{\sigma wtel^2}{8EI}, \tag{6}$$

while, for the case of uniformly applied confining pressure, $q$, the deflection of the column at the midpoint is going to be:

$$\delta' = \frac{5qwl^4}{384EI}. \tag{7}$$

Please note that for the case of uniformly applied confining pressure, the eccentricity does not enter into the calculation of the maximum deflection. If it is assumed for the sake of simplification that the role of the confining pressure is not to allow any deflection of the column, then it should be:

$$\delta_{\max} = \delta' \Rightarrow \frac{\sigma te}{8} = \frac{5ql^2}{384} \Rightarrow \frac{q}{\sigma} = 9.6\frac{te}{l^2}, \quad (for\colon \sigma < \frac{1}{10}\sigma_b). \tag{8}$$

Eq. (8) describes the required confining pressure $q$, required to stop the deflection of the column, as part of the average longitudinal stress, $\sigma$. The effect of the slenderness ratio and eccentricity on the $q/\sigma$ ratio are demonstrated in the parametric analysis presented in Fig. 4. It can be seen that for a slenderness ratio of 10, a confining pressure of no more than 5% of the axial stress is adequate to stop any

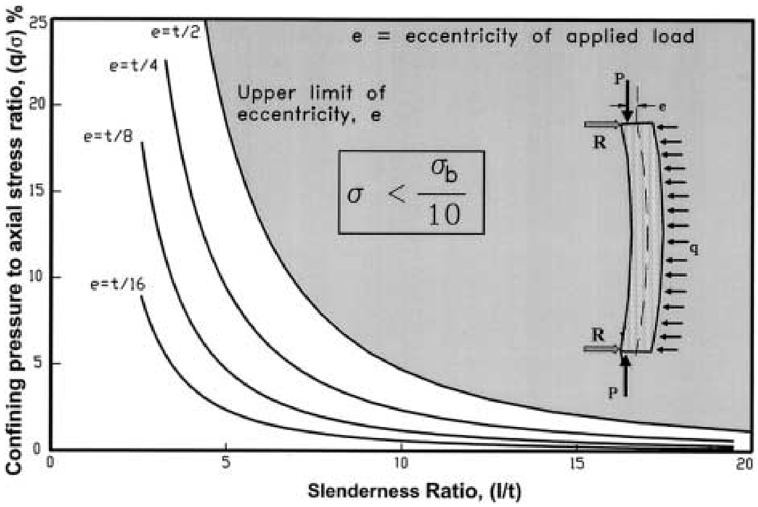


**Fig. 4.** Parametric analysis for the determination of the confinement effects using the maximum deflection analysis

deflection on the column. This analysis introduces the upper bound of confining pressure. The approach oversimplifies the role of confining pressure, and the effect of deflections on the stability of the column, especially in the case of large loads ($\sigma > 1/10\sigma_b$). However, it demonstrates the favourable effect that confining pressure can have on buckling failure and the importance of the eccentricity of the applied load.

In reality, some deflection of the column can be allowed, depending on the mechanical characteristics of the material, without jeopardizing the stability of the column. An analysis was carried out, where the maximum compressive and tensile stresses in the middle of the column were determined taking into account the effect of confining pressure.

The maximum bending moment in the eccentrically loaded column with confining pressure, $q$, occurs at the midpoint where the deflection is at maximum. Based on the superposition method for the equilibrium analysis, the bending moment will be the result of three acting loads: the loading force P, the reaction force R (expressed as a function of $q$), and the confining pressure, $q$, itself, as can be seen in Eq. (9).

$$M_{\max} = P(e + \delta - \delta') - qw\frac{l}{2}\frac{l}{4} + \frac{qwl}{2}\frac{l}{2}, \tag{9}$$

where: $\delta = e\left[\dfrac{1}{\cos\dfrac{\kappa l}{2}} - 1\right]; \quad \delta' = \dfrac{5qwl^4}{384EI};$ (Gere and Timoshenko, 1990)

The maximum compressive and tensile stresses are going to be:

$$\sigma_{\text{compr}}^{\max} = \sigma + \frac{M_{\max}}{S}; \quad \sigma_{\text{tens}}^{\max} = \sigma - \frac{M_{\max}}{S}. \tag{10}$$

After linking Eqs. (9) and (10) we get:

$$\sigma_{\max} = \sigma + \frac{6\sigma}{t}\left[\frac{e}{\cos\left(\frac{l}{t}\sqrt{\frac{3\sigma}{E}}\right)} - \frac{5qwl^4}{384EI}\right] - \frac{1}{S}\frac{qwl^2}{8}, \tag{11}$$

$$\sigma_{\min} = \sigma - \frac{6\sigma}{t}\left[\frac{e}{\cos\left(\frac{l}{t}\sqrt{\frac{3\sigma}{E}}\right)} - \frac{5qwl^4}{384EI}\right] + \frac{1}{S}\frac{qwl^2}{8}. \tag{12}$$

After substituting for $S = wt^2/6$ and $I = wt^3/12$, the final equations will be:

$$\sigma_{\max} = \sigma + \frac{6\sigma}{t}\left[\frac{e}{\cos\left(\frac{l}{t}\sqrt{\frac{3\sigma}{E}}\right)} - \frac{5ql^4}{32Et^3}\right] - \frac{3}{4}\frac{ql^2}{t^2}, \tag{13}$$

$$\sigma_{\min} = \sigma - \frac{6\sigma}{t}\left[\frac{e}{\cos\left(\frac{l}{t}\sqrt{\frac{3\sigma}{E}}\right)} - \frac{5ql^4}{32Et^3}\right] + \frac{3}{4}\frac{ql^2}{t^2}. \tag{14}$$

An example of the application of the approach is introduced in Fig. 5. It can be seen that for the normal range of slenderness ratios, as determined in Fig. 3, and for an applied load of 100 MPa, a confining pressure of 0.5 to 1 MPa is adequate to stop any development of compressive or tensile failure in the column. This confinement pressure can be provided partly by the fractured zone around an opening and by backfill or stored material as is the case in an ore pass. Small confining pressures have been calculated in the case of ore passes full of muck using bin load theories (Kazakidis and Morrison, 1994). Backfill pressure measurements at mines indicated that pressures as high as 2 MPa can develop (CRRP, 1995). The stabilizing effect of confinement in underground mines of the Sudbury Basin has been observed in many cases.

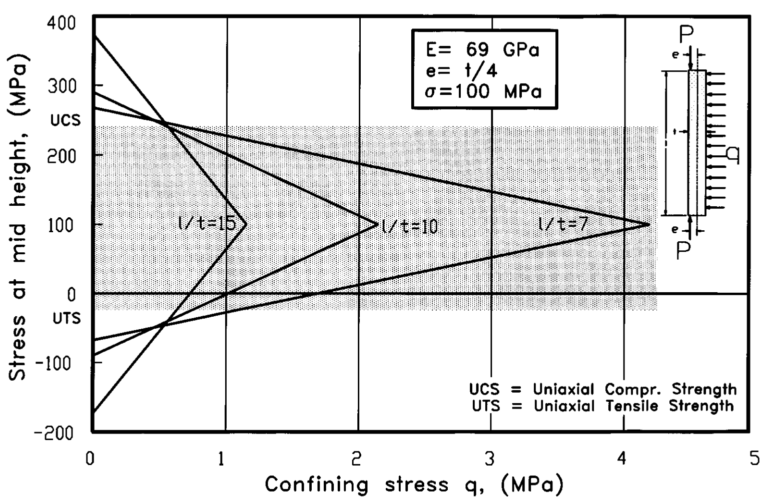


**Fig. 5.** Effect of confining pressure $q$, on stress build-up in a column under eccentric load

## 5. Energy Analysis

The strain energy stored in a column under eccentric load, prior to failure of the column, was then examined. The amount of stored strain energy is often used as indicator of the potential severity of failure that can occur in the vicinity of underground openings and can relate to the design of support systems, and, overall, to underground mine design. The storage of strain energy, $U$, will be due to longitudinal loading and to the bending of the column (Logan, 1991) and is described by the equation:

$$U = \frac{\sigma^2 A l}{2E} + \int_0^l \frac{M^2}{2EI}\, dx, \tag{15}$$

where:

$\sigma$: the average stress applied longitudinally;
$M$: the bending moment;
$E$: Young's Modulus;
$l$: the length of the column;
$A$: the cross section under loading, ($A = wt$);
$I$: the moment of inertia, ($I = wt^3/12$, for rectangular cross sections).

The calculations used in the strain energy analysis and the procedure followed are shown in Appendix I. The stored strain energy for a loading stress of 100 MPa, and a column length of 10 m ($l = 10$ m, $w = 1$ m, $t = 1$ m) was found to be approximately 1 MJ.

## 6. Conclusions

An eccentric loading approach was applied to quantify confinement effects and examine the energy balance in a slab under buckling load. It was indicated that confining pressure provided by fill or broken rock material can prove adequate to control buckling failure under certain loading conditions. The stored strain energy in a slab under eccentric loading conditions, in the vicinity of underground openings was determined to be in the order of 1 MJ.

## References

CRRP (1995): Canadian Rockburst Research Program 1990–1995, vol. 6, chapter 4. Mining Research Directorate, Sudbury, Ontario, 47–55.

Dyskin, A. V., Germanovish, L. N. (1993): Model of rockburst caused by cracks growing near free surface. Proc., 3rd Int. Symp. on Rockbursts and Seismicity in Mines, Kingston, Ontario, 169–174.

Fairhurst, C., Cook, N. G. W. (1966): The phenomenon or rock splitting parallel to the direction of the maximum compression in the neighbourhood of a surface. Proc., 1st ISRM Congress, Col. 1, Lisbon.

Gere, J. M., Timoshenko, S. P. (1990): Mechanics of materials, 3rd edn. PWS-Kent Publishing Co., Boston, 577–626.

Hedley, D. G. F. (1992): Rockburst handbook for Ontario Hardrock Mines. CANMET Special Report SP92-1E, 305.

Hoek, E., Brown, E. T. (1981): Underground excavations in Rock. IMM, 234–240.

Hu, Y. (1997): The buckling failure analysis of a cavern in jointed rock. Int. J. Rock Mech. Min. Sci. 34(3–4).

Jaeger, J. C., Cook, N. G. W. (1976): Fundamentals of rock mechanics, 2nd edn. Chapman and Hall, London, 132–133.

Kazakidis, V. N., Morrison, D. M. (1994): Analysis of key factors affecting ore pass stability in highly stressed rock masses: A case study. IV South American Congress, Workshop on Applied Rockburst Research, Santiago, Chile, 185–197.

Logan, D. L. (1991): Mechanics of materials. HarperCollins Publishers, New York, 601–613.

Papamichos, E., Vardoulakis, I. (1990): Buckling of layered elastic media: A cosserat-continuum approach and its valimation. Int. J. Numer. Analyt. Meth. Geomech. 14, 473–498.

Scholz, C. H. (1990): The mechanics of earhquakes and faulting. Cambridge University Press, Cambridge.

Swan, G., Semadeni, T. (1992): Rockbursts in a development drift: field observations. Workshop on Induced Seismicity, Santa Fe, 12 pp.

Timoshenko, S. (1976): Strength of materials – Part I: Elementary theory and problems, 3rd edn. R. E. Krieger Publ. Co., Huntington, N. Y., 258–280.

## Appendix I: Strain Energy Analysis in Buckling Failure

The deflection of a column is described by the general equation (Gere and Timoshenko, 1990):

$$u = e\left(\tan\frac{\kappa l}{2}\sin\kappa x + \cos\kappa x - 1\right), \quad \text{where } k = \sqrt{\frac{P}{EI}}.$$

The stored strain energy within a column is given by the general equation:

$$U = \int_v \frac{\sigma_x^2}{2E}\,dv, \quad \text{with } \sigma_x = \frac{P}{A} + \frac{M}{I}y, \quad (M = P(e+u))$$

Then it will be: ($I = \int y^2\,dA$, *for axial loading*: $dv = A\,dx$, *for bending*: $dv = dA\,dx$)

$$\begin{aligned} U &= \int_v \frac{1}{2E}\left(\frac{P}{A} + \frac{M}{I}y\right)^2 dv = \int_v \frac{1}{2E}\frac{P^2}{A^2}\,dv + \int_v \frac{1}{2E}\frac{M^2}{I^2}y^2\,dv + \int_v \frac{1}{E}\frac{PM}{AI}y\,dv \\ &= \int_0 \frac{1}{2E}\frac{P^2}{A^2}A\,dx + \int_0^l \int \frac{1}{2E}\frac{M^2}{I^2}(y^2\,dA)\,dx + \int_0^l \int \frac{1}{E}\frac{PM}{AI}(y\,dA)\,dx \\ &= \int_0^l \frac{P^2}{2EA}\,dx + \int_0^l \frac{1}{2E}\frac{M^2}{I^2}I\,dx + 0 = \frac{P^2 l}{2EA} + \int_0^l \frac{M^2}{2EI}\,dx = U_1 + U_2. \end{aligned}$$

With $U_1$ being the energy stored due to the axial stress and $U_2$ the energy due to the bending moment.

$$U_1 = \frac{P^2 l}{2EA}$$

$$U_2 = \int_0^l \frac{M^2}{2EI}\,dx = 2\int_0^{l/2} \frac{P^2(e+u)^2}{(2EI)}\,dx = \frac{P^2}{EI}\int_0^{l/2}(e^2+2eu+u^2)\,dx$$
$$= \frac{P^2}{EI}\left[\int_0^{l/2} e^2\,dx + \int_0^{l/2} 2eu\,dx + \int_0^{l/2} u^2\,dx\right].$$

The above equation consists of three parts: $U_2 = U_2^A + U_2^B + U_2^C$

$$U_2^A = \int_0^{l/2} e^2\,dx = \frac{e^2 l}{2},$$

$$U_2^B = \int_0^{l/2} 2eu\,dx = 2e^2\int_0^{l/2}\left(\tan\left(\frac{\kappa l}{2}\right)\sin(\kappa x) + \cos(\kappa x) - 1\right)$$
$$= 2e^2\left|\left(-\tan\left(\frac{\kappa l}{2}\right)\kappa\cos(\kappa x) + \kappa\sin(\kappa x) - x\right)\right|_0^{l/2}$$
$$= 2e^2\left[\left(-\tan\left(\frac{\kappa l}{2}\right)\kappa\cos\left(\kappa\frac{l}{2}\right) + \kappa\sin\left(\kappa\frac{l}{2}\right) - \frac{l}{2} + \kappa\tan\left(\kappa\frac{l}{2}\right)\right]\right.$$

$$U_2^C = \int_0^{l/2} u^2\,dx$$
$$= e^2\int_0^{l/2}\left[\tan^2\left(\kappa\frac{l}{2}\right)\sin^2(\kappa x) + \cos^2(\kappa x) + 1 + 2\tan\left(\kappa\frac{l}{2}\right)\sin(\kappa x)\cos(\kappa x)\right.$$
$$\left. - 2\tan\left(\kappa\frac{l}{2}\right)\sin(\kappa x) - 2\cos(\kappa x)\right]dx$$
$$= e^2\left|\tan^2\left(\kappa\frac{l}{2}\right)\left(\frac{x}{2} - \frac{1}{4x}\sin(2\kappa x)\right) + \frac{x}{2} + \frac{1}{4\kappa}\sin(2\kappa x) + x\right.$$
$$\left. - \tan\left(\kappa\frac{l}{2}\right)2\kappa\cos(2\kappa x) + \tan\left(\kappa\frac{l}{2}\right)2\kappa\cos(\kappa x) - 2\kappa\sin(\kappa x)\right|_0^{l/2}$$
$$= e^2\left[\left[\tan^2\left(\kappa\frac{l}{2}\right)\left(\frac{l}{4} - \frac{1}{4\kappa}\sin(\kappa l)\right) + \frac{3l}{4} + \frac{1}{4\kappa}\sin(\kappa l) - \tan\left(\kappa\frac{l}{2}\right)2\kappa\cos(\kappa l)\right.\right.$$
$$\left.\left. + \tan\left(\kappa\frac{l}{2}\right)2\kappa\cos\left(\kappa\frac{l}{2}\right) - 2\kappa\sin\left(\kappa\frac{l}{2}\right)\right] - \left[-\tan\left(\kappa\frac{l}{2}\right)2\kappa + \tan\left(\kappa\frac{l}{2}\right)2\kappa\right]\right].$$

After completing the substitutions it is found that:

$$U_2 = \frac{P^2 e^2}{EI}\left\{\frac{l}{2} + 2\left[-\tan\left(\kappa\frac{l}{2}\right)\kappa\cos\left(\kappa\frac{l}{2}\right) + \kappa\sin\left(\kappa\frac{l}{2}\right) - \frac{l}{2} + \kappa\tan\left(\kappa\frac{l}{2}\right)\right]\right.$$
$$+\left[\tan^2\left(\kappa\frac{l}{2}\right)\left(\frac{l}{4} - \frac{\sin(\kappa l)}{4\kappa}\right) + \frac{3l}{4} + \frac{\sin(\kappa l)}{4\kappa} - \tan\left(\frac{\kappa l}{2}\right)2\kappa\cos(\kappa l)\right.$$
$$\left.\left.+\tan\left(\frac{\kappa l}{2}\right)2\kappa\cos\left(\frac{\kappa l}{2}\right) - 2\kappa\sin\left(\frac{\kappa l}{2}\right)\right]\right\}$$

After setting for *average* axially loading stress $\sigma = P/A$, and substituting for $U_1$ and $U_2$ we receive:

Total strain energy:

$$U = U_1 + U_2$$
$$= \frac{\sigma^2 A l}{2E} + \frac{\sigma^2 A^2 e^2}{EI}\left\{\frac{l}{2} + 2\left[-\tan\left(\kappa\frac{l}{2}\right)\kappa\cos\left(\kappa\frac{l}{2}\right)\right.\right.$$
$$\left.+\kappa\sin\left(\kappa\frac{l}{2}\right) - \frac{l}{2} + \kappa\tan\left(\kappa\frac{l}{2}\right)\right]$$
$$+\left[\tan^2\left(\kappa\frac{l}{2}\right)\left(\frac{l}{4} - \frac{\sin(\kappa l)}{4\kappa}\right) + \frac{3l}{4} + \frac{\sin(\kappa l)}{4\kappa} - \tan\left(\frac{\kappa l}{2}\right)2\kappa\cos(\kappa l)\right.$$
$$\left.\left.+\tan\left(\frac{\kappa l}{2}\right)2\kappa\cos\left(\frac{\kappa l}{2}\right) - 2\kappa\sin\left(\frac{\kappa l}{2}\right)\right]\right\}.$$

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