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Reflection and transmission operator for irregular interfaces derived from the indirect boundary element method

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SUMMARY

A method is presented to turn into reality the idea of the reflection and transmission operators in the space–frequency domain for infinitely spread irregular interface in terms of the indirect boundary element method (abbreviated to I-BEM), and to apply them to the wavefield in irregularly stratified media. This method is a hybrid between the strategy of the reflection and transmission matrices method (abbreviated to R/T-MM) developed in seismology and that of I-BEM developing in the engineering community. The I-BEM is one of the ways to disassemble the wavefield into up and down going waves in layers sandwiched between irregular interfaces. Green's function matrices of I-BEM play the role of the wave function of R/T-MM, and the imaginary forces distributed along both faces of interfaces that of the coefficients vectors. The usage of the reference solution, that is the wavefield in the corresponding horizontally stratified media, gives us a good approximation to handle infinitely spread interfaces.

This method of calculation can stack the effect of reflection and transmission at irregular interfaces on the wavefield as the wave goes by. Therefore, for example, the waves coming directly from the seismic source can be extracted from latter phases. The test case for a 2-D homogeneous basin shows the incident wave and reflected waves that bounce up and down in the basin. It is expected that the formulation shown here can make consideration of the wavefield in complex velocity structures and the search for appropriate and efficient approximations easier.

Key words: boundary element method, irregular interface, reference solution, reflection and transmission, synthetic seismograms, wave propagation.

1 INTRODUCTION

The calculation of the seismic response of irregularly stratified media may be required, if we have to estimate seismic ground motion in an area with complicated geological setting such as, for instance, an alluvial basin. During the 1990s, a substantial development took place in numerical methods for this purpose. Domain methods such as finite element and finite difference methods have reached a point where they can be applied to real problems, whereas boundary methods as the boundary element and boundary integral equation methods are still in the research stage. Unfortunately, the capacity of computer available today is not enough to calculate the real size problems using boundary methods. The boundary methods, however, have not yet lost their appeal, and are still attracting researchers, because of the possibility for substantial improvement in cost performance and accuracy, not only by advances in computer technology, but also by the wisdom and effort of the researchers involved. The topic introduced here is not intended for immediate practical application, but is an attempt to find a path for the future development of boundary methods at the border region between seismology and earthquake engineering.

Some theoretical remarks for the wavefield in horizontally stratified media in the wavenumber–frequency domain are given in the following. It is an efficient strategy to disassemble the wavefield in each layer into up and down going waves and to stack the effect of reflection and transmission as waves propagate, in a similar manner to that proposed by Kennett (1983), who treats the reflection and transmission coefficients matrices in the frequency–wavenumber domain. Takenaka & Fujiwara (1994) have applied a similar idea to an irregular structure in terms of the boundary element method in the direct formulation. Cheng (1990, 1995, 1996) developed the approach called the method of global generalized reflection and transmission matrices in the frequency–wavenumber domain for wave propagation in irregularly stratified media. There, the disassembling of the wavefield into up and down going plane waves is also employed. It is expected that formulation of reflection and transmission is possible in terms of I-BEM in the space-frequency domain, because up and down going waves are givn independently by the boundary integral of the product of Green's function matrix and the vectors of the imaginary forces distributed along the interfaces (Sánchez-Sesma & Campillo 1991). Note that this Green's function corresponds to the wave function in the theory of reflection and transmission matrices method (abbreviated to R/T-MM) for horizontally stratified media, and that the distributed force vector corresponds to the wave vector (Kennett 1983). This implies that the reflection and transmission operators may be defined by the relation between these distributed force vectors.

This paper is aimed at showing the procedure to define these reflection and transmission operators for an irregular interface in the space–frequency domain, and to show a good approximation to estimate the reflected and transmitted waves with examples of application. Calculation of a complete seismogram including surface waves, however, is not included in the scope of this paper. The benefit given by these operators is the separation of component waves, e.g., those corresponding to each reflection in reverberation.

2 APPLICATION OF I-BEM IN THE SPACE–FREQUENCY DOMAIN TO HANDLE REFLECTION AND TRANSMISSION OPERATORS

Consider the irregularly stratified media as shown in Fig. 1(a). For the stack of L layers, the free surface is named (0)-th interface and the deeper interface has the bigger number. The shallowest layer is named (1)-st layer and the deeper layer has the bigger number. The underlying half space, i.e. the deepest layer is named $(L+1)$-th layer. In fact $(l+1)$-th layer is sandwiched between (l)-th and $(l+1)$-th interfaces.

Hereafter, the symbols $G^{l}_{l',l''}(x;\,\xi)$ and $H^{l}_{l',l''}(x;\,\xi)$ are used for the displacement and traction Green's function tensors for the full space in (l)-th layer, respectively, the source of which is located at ξ on (l')-th interface and the receiver of which is located at x on (l'')-th interface. The distributed force vectors considered along the upper and lower faces of the (l)-th interface are denoted by the symbols $\bar{\phi}^{l}(\xi)$ and $\underline{\phi}^{l}(\xi)$, respectively.

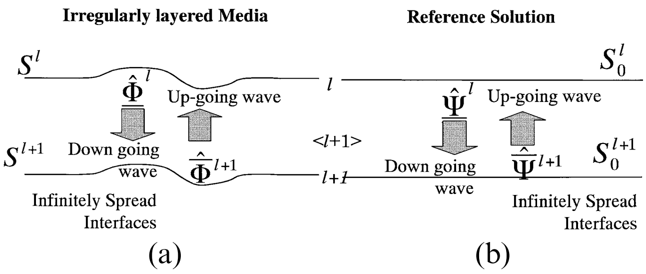


Figure 1. (a) Separation of up- and down going waves in irregularly stratified media in terms of I-BEM. (b) The corresponding horizontally stratified media for the reference solution. Note that the interfaces for both media are spread infinitely long. This is why the symbols of the distributed force vectors $\hat{\bar{\Phi}}^{l+1}(\xi)$, $\hat{\underline{\Phi}}^{l}(\xi)$, $\hat{\bar{\Psi}}^{l+1}$ and $\hat{\underline{\Psi}}^{l}$ are used.

2.1 Up and down going waves in terms of the I-BEM

The displacement and traction wavefield in $(l+1)$-th layer between (l)- and $(l+1)$-interfaces are given as follows in terms of I-BEM (Fig. 1a),

$$\begin{cases} u^{l+1}_{l}(x) = \int_{S^{l+1}} G^{l+1}_{l,l+1}(x;\,\xi)\bar{\phi}^{l+1}(\xi)d\xi + \int_{S^{l}} G^{l+1}_{l,l}(x;\,\xi)\underline{\phi}^{l}(\xi)d\xi\,, \\ t^{l+1}_{l}(x) = \int_{S^{l+1}} H^{l+1}_{l,l+1}(x;\,\xi)\bar{\phi}^{l+1}(\xi)d\xi + \int_{S^{l}} H^{l+1}_{l,l}(x;\,\xi)\underline{\phi}^{l}(\xi)d\xi\,, \end{cases} \tag{1}$$

Note that these boundary integrals have to be performed all along the infinitely spread interfaces.

Suppose that these boundary integrals can be discretized, e.g. as described in Sánchez-Sesma & Campillo (1991) for in-plane problems or by Yokoi & Sánchez-Sesma (1998) for 3-D problems. The symbol $\langle\,\rangle$, denotes the product with Green's function tensor for the full space and integration along the infinitely long boundaries in discrete form, eq. (1) is written as follows.

$$\begin{bmatrix} u^{l+1}_{l} \\ t^{l+1}_{l} \end{bmatrix} = \left\langle \begin{matrix} G^{l+1}_{l,l+1} \\ H^{l+1}_{l,l+1} \end{matrix} \right\rangle \hat{\bar{\mathbf{\Phi}}}^{l+1} + \left\langle \begin{matrix} G^{l+1}_{l,l} \\ H^{l+1}_{l,l} \end{matrix} \right\rangle \hat{\underline{\mathbf{\Phi}}}^{l} \tag{2}$$

The symbols $\hat{\bar{\mathbf{\Phi}}}^{l}(\xi)$ and $\hat{\underline{\mathbf{\Phi}}}^{l}(\xi)$ indicate the force vectors, i.e. the series of the distributed force vectors $\bar{\phi}^{l}(\xi)$ and $\underline{\phi}^{l}(\xi)$ that correspond to discretized segments of (l)-th interface spread away to infinity. Therefore, these are column vectors with an infinite number of elements. Note that the first term of the right side of eq. (2) corresponds to the contribution from the lower interface that is composed mostly of up going waves. The second term corresponds to the contribution from the upper interface that is composed mostly of down going waves.

It might be better to use the term 'contribution of the upper/lower interface' in order to maintain the accuracy of the description. The upper interface of a layer can emit up going wave, the lower down going wave if their shape is abrupt enough. This situation is possible because I-BEM does not put any limitation on the shape of interfaces. In this paper, however, the term 'up/down going wave', is used instead, for simplicity of expression and for the emphasis of the similarity of the theory for irregular interfaces to that for a flat interface.

As shown here, the I-BEM is originally based on the separation of up and down going waves. These are the sets of contribution from distributed forces, that are described by Green's functions for the full space.

2.2 Reference solution

The boundary methods based on the Green's functions for half space and layered media given in the wavenumber domain can be applied directly to infinitely spread interfaces and require the integral over the wavenumber ($\int\,()dk$, where k denotes the wavenumber) (e.g. Campillo 1987; Kim & Papageorgiou 1993 and Hisada & Aki & Teng 1993). In contrast those based on the Green's functions for the full space given in the space domain can handle only finite interfaces and does not need the integral over the wavenumber. After Fujiwara & Takenaka (1993) introduced the basic idea itself, the technique in I-BEM that can handle infinitely spread interfaces is introduced by Yokoi &

Takenaka (1995) for the 2-D free surface and by Yokoi & Sánchez-Sesma (1998) for a 3-D surface. The reference solutions for the flat free surface in these cases are calculated by I-BEM in the wavenumber domain. Yokoi (1996) has applied this technique to irregularly stratified media with the reference solution computed by the Thomson–Haskell matrix method (Thomson 1950; Haskell 1953).

Consider a horizontally stratified media where the material properties of each layer are the same as those of the irregularly layered media shown in Fig. 1(b). The wavefield in the horizontally stratified media is called the reference solution and can be obtained in the wavenumber domain by such methods as the Thomson–Haskell matrix method, R/T-MM (Kennett 1983), the method given by Luco & Apsel (1983) and so on.

Displacement and traction vectors of the down going wavefield observed at (l')-th interface in the (l)-th layer is denoted ($\underline{u}^l_{l'}(x)$, $\underline{t}^l_{l'}(x)$) and that of up going wave ($\bar{u}^l_{l'}(x)$, $\bar{t}^l_{l'}(x)$) for the reference solution. These waves also have their corresponding forces distributed along the horizontal plane interfaces. The symbols $\bar{\psi}^l(\xi)$ and $\underline{\psi}^l(\xi)$ are used for these force vectors along upper and lower faces of (l)-th interface. The wavefield in ($l+1$)-th layer is described by the following.

$$\begin{cases} \bar{u}^{l+1}_l(x) + \underline{u}^{l+1}_l(x) = \int_{S_0^{l+1}} G^{l+1}_{l,l+1}(x;\xi)\bar{\psi}^{l+1}(\xi)d\xi \\ \qquad + \int_{S_0^l} G^{l+1}_{l,l}(x;\xi)\underline{\psi}^l(\xi)d\xi\,, \\ \bar{t}^{l+1}_l(x) + \underline{t}^{l+1}_l(x) = \int_{S_0^{l+1}} H^{l+1}_{l,l+1}(x;\xi)\bar{\psi}^{l+1}(\xi)d\xi \\ \qquad + \int_{S_0^l} H^{l+1}_{l,l}(x;\xi)\underline{\psi}^l(\xi)d\xi\,. \end{cases} \tag{3}$$

The symbol S_0^l denotes (l)-th flat interface that is infinitely spread. Note that these boundary integrals also have to be performed all along the infinitely spread interfaces. The first terms in the both sides correspond to up going wave and the second ones to down going wave, independent of each other.

Eq. (3) suggests the following physical insight. The up going wave can be given by summing up $G^{l+1}_{l,l+1}(x;\xi)$, $H^{l+1}_{l,l+1}(x;\xi)$, i.e. the contribution of the point sources distributed along the upper face of the lower interface, with the weight coefficient $\bar{\psi}^{l+1}(\xi)$. A similar consideration is given for the down going wave with the second term of the right side. In other words I-BEM is based on the decomposition of the wavefield into two sets of contributions of point sources, one is up going and another is down going waves. R/T-MM (Kennett 1983) is also based on the separation of up and down going waves, but these are decomposed into plane waves in the frequency–wavenumber domain, that is, wave functions with coefficients called wave vectors. Obviously, Green's functions in I-BEM correspond to the wave functions in R/T-MM and the distributed forces to the wave vectors. Hereafter, it is shown that it is possible to make use of the distributed forces in I-BEM in a similar way as the wave vectors in R/T-MM, in spite of the difference in the ways the wavefield is decomposed.

Separate the interface S^l in eq. (1) into the finite part around the irregularity, S^l_i, which can be handled in numerical calculation directly, and the infinitely spread part, S^l_e (Fig. 2a). Separate the infinitely spread flat interface S^l_o in eq. (3) into the finite part, S^l_f, and the rest, S^l_e, spread infinitely (Fig. 2b). It is assumed that the infinite part of the interface S^l in eq. (1) is the same as S^l_e in eq. (3), because localized irregularity is usually considered. Then, the following approximation is employed, as shown clearly, e.g. by Yokoi & Sánchez-Sesma (1998),

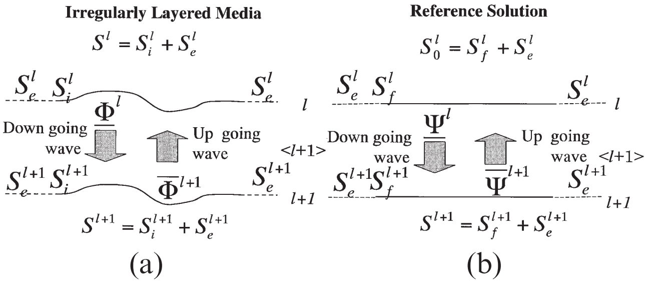


Figure 2. Division of interfaces for the wavefield in irregularly stratified media (a) and in the horizontally stratified ones for the reference solution (b). The contribution of the far part of interface S^l_e for irregularly stratified media is estimated approximately by using the reference solution. The interfaces handled in the boundary integral equation S^l_i and S^l_f are finite. This is why the symbols $\bar{\boldsymbol{\Phi}}^{l+1}(\xi)$, $\underline{\boldsymbol{\Phi}}^l(\xi)$, $\bar{\boldsymbol{\Psi}}^{l+1}$ and $\underline{\boldsymbol{\Psi}}^l$ are used for the distributed force vectors.

$$\bar{\phi}^{l+1}(\xi) \approx \bar{\psi}^{l+1}(\xi),\ \xi \in S^{l+1}_e,\ \underline{\phi}^l(\xi) \approx \underline{\psi}^l(\xi),\ \xi \in S^l_e \tag{4}$$

This approximation and eq. (3) give the following.

$$\begin{cases} \begin{cases} \int_{S^l_e} G^{l+1}_{l,l}(x;\xi)\underline{\phi}^l(\xi)d\xi \\ \quad \approx \int_{S^l_e} G^{l+1}_{l,l}(x;\xi)\underline{\psi}^l(\xi)d\xi \\ \quad = \underline{u}^{l+1}_l(x) - \int_{S^l_f} G^{l+1}_{l,l}(x;\xi)\underline{\psi}^l(\xi)d\xi\,, \\ \int_{S^{l+1}_e} G^{l+1}_{l,l+1}(x;\xi)\bar{\phi}^{l+1}(\xi)d\xi \\ \quad \approx \int_{S^{l+1}_e} G^{l+1}_{l,l+1}(x;\xi)\bar{\psi}^{l+1}(\xi)d\xi \\ \quad = \bar{u}^{l+1}_l(x) - \int_{S^{l+1}_f} G^{l+1}_{l,l+1}(x;\xi)\bar{\psi}^{l+1}(\xi)d\xi\,, \end{cases} \\ \begin{cases} \int_{S^l_e} H^{l+1}_{l,l}(x;\xi)\underline{\phi}^l(\xi)d\xi \\ \quad \approx \int_{S^l_e} H^{l+1}_{l,l}(x;\xi)\underline{\psi}^l(\xi)d\xi \\ \quad = \underline{t}^{l+1}_l(x) - \int_{S^l_f} H^{l+1}_{l,l}(x;\xi)\underline{\psi}^l(\xi)d\xi\,, \\ \int_{S^{l+1}_e} H^{l+1}_{l,l+1}(x;\xi)\bar{\phi}^{l+1}(\xi)d\xi \\ \quad \approx \int_{S^{l+1}_e} H^{l+1}_{l,l+1}(x;\xi)\bar{\psi}^{l+1}(\xi)d\xi \\ \quad = \bar{t}^{l+1}_l(x) - \int_{S^{l+1}_f} G^{l+1}_{l,l+1}(x;\xi)\bar{\psi}^{l+1}(\xi)d\xi\,. \end{cases} \end{cases} \tag{5}$$

These allow handling eq. (1) in numerical calculation, because the integrals along the infinitely spread parts of the interfaces S^l_e and S^{l+1}_e can be approximated by the known vectors and the integrals over the finite parts, S_f^l and S_f^{l+1}. Following the notation used in eq. (2) gives the discrete form of eq. (5) as follows.

$$\begin{cases} \left\langle \begin{matrix} G_{l,l}^{l+1} \\ H_{l,l}^{l+1}\end{matrix}\right\rangle \underline{\hat{\mathbf{\Phi}}}_e^l \approx \left\langle \begin{matrix} G_{l,l}^{l+1} \\ H_{l,l}^{l+1}\end{matrix}\right\rangle \underline{\hat{\mathbf{\Psi}}}_e^l = \begin{bmatrix} \underline{u}_l^{l+1} \\ \underline{t}_l^{l+1}\end{bmatrix} - \begin{bmatrix} G_{l,l}^{l+1} \\ H_{l,l}^{l+1}\end{bmatrix}\underline{\mathbf{\Psi}}^l, \\ \left\langle \begin{matrix} G_{l,l+1}^{l+1} \\ H_{l,l+1}^{l+1}\end{matrix}\right\rangle \hat{\bar{\mathbf{\Phi}}}_e^{l+1} \approx \left\langle \begin{matrix} G_{l,l+1}^{l+1} \\ H_{l,l+1}^{l+1}\end{matrix}\right\rangle \hat{\bar{\mathbf{\Psi}}}_e^{l+1} = \begin{bmatrix} \bar{u}_l^{l+1} \\ \bar{t}_l^{l+1}\end{bmatrix} - \begin{bmatrix} G_{l,l+1}^{l+1} \\ H_{l,l+1}^{l+1}\end{bmatrix}\bar{\mathbf{\Psi}}^{l+1}, \end{cases} \tag{6}$$

where the symbols $\underline{\hat{\mathbf{\Phi}}}_e^l$ and $\hat{\bar{\mathbf{\Phi}}}_e^{l+1}$ denote the infinitely long vectors composed of the forces $\underline{\phi}^l(\xi)$ and $\underline{\phi}^{l+1}(\xi)$ distributed along the lower face of S_e^l and the upper face of S_e^{l+1}, respectively, that correspond to the wavefield in the irregularly stratified media, whereas the symbols $\underline{\hat{\mathbf{\Psi}}}_e^l$ and $\hat{\bar{\mathbf{\Psi}}}_e^{l+1}$ denote those of the reference solution that are composed of $\underline{\psi}^l(\xi)$ and $\bar{\psi}^{l+1}(\xi)$ distributed along the lower face of S_e^l and the upper face of S_e^{l+1}, respectively. The symbols $\bar{\mathbf{\Psi}}^l$ and $\underline{\mathbf{\Psi}}^l$ are used for the force vectors, i.e. the finite series of the distributed force vectors $\bar{\psi}^l(\xi)$ and $\underline{\psi}^l(\xi)$ that correspond to discretized segments of (l)-th flat interface S_f^l. Note that all terms in the right side of eq. (6) are of the reference solution, known and composed of the vectors and matrix of finite size. This is an approximation to estimate the contribution from the infinitely spread part of the interfaces S_e^l and S_e^{l+1} by the calculation with vectors and matrices of finite size.

The approximation in eq. (6) can suppress the non-physical waves, i.e. the artificial noises that appear with substantial amplitude in case of simple truncation of interfaces. The efficiency of suppression is discussed for inplane problems by Yokoi & Takenaka (1995) and for 3-D ones by Yokoi & Sánchez-Sesma (1998). The use of the reference solution allows using modelled interfaces as short as a few tens per cent longer than the length of irregularity considered as a good approximation.

Besides the problem of infinitely spread interface, eqs (1) and (5) have the problem of singularity in the boundary integrals. This difficulty, however, has been solved by Sánchez-Sesma & Campillo (1991) for in-plane problems, and by Sánchez-Sesma & Luzón (1995) for 3-D, using the analytical expressions of Green's functions of the full space.

The approximation in eq. (6) is used in the next Section in order to turn the theoretical consideration of wave reflection and transmission into practical numerical calculation.

2.3 Reflection and transmission operators for an irregular interface

Pay attention to (l)-th irregular interface (l)-th and ($l+1$)-th layers and the wavefield in these two layers in irregularly stratified media in order to obtain the adequate definition of reflection and transmission operators for irregular interface.

2.3.1 Upcoming incident wave to an interface

Consider an upcoming incident wave to the (l)-th interface from beneath (Fig. 3a). The reflection and transmission at (l)-th interface are taken into account, but the reflection at the ($l-1$)-th interface of the transmitted wave is neglected here. The reflection at ($l+1$)-th interface of the reflected wave is

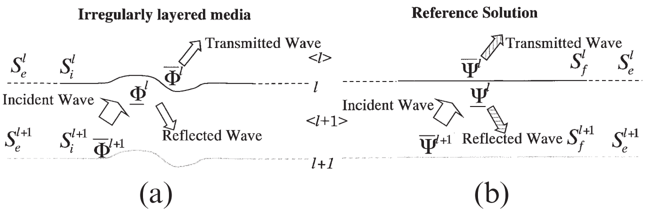


Figure 3. Configuration for the reflection and transmission at (l)-th interface for upcoming wave with consideration on the reference solution. The incident wave comes from ($l+1$)-th interface. The reflection at ($l+1$)-th interface for down going wave is neglected.

also neglected. This means that we concentrate on the wave phenomena occurring at the (l)-th interface.

The boundary conditions along the (l)-th interface are those of continuity of both displacement and traction and they can be formulated as follows. The wavefield in ($l+1$)-th layer is the sum of the contribution from the distributed force along the upper face of ($l+1$)-th interface $\bar{\phi}^{l+1}(\xi)$ and that of the distributed force along the lower face of the (l)-th interface $\underline{\phi}^l(\xi)$. Whereas the wavefield in the (l)-th layer is the contribution from the distributed force along the upper face of the (l)-th interface $\bar{\phi}^l(\xi)$. Therefore, the following boundary integral equations are obtained,

$$\begin{cases} \displaystyle\int_{S^{l+1}} G_{l,l+1}^{l+1}(x;\xi)\bar{\phi}^{l+1}(\xi)d\xi + \int_{S^l} G_{l,l}^{l+1}(x;\xi)\underline{\phi}^l(\xi)d\xi \\ \displaystyle\quad = \int_{S^l} G_{l,l}^l(x;\xi)\bar{\phi}^l(\xi)d\xi, \\ \displaystyle\int_{S^{l+1}} H_{l,l+1}^{l+1}(x;\xi)\bar{\phi}^{l+1}(\xi)d\xi + \int_{S^l} H_{l,l}^{l+1}(x;\xi)\underline{\phi}^l(\xi)d\xi \\ \displaystyle\quad = \int_{S^l} H_{l,l}^l(x;\xi)\bar{\phi}^l(\xi)d\xi, \end{cases} \quad \text{for } x \in S^l \tag{7}$$

The first term of the left side of eq. (7) corresponds to incident wave, the second term to reflected wave and the right side to transmitted wave. By using the expression for the operator of boundary integral along the infinitely spread interfaces, eq. (7) is discretized as follows,

$$\left\langle \begin{matrix} G_{l,l+1}^{l+1} \\ H_{l,l+1}^{l+1}\end{matrix}\right\rangle \hat{\bar{\mathbf{\Phi}}}^{l+1} + \left\langle \begin{matrix} G_{l,l}^{l+1} \\ H_{l,l}^{l+1}\end{matrix}\right\rangle \underline{\hat{\mathbf{\Phi}}}^l = \left\langle \begin{matrix} G_{l,l}^{l} \\ H_{l,l}^{l}\end{matrix}\right\rangle \hat{\bar{\mathbf{\Phi}}}^l \quad \text{or}$$

$$\left\langle \begin{matrix} G_{l,l}^l & -G_{l,l}^{l+1} \\ H_{l,l}^l & -H_{i,l}^{l+1}\end{matrix}\right\rangle \left\langle \begin{matrix} \hat{\bar{\mathbf{\Phi}}}^l \\ \underline{\hat{\mathbf{\Phi}}}^l \end{matrix}\right\rangle = \left\langle \begin{matrix} G_{l,l+1}^{l+1} \\ H_{l,l+1}^{l+1}\end{matrix}\right\rangle \hat{\bar{\mathbf{\Phi}}}^{l+1} \tag{8}$$

The two infinitely long force vectors $\underline{\hat{\mathbf{\Phi}}}^l$, $\hat{\bar{\mathbf{\Phi}}}^l$ are combined together in the last abstract expression. The unknowns $\underline{\hat{\mathbf{\Phi}}}^l$, $\hat{\bar{\mathbf{\Phi}}}^l$ might be given explicitly, if there would be an inverse operator to the operator of

$$\left\langle \begin{matrix} G_{l,l}^l & -G_{l,l}^{l+1} \\ H_{l,l}^l & -H_{l,l}^{l+1}\end{matrix}\right\rangle.$$

Unfortunately, this does not exist. These vectors might be given as the results of application of some operators to the force vector $\hat{\bar{\mathbf{\Phi}}}^{l+1}$ that represents the incident wave. These operators may be able to play the role of the reflection and transmission operators. Therefore, the reflection and transmission operators for the upcoming incident wave $\langle R^U_{l,l+1}\rangle$ and $\langle T^U_{l,l+1}\rangle$, respectively, are defined by the relation of infinitely long force vectors $\hat{\underline{\boldsymbol{\Phi}}}^l$, $\hat{\overline{\boldsymbol{\Phi}}}^l$ and $\hat{\overline{\boldsymbol{\Phi}}}^{l+1}$.

$$\hat{\underline{\boldsymbol{\Phi}}}^l = \langle R^U_{l,l+1}\rangle \hat{\overline{\boldsymbol{\Phi}}}^{l+1}, \quad \hat{\overline{\boldsymbol{\Phi}}}^l = \langle T^U_{l,l+1}\rangle \hat{\overline{\boldsymbol{\Phi}}}^{l+1} . \tag{9}$$

The subscript l and $l+1$ of the operators denote the reflection and transmission between (l)-th and ($l+1$)-th layers. It might be easy to handle the reflected and transmitted waves directly, if eq. (8) could be solved directly by numerical calculation. Unfortunately, this is not possible, because eq. (8) is composed of vectors and matrices of infinite size. This difficulty, however, can be resolved by the approximation in eq. (6) using the reference solution that has been explained in the previous Section.

Note that our purpose is neither to calculate the values of $\hat{\underline{\boldsymbol{\Phi}}}^l$ and $\hat{\overline{\boldsymbol{\Phi}}}^l$, nor to formulate $\langle R^U_{l,l+1}\rangle$ and $\langle T^U_{l,l+1}\rangle$ explicitly, but to handle the reflected and transmitted wavefields. Therefore, a way to calculate displacement and traction of the reflected wave

$$\left\langle \begin{matrix} G^{l+1}_{l,l} \\ H^{l+1}_{l,l} \end{matrix} \right\rangle \hat{\underline{\boldsymbol{\Phi}}}^l$$

and those of the transmitted wave

$$\left\langle \begin{matrix} G^{l}_{l,l} \\ H^{l}_{l,l} \end{matrix} \right\rangle \hat{\overline{\boldsymbol{\Phi}}}^l .$$

is required.

As it is not possible to handle infinitely spread interfaces in numerical calculation, the interface S^l is divided into S^l_i, which can be discretized and handled in computation, and the rest of the interface S^l_e (Fig. 3a). Then, eq. (8) is changed as follows.

$$\left\{ \begin{bmatrix} G^{l+1}_{l,l+1} \\ H^{l+1}_{l,l+1} \end{bmatrix} \overline{\boldsymbol{\Phi}}^{l+1} + \left\langle \begin{matrix} G^{l+1}_{l,l+1} \\ H^{l+1}_{l,l+1} \end{matrix} \right\rangle \hat{\overline{\boldsymbol{\Phi}}}^{l+1}_e \right\} + \left\{ \begin{bmatrix} G^{l+1}_{l,l} \\ H^{l+1}_{l,l} \end{bmatrix} \underline{\boldsymbol{\Phi}}^{l} + \left\langle \begin{matrix} G^{l+1}_{l,l} \\ H^{l+1}_{l,l} \end{matrix} \right\rangle \hat{\underline{\boldsymbol{\Phi}}}^{l}_e \right\} = \left\{ \begin{bmatrix} G^{l}_{l,l} \\ H^{l}_{l,l} \end{bmatrix} \overline{\boldsymbol{\Phi}}^{l} + \left\langle \begin{matrix} G^{l}_{l,l} \\ H^{l}_{l,l} \end{matrix} \right\rangle \hat{\overline{\boldsymbol{\Phi}}}^{l}_e \right\} . \tag{10}$$

The symbols $\overline{\boldsymbol{\Phi}}^l$ and $\underline{\boldsymbol{\Phi}}^l$ are used for the force vectors, i.e. the finite series of the distributed force vectors $\bar{\phi}^l(\xi)$ and $\underline{\phi}^l(\xi)$ that correspond to discretized segments of (l)-th limited irregular interface S^l_i. The first terms of each { } are the contributions from S^l_i, whereas the second ones those from the infinitely spread outer part S^l_e.

Moving all known variables into the right side, the following simultaneous linear equations are obtained.

$$\begin{bmatrix} G^l_{l,l} & -G^{l+1}_{l,l} \\ H^l_{l,l} & -H^{l+1}_{l,l} \end{bmatrix} \begin{bmatrix} \overline{\boldsymbol{\Phi}}^l \\ \underline{\boldsymbol{\Phi}}^l \end{bmatrix} = \begin{bmatrix} G^{l+1}_{l,l+1} \\ H^{l+1}_{l,l+1} \end{bmatrix} \overline{\boldsymbol{\Phi}}^{l+1} + \left\langle \begin{matrix} G^{l+1}_{l,l+1} \\ H^{l+1}_{l,l+1} \end{matrix} \right\rangle \hat{\overline{\boldsymbol{\Phi}}}^{l+1}_e + \left\langle \begin{matrix} G^{l+1}_{l,l} \\ H^{l+1}_{l,l} \end{matrix} \right\rangle \hat{\underline{\boldsymbol{\Phi}}}^{l}_e - \left\langle \begin{matrix} G^{l}_{l,l} \\ H^{l}_{l,l} \end{matrix} \right\rangle \hat{\overline{\boldsymbol{\Phi}}}^{l}_e . \tag{11}$$

The easiest way to solve these equations is to neglect the last three terms of the right member that correspond to the infinitely spread part of interface. This, however, may cause a substantial distortion of the calculated wavefield as mentioned above. Following the approximation in eq. (6), these three terms can be substituted with the known ones.

$$\begin{bmatrix} G^l_{l,l} & -G^{l+1}_{l,l} \\ H^l_{l,l} & -H^{l+1}_{l,l} \end{bmatrix} \begin{bmatrix} \overline{\boldsymbol{\Phi}}^l \\ \underline{\boldsymbol{\Phi}}^l \end{bmatrix} \cong \begin{bmatrix} G^{l+1}_{l,l+1} \\ H^{l+1}_{l,l+1} \end{bmatrix} \overline{\boldsymbol{\Phi}}^{l+1} + \left(\begin{bmatrix} \bar{u}^{l+1}_l \\ \bar{t}^{l+1}_l \end{bmatrix} - \begin{bmatrix} G^{l+1}_{l,l+1} \\ H^{l+1}_{l,l+1} \end{bmatrix} \overline{\boldsymbol{\Psi}}^{l+1} \right) + \left(\begin{bmatrix} \underline{u}^{l+1}_l \\ \underline{t}^{l+1}_l \end{bmatrix} - \begin{bmatrix} G^{l+1}_{l,l} \\ H^{l+1}_{l,l} \end{bmatrix} \underline{\boldsymbol{\Psi}}^{l} \right) - \left(\begin{bmatrix} \bar{u}^{l}_l \\ \bar{t}^{l}_l \end{bmatrix} - \begin{bmatrix} G^{l}_{l,l} \\ H^{l}_{l,l} \end{bmatrix} \overline{\boldsymbol{\Psi}}^{l} \right) . \tag{12}$$

Note that all terms of the right member are known or given and that all vectors and matrices are of finite size in eq. (12). Therefore, numerical calculation can handle these simultaneous linear equations.

By using the solution of eq. (12) $\overline{\boldsymbol{\Phi}}^l$ and $\underline{\boldsymbol{\Phi}}^l$, the reflected and transmitted wavefield can be numerically estimated as follows.

$$\begin{cases} \left\langle \begin{matrix} G^{l+1}_{l,l} \\ H^{l+1}_{l,l} \end{matrix} \right\rangle \hat{\underline{\boldsymbol{\Phi}}}^l = \left\langle \begin{matrix} G^{l+1}_{l,l} \\ H^{l+1}_{l,l} \end{matrix} \right\rangle \langle R^U_{l,l+1}\rangle \hat{\overline{\boldsymbol{\Phi}}}^{l+1} \\ \qquad \approx \begin{bmatrix} \underline{u}^{l+1}_l \\ \underline{t}^{l+1}_l \end{bmatrix} - \begin{bmatrix} G^{l+1}_{l,l} \\ H^{l+1}_{l,l} \end{bmatrix} \underline{\boldsymbol{\Psi}}^l + \begin{bmatrix} G^{l+1}_{l,l} \\ H^{l+1}_{l,l} \end{bmatrix} \underline{\boldsymbol{\Phi}}^l , \\ \left\langle \begin{matrix} G^{l}_{l,l} \\ H^{l}_{l,l} \end{matrix} \right\rangle \hat{\overline{\boldsymbol{\Phi}}}^l = \left\langle \begin{matrix} G^{l}_{l,l} \\ H^{l}_{l,l} \end{matrix} \right\rangle \langle T^U_{l,l+1}\rangle \hat{\overline{\boldsymbol{\Phi}}}^{l+1} \\ \qquad \approx \begin{bmatrix} \bar{u}^{l}_l \\ \bar{t}^{l}_l \end{bmatrix} - \begin{bmatrix} G^{l}_{l,l} \\ H^{l}_{l,l} \end{bmatrix} \overline{\boldsymbol{\Psi}}^l + \begin{bmatrix} G^{l}_{l,l} \\ H^{l}_{l,l} \end{bmatrix} \overline{\boldsymbol{\Phi}}^l . \end{cases} \tag{13}$$

The left members show displacement and traction vectors along (l)-th interface due to the contribution of the force distributed along the lower and upper faces of (l)-th interface $\hat{\underline{\boldsymbol{\Phi}}}^l$ and $\hat{\overline{\boldsymbol{\Phi}}}^l$, that correspond to reflected and transmitted waves, respectively. The middle ones show that the force distributed along upper face of ($l+1$)-th interface $\hat{\overline{\boldsymbol{\Phi}}}^{l+1}$ is the cause of these displacement and traction vectors via reflection and transmission at (l)-th interface. Only the displacement and traction vectors in the left members are used in the next step of computation. Therefore, it is neither necessary to obtain the values of $\hat{\underline{\boldsymbol{\Phi}}}^l$ and $\hat{\overline{\boldsymbol{\Phi}}}^l$, nor to formulate explicitly. Neither, explicit formulation for the operators of reflection and transmission. The right members show a practical way to estimate approximately these displacement and traction vectors, based on eq. (6).

2.3.2 Down coming incident wave to an interface

It is obvious that this case can be considered by turning Fig. 3 (a) and (b) upside down and revising the notation used in the previous Section.

Define the reflection and transmission operators for the down coming incident wave, $\langle T^D_{l,l+1}\rangle$, $\langle R^D_{l,l+1}\rangle$, respectively, by the relation of infinitely long force vectors.

$$\hat{\overline{\mathbf{\Phi}}}^l = \langle R^D_{l,l+1}\rangle \underline{\hat{\mathbf{\Phi}}}^{l-1}, \quad \underline{\hat{\mathbf{\Phi}}}^l = \langle T^D_{l,l+1}\rangle \underline{\hat{\mathbf{\Phi}}}^{l-1}. \tag{14}$$

The subscripts l and $l+1$ denote the reflection and transmission between (l)-th and ($l+1$)-th layers.

A similar consideration with the reference solution gives the simultaneous linear equations that we have to solve for this case as follows.

$$\begin{aligned}
&\begin{bmatrix} G^l_{l,l} & -G^{l+1}_{l,l} \\ H^l_{l,l} & -H^{l+1}_{l,l} \end{bmatrix} \begin{bmatrix} \overline{\mathbf{\Phi}}^l \\ \underline{\mathbf{\Phi}}^l \end{bmatrix} \\
&= -\begin{bmatrix} G^{l+1}_{l,l+1} \\ H^{l+1}_{l,l+1} \end{bmatrix} \underline{\mathbf{\Phi}}^{l-1} - \left(\begin{bmatrix} \underline{u}^l_l \\ \underline{t}^l_l \end{bmatrix} - \begin{bmatrix} G^l_{l,l-1} \\ H^l_{l,l-1} \end{bmatrix} \underline{\mathbf{\Psi}}^{l-1} \right) \\
&\quad + \left(\begin{bmatrix} \underline{u}^{l+1}_l \\ \underline{t}^{l+1}_l \end{bmatrix} - \begin{bmatrix} G^{l+1}_{l,l} \\ H^{l+1}_{l,l} \end{bmatrix} \underline{\mathbf{\Psi}}^l \right) - \left(\begin{bmatrix} \bar{u}^l_l \\ \bar{t}^l_l \end{bmatrix} - \begin{bmatrix} G^l_{l,l} \\ H^l_{l,l} \end{bmatrix} \overline{\mathbf{\Psi}}^l \right)
\end{aligned} \tag{15}$$

By using the solution of eq. (15), the reflected and transmitted wavefield can be calculated as follows.

$$\begin{cases}
\left\langle \begin{matrix} G^l_{l,l} \\ H^l_{l,l} \end{matrix} \right\rangle \hat{\overline{\mathbf{\Phi}}}^l = \left\langle \begin{matrix} G^l_{l,l} \\ H^l_{l,l} \end{matrix} \right\rangle \langle R^D_{l,l+1}\rangle \underline{\hat{\mathbf{\Phi}}}^l \\
\qquad \approx \begin{bmatrix} \bar{u}^l_l \\ \bar{t}^l_l \end{bmatrix} - \begin{bmatrix} G^l_{l,l} \\ H^l_{l,l} \end{bmatrix} \overline{\mathbf{\Psi}}^l + \begin{bmatrix} G^l_{l,l} \\ H^l_{l,l} \end{bmatrix} \overline{\mathbf{\Phi}}^l, \\
\left\langle \begin{matrix} G^{l+1}_{l,l} \\ H^{l+1}_{l,l} \end{matrix} \right\rangle \underline{\hat{\mathbf{\Phi}}}^l = \left\langle \begin{matrix} G^{l+1}_{l,l} \\ H^{l+1}_{l,l} \end{matrix} \right\rangle \langle T^D_{l,l+1}\rangle \underline{\hat{\mathbf{\Phi}}}^{l-1} \\
\qquad \approx \begin{bmatrix} \underline{u}^{l+1}_l \\ \underline{t}^{l+1}_l \end{bmatrix} - \begin{bmatrix} G^{l+1}_{l,l} \\ H^{l+1}_{l,l} \end{bmatrix} \underline{\mathbf{\Psi}}^l + \begin{bmatrix} G^{l+1}_{l,l} \\ H^{l+1}_{l,l} \end{bmatrix} \underline{\mathbf{\Phi}}^l.
\end{cases} \tag{16}$$

$\overline{\mathbf{\Phi}}^l$ and $\underline{\mathbf{\Phi}}^l$ are parts of the solution of eq. (15). The left side of both formulae show displacement and traction vectors along (l)-th interface due to the contribution of the force distributed along the upper and lower face of (l)-th interface $\hat{\overline{\mathbf{\Phi}}}^l$ and $\underline{\hat{\mathbf{\Phi}}}^l$, that correspond to reflected and transmitted waves, respectively. The middle ones show that the force distributed along lower face of ($l-1$)-th interface $\underline{\hat{\mathbf{\Phi}}}^{l-1}$ is the cause of these displacement and traction vectors. The right ones show a practical way to estimate approximately these displacement and traction vectors, based on eq. (6).

The reflection and transmission operators for an interface in the frequency–space domain for up- and down-coming incident wave, $\langle R^U_{l,l+1}\rangle$, $\langle T^U_{l,l+1}\rangle$, $\langle T^D_{l,l+1}\rangle$ and $\langle R^D_{l,l+1}\rangle$ are defined in terms of I-BEM. Reflection and transmission operators for a layer are discussed in the next Section by making use of these four operators for an interface.

2.4 Reflection and transmission operators for a layer sandwiched between irregular interfaces

Pay attention to (l)-th layer in irregularly stratified media. (l)-th layer is sandwiched between ($l-1$)-th and (l)-th interfaces. Hereafter, the description for the method of using the reference

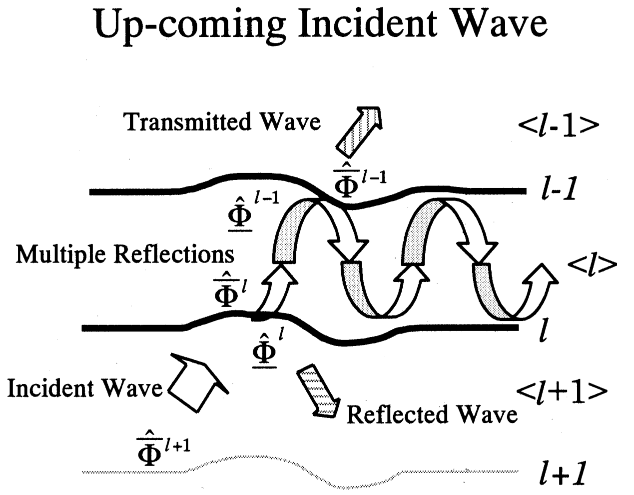


Figure 4. Configuration for the reflection and transmission for (l)-th layer for upcoming incident wave. The incident wave comes up from ($l+1$)-th interfaces. The reverberation in (l)-th layer corresponds to the ascending polynomials of operators in eq. (19).

solution is omitted in the description and also in figures, except for the case of the surface reflection, because eqs (13) and (16) have shown the method sufficiently. Namely, simultaneous linear eqs (12) and (15) have to be solved for reflection/transmission at each interface, as the wave goes by.

2.4.1 *Upcoming incident wave to a layer*

Consider the reflection and transmission between ($l-1$)-th and ($l+1$)-th layers due to up and down coming incident wave as shown in Fig. 4. Define the reflection operator for the upcoming incident wave and the reflected wave in ($l+1$)-th layer, and transmission operator for the incident wave and the transmitted wave in ($l-1$)-th layer (Fig. 5) by the relation of the forces distributed along the infinitely spread irregular interfaces as follows,

$$\underline{\hat{\mathbf{\Phi}}}^l = \langle R^U_{l-1,l+1}\rangle \hat{\overline{\mathbf{\Phi}}}^{l+1}, \quad \hat{\overline{\mathbf{\Phi}}}^{l-1} = \langle T^U_{l-1,l+1}\rangle \hat{\overline{\mathbf{\Phi}}}^{l+1}. \tag{17}$$

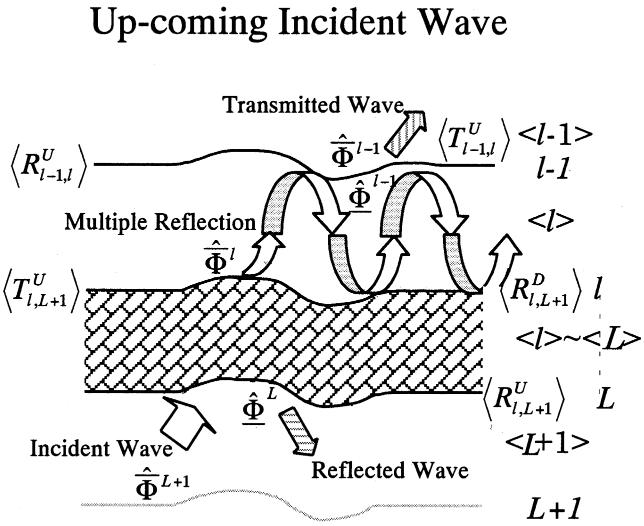


Figure 5. Configuration for the reflection and transmission for the stacked layers between (l)-th and (L)-th layer for upcoming incident wave. The incident wave comes up from ($l+1$)-th interface.

Consider the process of wave propagation in the (l)-th layer. The wave transmitted at the (l)-th interface penetrates into (l)-th layer. A part of this wave is transmitted at the ($l-1$)-th interface and penetrates into ($l-1$)-th layer. By using reflection and transmission operators for an interface defined above, this double transmission is described as follows.

$$\langle T^U_{l-1,l}\rangle\langle T^U_{l,l+1}\rangle\hat{\overline{\mathbf{\Phi}}}^{l+1}\,. \tag{18.1}$$

The rest is reflected at the ($l-1$)-th interface, propagates downward, reflected again at the (l)-th interface, propagates upward and is transmitted at the ($l-1$)-th interface. This wave propagation is described as follows.

$$\langle T^U_{l-1,l}\rangle\langle R^D_{l,l+1}\rangle\langle R^U_{l-1,l}\rangle\langle T^U_{l,l+1}\rangle\hat{\overline{\mathbf{\Phi}}}^{l+1}\,. \tag{18.2}$$

The wave that is reflected again in (l)-th layer is described as follows.

$$\langle T^U_{l-1,l}\rangle\langle R^D_{l,l+1}\rangle\langle R^U_{l-1,l}\rangle\langle R^D_{l,l+1}\rangle\langle R^U_{l-1,l}\rangle\langle T^U_{l,l+1}\rangle\hat{\overline{\mathbf{\Phi}}}^{l+1}\,. \tag{18.3}$$

The wave propagating upward in the ($l-1$)-th layer can be evaluated by the sum of these waves. The wave transmitted into the ($l+1$)-th layer can be evaluated in a similar way.

Consideration of the multiple reflection in (l)-th layer gives the following relations.

$$\begin{cases} \hat{\overline{\mathbf{\Phi}}}^{l-1} = \langle T^U_{l-1,l}\rangle\{\langle I\rangle + \langle R^D_{l,l+1}\rangle\langle R^U_{l-1,l}\rangle + (\langle R^D_{l,l+1}\rangle\langle R^U_{l-1,l}\rangle)^2 \\ \qquad +(\langle R^D_{l,l+1}\rangle\langle R^U_{l-1,l}\rangle)^3 + \cdots\}\langle T^U_{l,l+1}\rangle\hat{\overline{\mathbf{\Phi}}}^{l+1} \\ \qquad = \langle T^U_{l-1,l}\rangle(\langle I\rangle - \langle R^D_{l,l+1}\rangle\langle R^U_{l-1,l}\rangle)^{-1}\langle T^U_{l,l+1}\rangle\hat{\overline{\mathbf{\Phi}}}^{l+1}\,, \\ \hat{\underline{\mathbf{\Phi}}}^{l} = \langle R^U_{l,l+1}\rangle\hat{\overline{\mathbf{\Phi}}}^{l+1} + \langle T^D_{l,l+1}\rangle\langle R^U_{l-1,l}\rangle \\ \qquad \times\{\langle I\rangle + \langle R^D_{l,l+1}\rangle\langle R^U_{l-1,l}\rangle + (\langle R^D_{l,l+1}\rangle\langle R^U_{l-1,l}\rangle)^2 \\ \qquad +(\langle R^D_{l,l+1}\rangle\langle R^U_{l-1,l}\rangle)^3 + \cdots\}\langle T^U_{l,l+1}\rangle\hat{\overline{\mathbf{\Phi}}}^{l+1} \\ \qquad = \{\langle R^U_{l,l+1}\rangle + \langle T^D_{l,l+1}\rangle\langle R^U_{l-1,l}\rangle \\ \qquad \times(\langle I\rangle - \langle R^D_{l,l+1}\rangle\langle R^U_{l-1,l}\rangle)^{-1}\langle T^U_{l,l+1}\rangle\}\hat{\overline{\mathbf{\Phi}}}^{l+1}\,. \end{cases} \tag{19}$$

A conventional expression is used for the ascending polynomial series in the right side of both formulae just to make the expressions shorter. The symbol $\langle I\rangle$ denotes the identity operator. Actually, it is not possible to obtain an explicit formulation of this inverse operator. In practice, calculation is conducted by using the ascending polynomial series.

Eq. (19) gives the following relation of the reflection and transmission operators.

$$\begin{cases} \langle T^U_{l-1,l+1}\rangle = \langle T^U_{l-1,l}\rangle(\langle I\rangle - \langle R^D_{l,l+1}\rangle\langle R^U_{l-1,l}\rangle)^{-1}\langle T^U_{l,l+1}\rangle\,, \\ \langle R^U_{l-1,l+1}\rangle = \langle R^U_{l,l+1}\rangle + \langle T^D_{l,l+1}\rangle\langle R^U_{l-1,l}\rangle \\ \qquad \times(\langle I\rangle - \langle R^D_{l,l+1}\rangle\langle R^U_{l-1,l}\rangle)^{-1}\langle T^U_{l,l+1}\rangle\,. \end{cases} \tag{20}$$

2.4.2 *Down coming incident wave to a layer*

It is obvious that this case can be considered by turning Fig. 4 upside down and revising the notation used in the previous Section.

The definition of the operators is given by the following.

$$\hat{\overline{\mathbf{\Phi}}}^{l} = \langle R^D_{l-1,l+1}\rangle\hat{\underline{\mathbf{\Phi}}}^{l-1}\,,\quad \hat{\underline{\mathbf{\Phi}}}^{l+1} = \langle T^D_{l-1,l+1}\rangle\hat{\underline{\mathbf{\Phi}}}^{l-1}\,. \tag{21}$$

The following relation is given by a similar process to that described by eqs (19) and (20).

$$\begin{cases} \langle R^D_{l-1,l+1}\rangle = \langle R^D_{l-1,l}\rangle + \langle T^U_{l-1,l}\rangle\langle R^D_{l,l+1}\rangle \\ \qquad \times(\langle I\rangle - \langle R^U_{l-1,l}\rangle\langle R^D_{l,l+1}\rangle)^{-1}\langle T^D_{l-1,l}\rangle, \\ \langle T^D_{l-1,l+1}\rangle = \langle T^D_{l,l+1}\rangle(\langle I\rangle - \langle R^U_{l-1,l}\rangle\langle R^D_{l,l+1}\rangle)^{-1}\langle T^D_{l-1,l}\rangle\,, \end{cases} \tag{22}$$

Note that eqs (20) and (22) have similar structure as those of Kennett (1983) for horizontally stratified media.

2.5 Reflection and transmission for stacked layers

Suppose that the reflection and transmission operators are already defined for irregularly stratified media between (l)-th and (L)-th interfaces, where L denotes the total number of layers except the deepest half space. Therefore, the upper layer of these media is (l)-th layer, and the lower one is the deepest half space, i.e. ($L+1$)-th layer. Therefore, for upcoming incident wave, the following operators are defined:

$$\hat{\underline{\mathbf{\Phi}}}^{L} = \langle R^U_{l,L+1}\rangle\hat{\overline{\mathbf{\Phi}}}^{L+1}\,,\quad \hat{\overline{\mathbf{\Phi}}}^{l} = \langle T^U_{l,L+1}\rangle\hat{\overline{\mathbf{\Phi}}}^{L+1}\,. \tag{23}$$

and for down coming incident wave,

$$\hat{\overline{\mathbf{\Phi}}}^{l} = \langle R^D_{l,L+1}\rangle\hat{\underline{\mathbf{\Phi}}}^{l-1}\,,\quad \hat{\underline{\mathbf{\Phi}}}^{L} = \langle T^D_{l,L+1}\rangle\hat{\underline{\mathbf{\Phi}}}^{l-1}\,. \tag{24}$$

The symbol $\hat{\overline{\mathbf{\Phi}}}^{L+1}$ just represents the real source located in the deepest layer. The way to handle its contribution is discussed below.

The reflection and transmission operators for ($l-1$)-th interface are defined eqs (9) and (14), i.e.

$$\begin{cases} \hat{\underline{\mathbf{\Phi}}}^{l-1} = \langle R^U_{l-1,l}\rangle\hat{\overline{\mathbf{\Phi}}}^{l}\,, \\ \hat{\overline{\mathbf{\Phi}}}^{l-1} = \langle T^U_{l-1,l}\rangle\hat{\overline{\mathbf{\Phi}}}^{l}\,, \end{cases} \quad \begin{cases} \hat{\overline{\mathbf{\Phi}}}^{l-1} = \langle R^D_{l-1,l}\rangle\hat{\underline{\mathbf{\Phi}}}^{l-2}\,, \\ \hat{\underline{\mathbf{\Phi}}}^{l-1} = \langle T^D_{l-1,l}\rangle\hat{\underline{\mathbf{\Phi}}}^{l-2}\,. \end{cases} \tag{25}$$

2.5.1 *Upcoming incident wave to stacked layers*

Consider the stratified media between ($l-1$)-th and (L)-th interfaces and the upcoming incident wave represented by $\hat{\overline{\mathbf{\Phi}}}^{L+1}$ (Fig. 5). Define the reflection and transmission operators for the layers between ($l-1$)-th layer and ($L+1$)-th layer in the following way.

$$\hat{\underline{\mathbf{\Phi}}}^{L} = \langle R^U_{l-1,L+1}\rangle\hat{\overline{\mathbf{\Phi}}}^{L+1}\,,\quad \hat{\overline{\mathbf{\Phi}}}^{l-1} = \langle T^U_{l-1,L+1}\rangle\hat{\overline{\mathbf{\Phi}}}^{L+1}\,. \tag{26}$$

Consideration of the multiple reflections in (l)-th layer gives the following relation.

$$\begin{cases} \hat{\bar{\boldsymbol{\Phi}}}^{l-1} = \langle T^U_{l-1,l}\rangle\{\langle I\rangle + \langle R^D_{l,L+1}\rangle\langle R^U_{l-1,l}\rangle(\langle R^D_{l,L+1}\rangle\langle R^U_{l-1,l}\rangle)^2 \\ \qquad +(\langle R^D_{l,L+1}\rangle\langle R^U_{l-1,l}\rangle)^3 + \cdots\}\langle T^U_{l,L+1}\rangle\hat{\bar{\boldsymbol{\Phi}}}^{L+1} \\ \quad = \langle T^U_{l-1,l}\rangle(\langle I\rangle - \langle R^D_{l,L+1}\rangle\langle R^U_{l-1,l}\rangle)^{-1}\langle T^U_{l,L+1}\rangle\hat{\bar{\boldsymbol{\Phi}}}^{L+1}, \\ \hat{\underline{\boldsymbol{\Phi}}}^{l} = \langle R^U_{l,L+1}\rangle\hat{\bar{\boldsymbol{\Phi}}}^{L+1} + \langle T^D_{l,L+1}\rangle\langle R^U_{l-1,l}\rangle \\ \qquad \times\{\langle I\rangle + \langle R^D_{l,L+1}\rangle\langle R^U_{l-1,l}\rangle + (\langle R^D_{l,L+1}\rangle\langle R^U_{l-1,l}\rangle)^2 \\ \qquad +(\langle R^D_{l,L+1}\rangle\langle R^U_{l-1,l}\rangle)^3 + \cdots\}\langle T^U_{l,L+1}\rangle\hat{\bar{\boldsymbol{\Phi}}}^{L+1} \\ \quad = \{\langle R^U_{l,L+1}\rangle + \langle T^D_{l,L+1}\rangle\langle R^U_{l-1,l}\rangle \\ \qquad \times(\langle I\rangle - \langle R^D_{l,L+1}\rangle\langle R^U_{l-1,l}\rangle)^{-1}\langle T^U_{l,L+1}\rangle\}\hat{\bar{\boldsymbol{\Phi}}}^{L+1}. \end{cases} \tag{27}$$

This gives the following relation of reflection and transmission operator.

$$\begin{cases} \langle R^U_{l-1,L+1}\rangle = \langle R^U_{l,L+1}\rangle + \langle T^D_{l,L+1}\rangle\langle R^U_{l-1,l}\rangle \\ \qquad \times(\langle I\rangle - \langle R^D_{l,L+1}\rangle\langle R^U_{l-1,l}\rangle)^{-1}\langle T^U_{l,L+1}\rangle, \\ \langle T^U_{l-1,L+1}\rangle = \langle T^U_{l-1,l}\rangle(\langle I\rangle - \langle R^D_{l,L+1}\rangle\langle R^U_{l-1,l}\rangle)^{-1}\langle T^U_{l,L+1}\rangle. \end{cases} \tag{28}$$

2.5.2 *Down coming incident wave to stacked layers*

It is obvious that this case can be considered by turning Fig. 5 upside down and revising the notation used in the previous section. The definition of operators is given by,

$$\hat{\bar{\boldsymbol{\Phi}}}^{l-1} = \langle R^D_{l-1,L+1}\rangle\hat{\underline{\boldsymbol{\Phi}}}^{l-2}, \quad \hat{\underline{\boldsymbol{\Phi}}}^{L} = \langle T^D_{l-1,L+1}\rangle\hat{\underline{\boldsymbol{\Phi}}}^{l-2}. \tag{29}$$

The following relation of reflection and transmission operators for stacked layers is obtained by a similar process as that used in eqs (27) and (28),

$$\begin{cases} \langle R^D_{l-1,L+1}\rangle = \langle R^D_{l-1,l}\rangle + \langle T^U_{l-1,l}\rangle\langle R^D_{l,L+1}\rangle \\ \qquad \times(\langle I\rangle - \langle R^U_{l-1,l}\rangle\langle R^D_{l,L+1}\rangle)^{-1}\langle T^D_{l-1,l}\rangle, \\ \langle T^D_{l-1,L+1}\rangle = \langle T^D_{l,L+1}\rangle(\langle I\rangle - \langle R^U_{l-1,l}\rangle\langle R^D_{l,L+1}\rangle)^{-1}\langle T^D_{l-1,l}\rangle. \end{cases} \tag{30}$$

2.6 Reflection at the irregular free surface

Suppose that L layers are already stacked on the half space. The next step is to install the free surface on them by considering the wave reflected at the surface.

The traction free condition along the free surface is given by the following boundary integral equation that means the sum of the traction due to incident wave and that due to reflected wave is equal to zero at the free surface (Fig. 6).

$$\int_{S^1} H^1_{0,1}(x;\xi)\bar{\phi}^1(\xi)d\xi + \int_{S^0} H^1_{0,0}(x;\xi)\underline{\phi}^0(\xi)d\xi = 0 \quad \text{for } x\in S^0. \tag{31}$$

By using the integral operator, this is written as follows,

$$\langle H^1_{0,1}\rangle\hat{\bar{\boldsymbol{\Phi}}}^1 + \langle H^1_{0,0}\rangle\hat{\underline{\boldsymbol{\Phi}}}^0 = 0 \quad \text{for } x\in S^0. \tag{32}$$

Define the reflection operator at the surface as

$$\hat{\underline{\boldsymbol{\Phi}}}^0 = \langle R^U_{0,1}\rangle\hat{\bar{\boldsymbol{\Phi}}}^1. \tag{33}$$

The approximation using the reference solution, i.e., that given by eq. (16), with l equal to zero, as written in Section 2.3.1, gives the following simultaneous linear equations

$$H^1_{0,0}\underline{\boldsymbol{\Phi}}^0 = -(\bar{t}^1_0 + H^1_{0,1}\bar{\boldsymbol{\Phi}}^1 - H^1_{0,1}\bar{\boldsymbol{\Psi}}^1) - (\underline{t}^1_0 - H^1_{0,0}\underline{\boldsymbol{\Psi}}^0). \tag{34}$$

By using this solution, the wavefield reflected at the surface can be numerically estimated as follows,

$$\left\langle\begin{matrix} G^1_{0,0} \\ H^1_{0,0}\end{matrix}\right\rangle\hat{\underline{\boldsymbol{\Phi}}}^0 = \left\langle\begin{matrix} G^1_{0,0} \\ H^1_{0,0}\end{matrix}\right\rangle\langle R^U_{0,1}\rangle\hat{\bar{\boldsymbol{\Phi}}}^1$$
$$\approx \begin{bmatrix}\underline{u}^1_0 \\ \underline{t}^1_0\end{bmatrix} - \begin{bmatrix} G^1_{0,0} \\ H^1_{0,0}\end{bmatrix}\underline{\boldsymbol{\Psi}}^0 + \begin{bmatrix} G^1_{0,0} \\ H^1_{0,0}\end{bmatrix}\underline{\boldsymbol{\Phi}}^0. \tag{35}$$

The left side shows displacement and traction vectors in the shallowest layer due to the contribution of the force distributed along the lower face of the surface, i.e., the wave reflected at the surface. The middle one shows that the force distributed along the upper face of (1)-th interface is the cause of these displacement and traction vectors. Only these vectors are used in the next step of computation. The right side shows a practical way to calculate them. The first and second terms correspond to the contribution from S^0_e, and the third that from S^0_i.

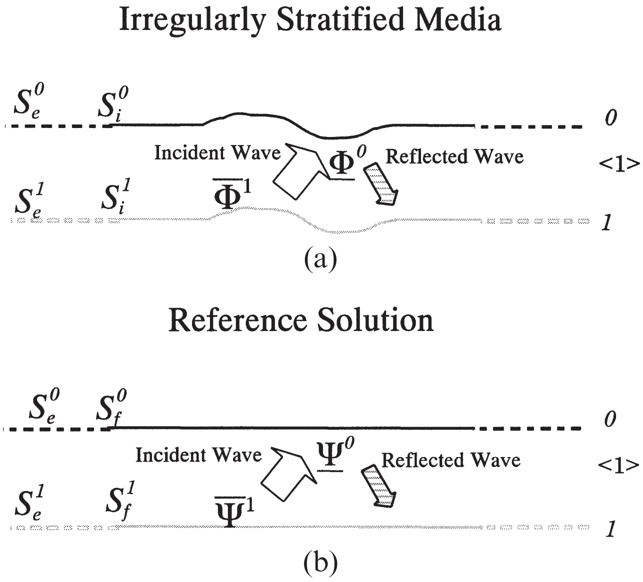


Figure 6. Configuration for the reflection at the free surface (a) with consideration on the reference solution (b). The incident wave comes up from (1)-th interface.

Consideration of the multiple reflections in the shallowest layer in the similar way as eqs (18.1), (18.2), and (18.3) give the following (Fig. 7),

$$\begin{aligned}\hat{\bar{\boldsymbol{\Phi}}}^1 &= \langle T^U_{1,L+1}\rangle\hat{\bar{\boldsymbol{\Phi}}}^{L+1} + \langle R^D_{1,L+1}\rangle\langle R^U_{0,1}\rangle\langle T^U_{1,L+1}\rangle\hat{\bar{\boldsymbol{\Phi}}}^{L+1}\\ &\quad + \langle R^D_{1,L+1}\rangle\langle R^U_{0,1}\rangle\langle R^D_{1,L+1}\rangle\langle R^U_{0,1}\rangle\langle T^U_{1,L+1}\rangle\hat{\bar{\boldsymbol{\Phi}}}^{L+1} + \cdots\\ &= (\langle I\rangle - \langle R^D_{1,L+1}\rangle\langle R^U_{0,1}\rangle)^{-1}\langle T^U_{1,L+1}\rangle\hat{\bar{\boldsymbol{\Phi}}}^{L+1}. \end{aligned} \tag{36}$$

Eqs (33) and (36) give the following for displacement at the free surface,

$$\begin{aligned} u^1_0 &= \langle G^1_{0,0}\rangle\hat{\underline{\boldsymbol{\Phi}}}^0 + \langle G^1_{0,1}\rangle\hat{\bar{\boldsymbol{\Phi}}}^1 = (\langle G^1_{0,0}\rangle\langle R^U_{0,1}\rangle + \langle G^1_{0,1}\rangle)\hat{\bar{\boldsymbol{\Phi}}}^1\\ &= (\langle G^1_{0,0}\rangle\langle R^U_{0,1}\rangle + \langle G^1_{0,1}\rangle)(\langle I\rangle - \langle R^D_{1,L+1}\rangle\langle R^U_{0,1}\rangle)^{-1}\\ &\quad \times\langle T^U_{1,L+1}\rangle\hat{\bar{\boldsymbol{\Phi}}}^{L+1}. \end{aligned} \tag{37}$$

The symbol $\hat{\bar{\boldsymbol{\Phi}}}^{L+1}$ represents the contribution of the real source located at $(L+1)$-th layer.

2.7 Contribution of the real source in the deepest layer

In the consideration described above, the real source of the upcoming wave in $(L+1)$-th layer is represented by $\hat{\bar{\boldsymbol{\Phi}}}^{L+1}$ that implies the image of an additional interface in the deepest half space. It is, however, easier to give the way to handle the contribution from the real source rather than to give an equivalent force distribution along $(L+1)$-th interface to any type of the real source. Hereafter $[\bar{u}^{L+1}_{INC}(x),\ \bar{t}^{L+1}_{INC}(x)]$ denotes the real source contribution to the wavefield in $(L+1)$-th layer.

Consider the upcoming incident wave to the (L)-th interface from beneath (Fig. 8). The reflection and transmission at the interface are taken into account, but the reflection at the $(L–1)$-th interface of the transmitted wave is omitted here. This configuration can be obtained by replacing l with L in the description of Section 2.3.1.

The boundary condition along the (L)-th interface is given by replacement of the first terms of the left member of eq. (7) with $\bar{u}^{L+1}_{INC}(x)$ and $\bar{t}^{L+1}_{INC}(x)$. Similar replacement of the terms

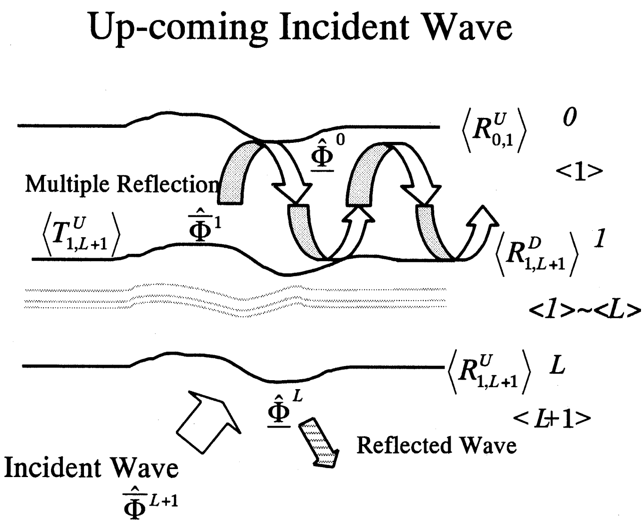


Figure 7. Configuration for the combination of the reverberation in the stacked layers between (2)-th and (L)-th layer and the reflection at the free surface.

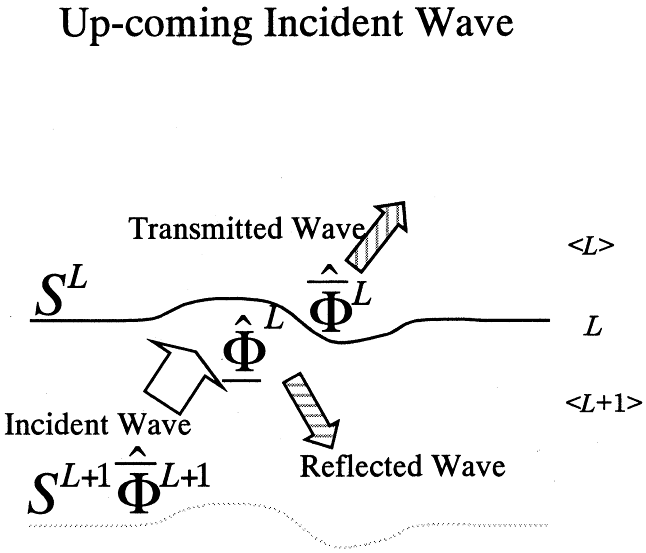


Figure 8. Configuration for the reflection and transmission at the deepest interface for upcoming incident wave. $\hat{\bar{\Phi}}^{L+1}$ represents the source of upcoming incident wave. Then, $[\bar{u}^{L+1}_{INC}(x),\ \bar{t}^{L+1}_{INC}(x)]$ can be used in place of $\hat{\bar{\Phi}}^{L+1}$ as written in eq. (40).

in eq. (8) with

$$\begin{bmatrix}\bar{u}^{L+1}_{INC}\\ \bar{t}^{L+1}_{INC}\end{bmatrix}$$

gives the expression for the operators.

Then, the following simultaneous linear equations are obtained.

$$\begin{bmatrix} G^L_{L,L} & -G^{L+1}_L\\ H^L_{L,L} & -H^{L+1}_L\end{bmatrix}\begin{bmatrix}\bar{\boldsymbol{\Phi}}^L\\ \underline{\boldsymbol{\Phi}}^L\end{bmatrix} = \begin{bmatrix}\bar{u}^{L+1}_{INC}\\ \bar{t}^{L+1}_{INC}\end{bmatrix} + \left(\begin{bmatrix}\underline{u}^{L+1}_L\\ \underline{t}^{L+1}_L\end{bmatrix} - \begin{bmatrix}G^{L+1}_L\\ H^{L+1}_L\end{bmatrix}\underline{\boldsymbol{\Psi}}^L\right) - \left(\begin{bmatrix}\bar{u}^L_L\\ \bar{t}^L_L\end{bmatrix} - \begin{bmatrix}G^L_{L,L}\\ H^L_{L,L}\end{bmatrix}\bar{\boldsymbol{\Psi}}^L\right). \tag{38}$$

The reflection and transmission operators for the upcoming wave $\langle R^U_{L,L+1}\rangle$ and $\langle T^U_{L,L+1}\rangle$, respectively, are defined by the relation of infinitely long force vectors.

$$\hat{\underline{\boldsymbol{\Phi}}}^L = \langle R^U_{L,L+1}\rangle\hat{\bar{\boldsymbol{\Phi}}}^{L+1},\quad \hat{\bar{\boldsymbol{\Phi}}}^L = \langle T^U_{L,L+1}\rangle\hat{\bar{\boldsymbol{\Phi}}}^{L+1}. \tag{39}$$

By using the solution of eq. (38), the reflected and transmitted wavefield can be numerically estimated as follows,

$$\left\{\begin{aligned} \left\langle\begin{matrix}G^{L+1}_{L,L}\\ H^{L+1}_{L,L}\end{matrix}\right\rangle\hat{\underline{\boldsymbol{\Phi}}}^L &= \left\langle\begin{matrix}G^{L+1}_{L,L}\\ H^{L+1}_{L,L}\end{matrix}\right\rangle\langle R^U_{L,L+1}\rangle\hat{\bar{\boldsymbol{\Phi}}}^{L+1}\\ &\approx \begin{bmatrix}\underline{u}^{L+1}_L\\ \underline{t}^{L+1}_L\end{bmatrix} - \begin{bmatrix}G^{L+1}_{L,L}\\ H^{L+1}_{L,L}\end{bmatrix}\underline{\boldsymbol{\Psi}}^L + \begin{bmatrix}G^{L+1}_{L,L}\\ H^{L+1}_{L,L}\end{bmatrix}\underline{\boldsymbol{\Phi}}^L,\\ \left\langle\begin{matrix}G^{L}_{L,L}\\ H^{L}_{L,L}\end{matrix}\right\rangle\hat{\bar{\boldsymbol{\Phi}}}^L &= \left\langle\begin{matrix}G^{L}_{L,L}\\ H^{L}_{L,L}\end{matrix}\right\rangle\langle T^U_{L,L+1}\rangle\hat{\bar{\boldsymbol{\Phi}}}^{L+1}\\ &\approx \begin{bmatrix}\bar{u}^{L}_L\\ \bar{t}^{L}_L\end{bmatrix} - \begin{bmatrix}G^{L}_{L,L}\\ H^{L}_{L,L}\end{bmatrix}\bar{\boldsymbol{\Psi}}^L + \begin{bmatrix}G^{L}_{L,L}\\ H^{L}_{L,L}\end{bmatrix}\bar{\boldsymbol{\Phi}}^L. \end{aligned}\right. \tag{40}$$

The physical meaning of eq. (40) is given by changing l in the description just after eq. (13) to L.

Replacing l with L in the description in Section 2.3.2., the discussion for the case of down-coming incident wave to the deepest interface can be discussed, sufficiently.

2.8 Disassembled transmission operator and displacement at the surface

In Section 2.5, the process of adding a layer above the stacked layers is discussed. Here, the process of adding a layer below the stacked layers is discussed.

Consider the stacked layers sandwiched by (1)-th and $(L-1)$-th interfaces, and (L)-th interface below them (Fig. 9). Similar as shown in eqs (23) and (24), the following reflection and transmission operators for these stacked layers are defined. For upcoming incident wave,

$$\underline{\hat{\Phi}}^{L-1} = \langle R^U_{1,L}\rangle \overline{\hat{\Phi}}^{L}, \quad \overline{\hat{\Phi}}^{1} = \langle T^U_{1,L}\rangle \overline{\hat{\Phi}}^{L}. \tag{41}$$

By considering the multiple reflections in (L)-th layer, the following is obtained,

$$\begin{aligned}\overline{\hat{\Phi}}^{1} &= \langle T^U_{1,L}\rangle\{\langle I\rangle + \langle R^D_{L,L+1}\rangle\langle R^U_{1,L}\rangle + (\langle R^D_{L,L+1}\rangle\langle R^U_{1,L}\rangle)^2 \\ &\quad + (\langle R^D_{L,L+1}\rangle\langle R^U_{1,L}\rangle)^3 + \cdots\}\langle T^U_{L,L+1}\rangle\overline{\hat{\Phi}}^{L+1} \\ &= \langle T^U_{1,L}\rangle(\langle I\rangle - \langle R^D_{L,L+1}\rangle\langle R^U_{1,L}\rangle)^{-1}\langle T^U_{L,L+1}\rangle\overline{\hat{\Phi}}^{L+1},\end{aligned} \tag{42}$$

This implies the following relation.

$$\langle T^U_{1,L+1}\rangle = \langle T^U_{1,L}\rangle(\langle I\rangle - \langle R^D_{L,L+1}\rangle\langle R^U_{1,L}\rangle)^{-1}\langle T^U_{L,L+1}\rangle. \tag{43}$$

Therefore, eq. (37) becomes as follows.

$$\begin{aligned}u^1_0 &= (\langle G^1_{0,0}\rangle\langle R^U_{0,1}\rangle + \langle G^1_{0,1}\rangle)(\langle I\rangle - \langle R^D_{1,L+1}\rangle\langle R^U_{0,1}\rangle)^{-1} \\ &\quad \times\langle T^U_{1,L+1}\rangle\overline{\hat{\Phi}}^{L+1} \\ &= (\langle G^1_{0,0}\rangle\langle R^U_{0,1}\rangle + \langle G^1_{0,1}\rangle)(\langle I\rangle - \langle R^D_{1,L+1}\rangle\langle R^U_{0,1}\rangle)^{-1} \\ &\quad \times\langle T^U_{1,L}\rangle(\langle I\rangle - \langle R^D_{L,L+1}\rangle\langle R^U_{1,L}\rangle)^{-1}(\langle T^U_{L,L+1}\rangle\overline{\hat{\Phi}}^{L+1}).\end{aligned} \tag{44}$$

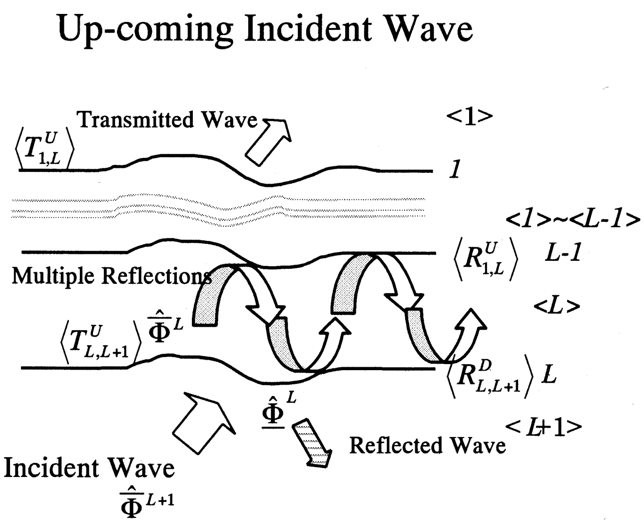


Figure 9. Configuration for adding a layer below the stacked layers. This corresponds to procedure to derive eq. (44).

Note that application of the operators $\langle R^D_{1,L+1}\rangle$, $\langle T^U_{1,L}\rangle$, $\langle R^U_{1,L}\rangle$ do not have a simple form if L is not equal to one, whereas solving simultaneous linear eqs (34) and (38) can give the simple way to handle $\langle R^U_{0,1}\rangle$, $\langle T^U_{L,L+1}\rangle$, $\langle R^D_{L,L+1}\rangle$, respectively. Therefore, eq (44) is not a convenient formulation for computation of wavefield in total. The benefit of eq. (44) can be found in extracting certain wave types as shown in the following Section.

3 EXAMPLES

In order to calculate the wavefield observed at the surface, it is necessary to solve simultaneous linear equations as shown above. It is obvious that computation of complete the wavefield is not always necessary. Namely, it is expected that extracted principle parts of wavefield can work well in some practical problems as a good approximation. The advantage given by usage of reflection and transmission operators is that these help us to find systematically good approximations and its actual algorithm based on the wave propagation. Some simple examples are shown below.

3.1 Separation of waves directly coming from the real source

Neglecting reflections for both up and down going waves for every interfaces, i.e.

$$\langle R^U_{l,l+1}\rangle \Rightarrow \langle 0\rangle, \quad \langle R^D_{l,l+1}\rangle \Rightarrow \langle 0\rangle \quad \text{for } l = 1, \ldots L, \tag{45}$$

the waves transmitted just once at each interface is obtained. This means the waves directly coming from the real source located at $(L+1)$-th layer.

Eqs (20) and (22) become

$$\langle T^D_{l-1,l+1}\rangle \approx \langle T^D_{l,l+1}\rangle\langle T^D_{l-1,l}\rangle, \quad \langle T^U_{l-1,l+1}\rangle \approx \langle T^U_{l-1,l}\rangle\langle T^U_{l,l+1}\rangle \tag{46}$$

Eq. (28) shows that the stack of transmission operators for each interface can give the transmission operator for stacked layers.

$$\begin{aligned}\langle T^U_{l-1,L+1}\rangle &= \langle T^U_{l-1,l}\rangle\{\langle I\rangle - \langle R^D_{l,L+1}\rangle\langle R^U_{l-1,l}\rangle\}^{-1}\langle T^U_{l,L+1}\rangle \\ &\approx \langle T^U_{l-1,l}\rangle\langle T^U_{l,L+1}\rangle \\ &= \left(\prod_{l'=l-1}^{L-1}\langle T^U_{l',l'+1}\rangle\right)(\langle T^U_{L,L+1}\rangle\overline{\hat{\Phi}}^{L+1})\end{aligned} \tag{47}$$

Therefore, eq. (44) becomes as simple as follows (Fig. 10).

$$\begin{aligned}u^1_0 &= (\langle G^1_{0,0}\rangle\langle R^U_{0,1}\rangle + \langle G^1_{0,1}\rangle)(\langle I\rangle - \langle R^D_{1,L}\rangle\langle R^U_{0,1}\rangle)^{-1} \\ &\quad \times\langle T^U_{1,L}\rangle(\langle I\rangle - \langle R^D_{L,L+1}\rangle\langle R^U_{1,L}\rangle)^{-1}(\langle T^U_{L,L+1}\rangle\overline{\hat{\Phi}}^{L+1}) \\ &\approx (\langle G^1_{0,0}\rangle\langle R^U_{0,1}\rangle + \langle G^1_{0,1}\rangle)\langle T^U_{1,L}\rangle(\langle T^U_{L,L+1}\rangle\overline{\hat{\Phi}}^{L+1}) \\ &= (\langle G^1_{0,0}\rangle\langle \hat{R}^U_{0,1}\rangle + \langle G^1_{0,1}\rangle)\left(\prod_{l=1}^{L-1}\langle T^U_{l,l+1}\rangle\right)(\langle T^U_{L,L+1}\rangle\overline{\hat{\Phi}}^{L+1}).\end{aligned} \tag{48}$$

The meaning of eq. (48) in terms of wave propagation is shown in Fig. 10. Putting $\langle\hat{R}^U_{0,1}\rangle \Rightarrow 0$ allows one to exclude the influence of the surface from the solution.

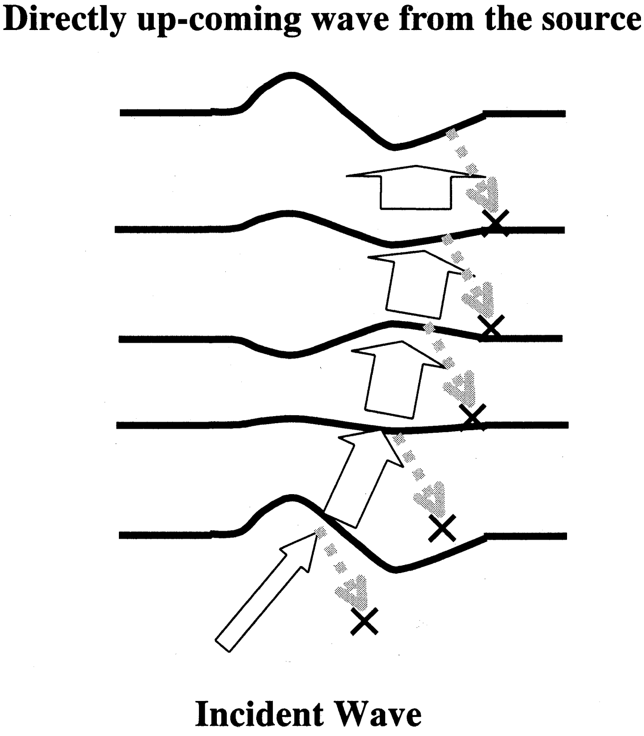


Figure 10. Schematic illustration for the wave directly coming from the source in the deepest layer that corresponds to eq. (48). The reflected waves at each interface are neglected.

This example corresponds to the following case. The contribution of surface waves into accelerograms obtained at sites near to the seismic source on firm ground, e.g. on tertiary deposit, is not as considerable as that of body waves in the frequency bands related to normal houses and low buildings. Moreover, it is said that the first big acceleration pulse is the main cause of damage to houses and buildings at a site close to a shallow seismic source. Eq. (48) can give an approximation for these cases in that the coda of *S* wave and surface waves do not matter.

3.2 Multiple reflections in a homogeneous basin

It may be necessary to obtain the solution to problems for which a solution has already been published, in order to check the validation of the method. This example is aimed to this purpose.

Consider a homogeneous surface layer on a homogeneous basement of different material property. Eq. (44) can be reduced by using $L=1$, $\langle T^U_{1,1}\rangle = \langle I\rangle$ and $\langle R^U_{1,1}\rangle = \langle 0\rangle$ as follows.

$$\begin{aligned} u^1_0 &= (\langle G^1_{0,0}\rangle\langle R^U_{0,1}\rangle + \langle G^1_{0,1}\rangle)(\langle I\rangle - \langle R^D_{1,2}\rangle\langle R^U_{0,1}\rangle)^{-1}(\langle T^U_{1,2}\rangle\hat{\bar{\Phi}}^2) \\ &= (\langle G^1_{0,0}\rangle\langle R^U_{0,1}\rangle + \langle G^1_{0,1}\rangle)\{\langle I\rangle + \langle R^D_{1,2}\rangle\langle R^U_{0,1}\rangle \\ &\quad + (\langle R^D_{1,2}\rangle\langle R^U_{0,1}\rangle)^2 + (\langle R^D_{1,2}\rangle\langle R^U_{0,1}\rangle)^3 + \cdots\}(\langle T^U_{1,2}\rangle\hat{\bar{\Phi}}^2 , \end{aligned} \tag{49}$$

where $(\langle I\rangle - \langle R^D_{1,2}\rangle\langle R^U_{0,1}\rangle)^{-1}$ denotes the reverberation in the surface layer (Fig. 11). The problem solved by Dravinski & Mossessian (1987) belongs to this type.

The operator for reverberation between the surface and the interface due to the upcoming wave $[\mathrm{Rev}^U_{0,1}] = (\langle I\rangle - \langle R^D_{1,2}\rangle\langle R^U_{0,1}\rangle)^{-1}$ can be calculated as follows. Eq. (19) shows

$$\begin{aligned} [\mathrm{Rev}^U_{0,1}] &= (\langle I\rangle - \langle R^D_{1,2}\rangle\langle R^U_{0,1}\rangle)^{-1} \\ &\equiv \langle I\rangle + \langle R^D_{1,2}\rangle\langle R^U_{0,1}\rangle \\ &\quad + \langle R^D_{1,2}\rangle\langle R^U_{0,1}\rangle\langle R^D_{1,2}\rangle\langle R^U_{0,1}\rangle + \cdots . \end{aligned} \tag{50}$$

This implies that $[\mathrm{Rev}^U_{0,1}]$ can be obtained in the following iterative way. Denote the result of the *n*-th iteration $[\mathrm{Rev}^U_{0,1}]_n$. Then,

$$\begin{aligned} [\mathrm{Rev}^U_{0,1}]_{n+1} &= \langle I\rangle + \langle R^D_{1,2}\rangle\langle R^U_{0,1}\rangle[\mathrm{Rev}^U_{0,1}]_n , \\ [\mathrm{Rev}^U_{0,1}]_1 &= \langle I\rangle + \langle R^D_{1,2}\rangle\langle R^U_{0,1}\rangle . \end{aligned} \tag{51}$$

In the actual calculation process, $\hat{\bar{\Phi}}^1 = \langle T^U_{1,2}\rangle\hat{\bar{\Phi}}^2$ is first obtained by eqs (38) and (40). This force vector is the input for the iterative process shown by eq. (51). Denote the initial value $[\hat{\bar{\Phi}}^1]_0$. Then, the first iteration gives

$$\begin{aligned} [\hat{\bar{\Phi}}^1]_1 &= (\langle I\rangle + \langle R^D_{1,2}\rangle\langle R^U_{0,1}\rangle)[\hat{\bar{\Phi}}^1]_0 \\ &= [\hat{\bar{\Phi}}^1]_0 + \langle R^D_{1,2}\rangle\langle R^U_{0,1}\rangle[\hat{\bar{\Phi}}^1]_0 . \end{aligned}$$

The higher order approximation is given as follows.

$$\begin{aligned} [\hat{\bar{\Phi}}^1]_{n+1} &= (\langle I\rangle + \langle R^D_{1,2}\rangle\langle R^U_{0,1}\rangle)[\hat{\bar{\Phi}}^1]_n \\ &= [\hat{\bar{\Phi}}^1]_n + \langle R^D_{1,2}\rangle\langle R^U_{0,1}\rangle[\hat{\bar{\Phi}}^1]_n . \end{aligned} \tag{52}$$

Eqs (34) and (35) show the way to calculate $\langle R^U_{0,1}\rangle[\hat{\bar{\Phi}}^1]_n$. Eqs (15) and (16) show that for $\langle R^D_{1,2}\rangle\langle R^U_{0,1}\rangle[\hat{\bar{\Phi}}^1]_n$. It is necessary to solve the simultaneous linear equations twice in each iterative

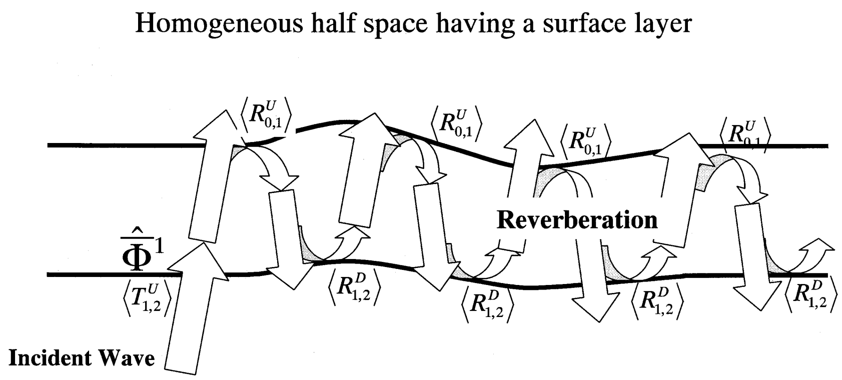


Figure 11. Schematic illustration that corresponds to eq. (49) for the waves that bounce up and down in the surface layer.

step. It is expected that a few times of reflection between the surface and the seismic bedrock of basin can provide a good approximation for accelerograms obtained at a site close to the shallow seismic source. Then, eq. (52) can be a good approximation. The idea of stacking the waves that bounce up and down in the surface layer has been used by Sánchez-Sesma *et al.* (1988) in the *origami* method for dipping layers.

Examples of computation for the seismic response of semicylindrical basin for plane wave incidence obtained by eq. (52) are shown in Figs 12 and 13. The P and S wave velocities and the density are 1.0 km s $^{-1}$, 0.5 km s $^{-1}$, 0.666 g cm^{-3} for the surface layer and 2.0 km s $^{-1}$, 1.0 km s $^{-1}$, 1.0 g cm^{-3} for the underlying half space, respectively. Yokoi (1996) has used the same structure. The thickness of the surface layer under both sides of the cylindrical basin, however, is set at zero for this example. The reference solution obtained by the Thomson–Haskell matrix method (Thomson 1950; Haskell 1953) considering only one interface or surface is used. The time dependency of incident waves is the Ricker wavelet, the normalized characteristic frequency of which is $\eta=\omega a/\pi\beta=$ 0.5, where a is the half-width of a basin. Fig. 12 shows clearly the efficiency of the reflection and transmission operators for the separation of waves with consideration of the reference solution. The responses for vertical P wave incidence are shown in Fig. 12(b) and those of vertical SV wave incidence in Fig. 12(a). The top panels show the waves directly reached to the surface, the middle ones these summed up with the contribution of the surface reflection, and the bottom ones these summed up with the contribution of the waves once bounced up from the interface. Fig. 13 shows the distribution of Fourier spectral amplitude at the normalized frequency $\eta=0.5$, for vertical P wave incidence (Fig. 13b) and for the vertical SV wave (Fig. 13a). This can be compared directly with those of Dravinski & Mossessian (1987). The solutions in Fig. 13 are the superposition of five iterations for the reverberation operator. Three iterations, however, are enough to reproduce the main part of S wave as shown in Fig. 12. More detailed discussion may be necessary, if we pay attention to the surface waves.

3.3 Multiple reflections in surface sediment that has several strata

This example is aimed to show the way in which an appropriate algorithm of computation can be derived by usage of reflection and transmission operators.

In general, the shallower sediments in alluvial basin has the lower S-wave velocity and density. Their impedance contrast is clear at the interfaces between hard rock and sediment, Tertiary and Quaternary or alluvial deposit. At these interfaces, the

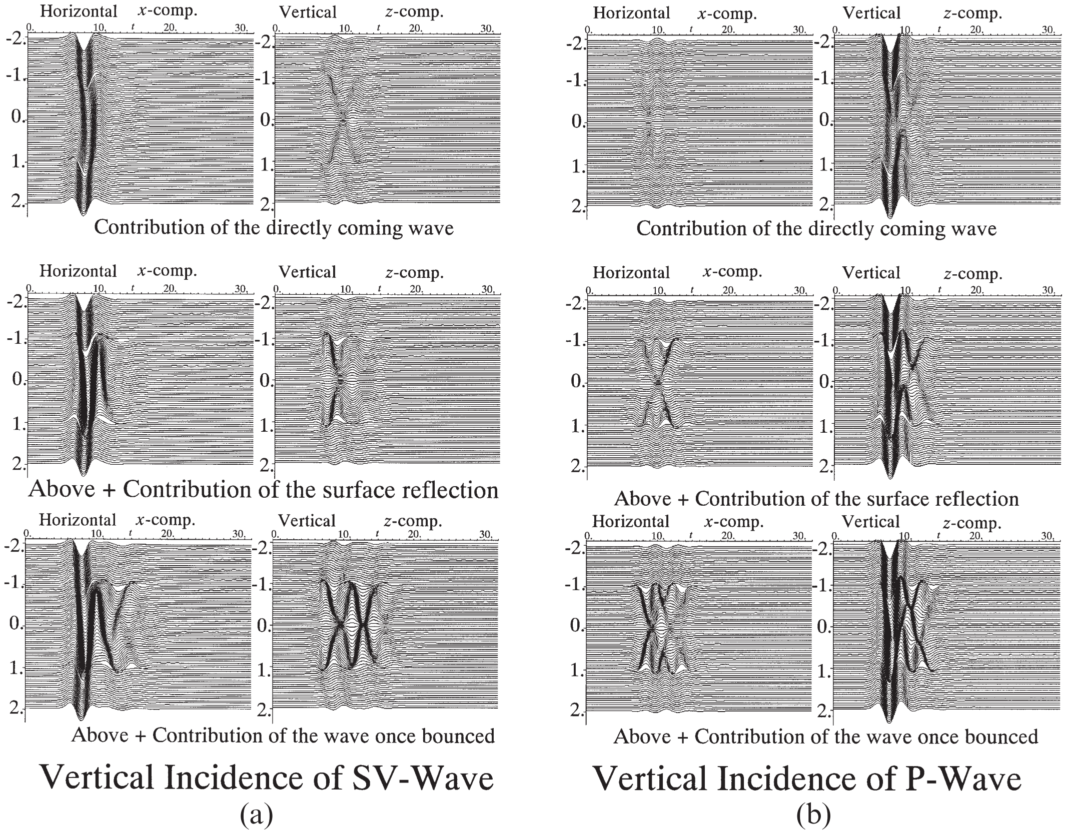
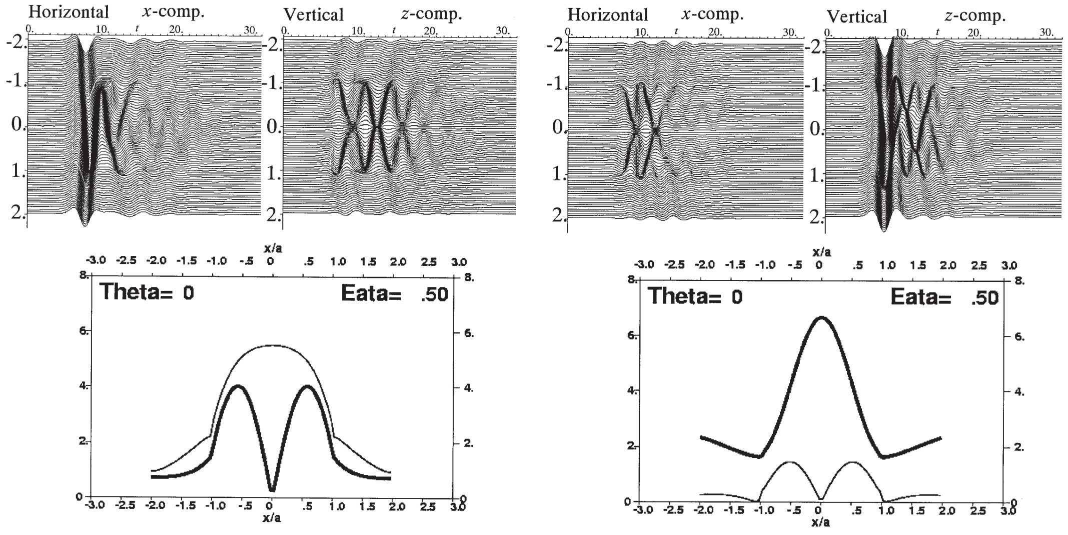


Figure 12. Examples of computation for the seismic response of semicylindrical homogeneous basin for vertical incidence of plane wave obtained by eq. (52). The material properties are explained in the main text (Section 3.2). The time dependency of incident wave is the Ricker wavelet, the normalized characteristic frequency of which is $\eta=\omega a/\pi\beta=0.5$, where a is the half-width of a basin. The responses for vertical P wave incidence are shown in the right panels and those of vertical SV wave incidence in the left panels. The top panels show the waves directly reached to the surface, the middle ones these summed up with the contribution of the surface reflection, and the bottom ones these summed up with the contribution of the waves once bounced up from the interface.



Vertical Incidence of SV Wave (a)

Vertical Incidence of P Wave (b)

Figure 13. Distribution of Fourier spectral amplitude at the normalized frequency $\eta = 0.5$, for vertical P wave incidence in the right panels and for vertical SV wave in the left panels. This can be compared directly with the results of Dravinski & Mossessian (1987). These are the superposition of five iterations for the reverberation operator.

waves can propagate from the deeper to the shallower layer easily, i.e. the transmission is dominant in comparison with the reflection for upcoming waves. In contrast, the waves that come from the shallower layer are reflected strongly by the high impedance contrast. In some special cases, the reflection for the upgoing wave represented by the operator $\langle R^U_{l,l+1} \rangle$ may be negligible for $l = 1 \ldots L$.

Using the approximation,

$$\langle R^U_{l,l+1} \rangle \approx 0 \quad \text{for } l = 2 \ldots L\,. \tag{53}$$

Eqs (28) and (30) can be changed as follows,

$$\begin{cases} \langle R^U_{l-1,L+1} \rangle \approx 0\,, \\ \langle T^U_{l-1,L+1} \rangle \approx \langle T^U_{l-1,l} \rangle \langle T^U_{l,L+1} \rangle\,, \end{cases}$$
$$\begin{cases} \langle R^D_{l-1,L+1} \rangle \approx \langle R^D_{l-1,l} \rangle + \langle T^U_{l-1,l} \rangle \langle R^D_{l,L+1} \rangle \langle T^D_{l-1,l} \rangle\,, \\ \langle T^D_{l-1,L+1} \rangle \approx \langle T^D_{l,L+1} \rangle \langle T^D_{l-1,l} \rangle\,. \end{cases} \tag{54}$$

The second one implies,

$$\langle T^U_{1,L} \rangle \approx \prod_{l=2}^{L} \langle T^U_{l-1,l} \rangle, \langle T^D_{1,L} \rangle \approx \prod_{l=2}^{L} \langle T^D_{l-1,l} \rangle \tag{55.1}$$

$$\begin{aligned} \langle R^D_{1,L+1} \rangle &\approx \langle R^D_{1,2} \rangle + \langle T^U_{1,2} \rangle \langle R^D_{2,L+1} \rangle \langle T^D_{1,2} \rangle \\ &\approx \langle R^D_{1,2} \rangle + \langle T^U_{1,2} \rangle (\langle R^D_{2,3} \rangle + \langle T^U_{2,3} \rangle \langle R^D_{3,L+1} \rangle \langle T^D_{2,3} \rangle) \\ &\quad \times \langle T^D_{1,2} \rangle \approx \cdots \\ &\approx \langle R^D_{1,2} \rangle + \sum_{l=2}^{L} \left\{ \prod_{k=1}^{l-1} \langle T^U_{k,k+1} \rangle \cdot \langle R^D_{l,l+1} \rangle \cdot \prod_{k=1}^{l-1} \langle T^D_{k,k+1} \rangle \right\}. \end{aligned} \tag{55.2}$$

Eq. (55.2) means that the reflection for down going wave in the first layer can be approximated by the stack of returning waves only once reflected at each interface.

Eq. (44) can be written in the following,

$$\begin{aligned} u^1_0 \approx (\langle G^1_{0,0} \rangle \langle R^U_{0,1} \rangle + \langle G^1_{0,1} \rangle)(\langle I \rangle - \langle R^D_{1,L+1} \rangle \langle R^U_{0,1} \rangle)^{-1} \\ \times \left(\prod_{l=2}^{L} \langle T^U_{l-1,l} \rangle \right) (\langle T^U_{L,L+1} \rangle \hat{\boldsymbol{\Phi}}^{L+1})\,, \end{aligned} \tag{56.1}$$

where $[\mathrm{Rev}^U_{0,L}] = (\langle I \rangle - \langle R^D_{1,L+1} \rangle \langle R^U_{0,1} \rangle)^{-1}$ can be calculated by the following,

$$\begin{aligned} [\mathrm{Rev}^U_{0,L}]_{n+1} &= \langle I \rangle + \langle R^D_{1,L+1} \rangle \langle R^U_{0,1} \rangle [\mathrm{Rev}^U_{0,L}]_n, \\ [\mathrm{Rev}^U_{0,L}]_1 &= \langle I \rangle + \langle R^D_{1,L+1} \rangle \langle R^U_{0,1} \rangle\,. \end{aligned} \tag{56.2}$$

Eqs (56.1) and (56.2) suggest the following algorithm.

(1) The displacement and traction vectors of the upcoming wave directly arrived from the real source to the first layer is calculated and kept in memory.

(2) The reflection at the surface is applied to obtain those of the down going reflected wave.

(3) The stack of returning waves reflected once at each interface for this down going wave is calculated and

(4) summed up with those of the wave kept in memory. For this sum, the surface reflection is applied as well as (2), then the procedure described above is repeated several times. Finally, the operator, namely, the first parenthesis of the right member of eq. (56.1) is applied and displacement at the surface is obtained.

In some special configurations, the algorithm mentioned above can give a good approximation for accelerograms that are composed mainly of body waves. More consideration, however, may be necessary for the surface waves.

This, however, is just an example. It is possible to design appropriate algorithms based on the approximation derived by consideration on impedance contrast at each interface and wave types requested, if the reflection and transmission operators for irregular interface are handled in computation.

4 DISCUSSION AND CONCLUSIONS

First, the reflection and transmission operators for an irregular interface are defined in the space–frequency domain in terms of I-BEM and the practical way to estimate approximately the reflected and transmitted wavefields, based on the approximation using the reference solution (eqs 6, 13 and 16). Second the reflection and transmission operators for a layer sandwiched between irregular interfaces are defined by using the operators for an irregular interface (eqs 20 and 22). Then, those for stacked layers with free surfaces are defined (eqs 28 and 39), and the wavefield at the surface due to real source in the deepest layer is formulated (eq. 37). Finally, some numerical examples for the separation of waves are shown.

As shown above, the formulation of reflection and transmission for irregular interface introduced here makes easier the search for an efficient approximation of wavefield with a clear theoretical back ground of wave propagation. As previously mentioned, the expressions giving the relations among reflection and transmission operators, i.e. eqs (20), (22), (28), (30), and (44) giving the displacement at the surface show a clear similarity with R/T-MM for horizontally stratified media (Kennett 1983). It may be allowed to say that the method presented here is an extension of R/T-MM to irregularly stratified media in terms of I-BEM, because R/T-MM is one of the efficient ways to obtain the reference solution. Also, it can be said that this is a hybrid method between R/T-MM and I-BEM in the space–frequency domain. The reference solution in the space–frequency domain can be calculated not only by R/T-MM but also by other methods, e.g. Luco & Apsel (1983). It is easy to imagine that the strategy used here may be useful in deriving the solution in the space–time domain by a combination of I-BEM in the time domain and the reference solution in the space–time domain.

As shown above, the discussions in this paper are on the reflection and transmission operators and on the calculation algorithm mainly for body waves by using these operators. The handling of surface waves is left for future studies. The method presented here shares the same problems with the ray theory including the generalized ray method and the same effect on extracting body waves and on excluding the influence of the free surface. This, however, has an advantage, against the ray theories, in that the integral on the ray parameter is not necessary and the amplitude of the body wave is kept without any special treatment.

The computation process is composed of the products of a Green's function matrix to force vector, sums among displacement–traction vectors and simultaneous linear equations. Therefore, there is no operation among the Green's function matrices in a computation based on the method presented here. This feature may make computation faster and required smaller main memory, in comparison with the recursive treatment of Green's function matrices (Yokoi 1996). Moreover, it is expected that this will allow us to apply the Fast Multipole Method (e.g. Fujiwara 2000), which can accelerate the calculation of the product of Green's function matrix to vectors. Therefore, it is expected that the calculation may be accelerated much more than the present performance.

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