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A Bayes filter in Friedland form for INS/GPS vector gravimetry

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SUMMARY

We introduce a dynamic linear model in which the observation equations are perturbed by a form that has constant (over time), non-random coefficients and may represent the disturbing gravity field under investigation. Because of its non-random behaviour, their form cannot be determined using Friedland's generalization of the Kalman filter. However, after putting it in dual form ('Bayes filter'), Friedland's approach can be further generalized to also cover the present case. This (apparently new) filter version is then employed to estimate the disturbing gravity vector from airborne INS/GPS data, following the ideas of Jekeli & Kwon (1999) for the combined analysis. Thus, the filter acts on the integration of INS and GPS acceleration vectors where the discrepancies are simultaneously modelled in terms of random system 'biases', i.e. self-calibration, and the local *non-random* disturbing gravity vector. We do not introduce a second filter step ('cascaded filter'), owing to problems with neglected correlations in a two-step procedure.

The new results are eventually compared with those of a related algorithm that may be interpreted as Kalman filtering with 'partial regularization', effectively using a stochastic gravity field representation. Improvements of between 10 per cent ('down' direction) and 60 per cent (north direction) were achieved, which we attribute in large part to the use of the disturbing gravity vector as a non-stochastic quantity.

Key words: generalized Kalman/Bayes filtering, INS/GPS integration, non-random gravity field representation, vector gravimetry.

INTRODUCTION

The integrated INS/GPS system has been investigated as a tool for vector gravimetry for more than 10 years (see, e.g. Eissfeller & Spietz 1989; Knickmeyer 1990). Recent results have been reported by Salychev & Schwarz (1995), Wei & Schwarz (1995) and Jekeli & Kwon (1999) among others. The respective approaches are mostly based on some form of Kalman-type filter, 'centralized' or 'cascaded' according to Wei & Schwarz (1990), using self-calibration techniques with random system 'biases' along with a random or non-random representation of the disturbing gravity vector. For more details, see Salychev & Schaffrin (1992), Salychev & Schwarz (1995), Schaffrin (1995) or Jekeli & Kwon (2000). Some simplified comparisons have been undertaken by Hammada (1996) who, in particular, looked into the 'wave algorithm' without system noise, and into a rather unspecific form of low-pass filtering based on finite impulse response (FIR) design.

The 'wave algorithm' uses a non-random gravity field representation; however, by not allowing any system noise in the model, these effects will show up later in the form of inexplicable jumps in the solution for the disturbing gravity vector. We, therefore, keep the system noise and introduce a functional model for the disturbing gravity field as a linear combination of certain spatial base functions. The degree of this expansion should depend on the extension of the area under investigation and the expected roughness/smoothness of the gravity field. If necessary, an adaptive technique similar to that considered by Wang *et al.* (1995) could be introduced as well.

We also decided to employ a centralized filter since the step procedures ('cascaded filters') regularly run into problems with the neglected correlations (see the remarks by Knickmeyer 1990, p. 53). In any case, we apply this filter at the level of INS and GPS accelerations, in the same way as Jekeli & Kwon (1999) and without the preprocessing recommended by Knickmeyer (1990, p. 51) for 'INS/GPS synthesis'.

This approach leads to a dynamic linear model with additional constant (over time) and non-random parameters in the observation and state equations. Such a model can no longer be treated by simply updating the ordinary Kalman filter in accordance with Friedland (1969). It is, however, possible to adopt Friedland's ideas to modify the dual form, commonly known as a 'Bayes filter'. After the general description of our mathematical model in the next section, we shall present the new filter algorithm in Section 2, followed by a discussion in Section 3 of how it is best applied in vector gravimetry with the integrated INS/GPS system. In Section 4, finally, we present a number of numerical results that will allow us to conclude the relative superiority of the new approach in many typical situations, measured by the respective standard deviations. We attribute this improvement mainly to the fact that the disturbing gravity vector is here modelled as non-stochastic quantity, and less to the rigorous way of our error/bias propagation.

1 THE DYNAMIC LINEAR MODEL WITH ADDITIONAL NON-RANDOM, CONSTANT PARAMETERS

In order to introduce our notation, we begin with the standard form of the dynamic linear model as in Schaffrin (1995), for instance, which for the interval from $t = t_{k-1}$ to $t = t_k$ consists of the following three parts:

$$y_k = A_k x_k + e_k \quad \text{'observation equations at } t = t_k\text{'}, \tag{1.1a}$$

$$x_k = \phi_{k-1} x_{k-1} + u_k \quad \text{'state equations between } t_{k-1} \text{ and } t_k\text{'}, \tag{1.1b}$$

$$\tilde{x}_{k-1} = x_{k-1} + e^0_{k-1} \quad \text{'initial condition at } t = t_{k-1}\text{'}, \tag{1.1c}$$

where the mean vector and the variance-covariance matrix of the random error vectors are specified as follows:

$$\begin{bmatrix} e_k \\ u_k \\ e^0_{k-1} \end{bmatrix} \sim \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \Sigma_k & 0 & 0 \\ 0 & \Theta_k & 0 \\ 0 & 0 & \Sigma^0_{k-1} \end{bmatrix} \right). \tag{1.1d}$$

We further do not allow any correlation over time. The dynamic linear model represents a linearized and discretized form of an initial-value problem for a set of ordinary differential equations whose parameters have simultaneously been observed indirectly, providing us with an (incremental) observation vector y_k at all time epochs $t = t_k$ which depends linearly on the (incremental) state vectors x_k through the coefficient matrices A_k, while e_k denotes the respective observational noise vector. In contrast, u_k denotes the so-called 'system (description) noise' vector for any $t = t_k$ with the transition matrix ϕ_{k-1} describing the essential relation between x_{k-1} and x_k. In addition, we consider the information $\tilde{x}_{k-1}$ about our state vector x_{k-1} at time epoch $t = t_{k-1}$ to be contaminated by some (independent) random noise e^0_{k-1} as well.

In this model the least-squares solution (LESS) is known to yield the best inhomogeneously linear prediction (inhom-BLIP) of x_k through the Kalman filter algorithm

$$K_k := \left(\Theta_k + \phi_{k-1}\Sigma^0_{k-1}\phi^{\mathrm{T}}_{k-1}\right) A^{\mathrm{T}}_k \left[\Sigma_k + A_k\left(\Theta_k + \phi_{k-1}\Sigma^0_{k-1}\phi^{\mathrm{T}}_{k-1}\right) A^{\mathrm{T}}_k\right]^{-1}, \tag{1.2a}$$

$$\breve{x}_k := \phi_{k-1}\tilde{x}_{k-1}, \quad z_k := y_k - A_k\breve{x}_k \quad \text{('innovation')}, \tag{1.2b}$$

$$\tilde{x}_k := \breve{x}_k + K_k \cdot z_k, \quad E\{\tilde{x}_k - x_k\} = 0 \quad \text{('weakly unbiased')}, \tag{1.2c}$$

$$\Sigma^0_k := MSPE\{\tilde{x}_k\} = D\{\tilde{x}_k - x_k\} = (I - K_k A_k) \cdot \left(\Theta_k + \phi_{k-1}\Sigma^0_{k-1}\phi^{\mathrm{T}}_{k-1}\right). \tag{1.2d}$$

The updated vector $\tilde{x}_k$ along with its Mean Square Prediction Error (MSPE) matrix Σ^0_k will be used as initial condition for the subsequent interval from $t = t_k$ to $t = t_{k+1}$. The above algorithm is most suitable in cases where, for all t_k, the number n_k of observation equations stays consistently below the number of random effects in the 'state vector' x_k. In all other cases, we may rather use its dual form which is sometimes called the 'Bayes filter', namely

$$\Sigma^0_k := \left[A^{\mathrm{T}}_k \Sigma^{-1}_k A_k + \left(\Theta_k + \phi_{k-1}\Sigma^0_{k-1}\phi^{\mathrm{T}}_{k-1}\right)^{-1}\right]^{-1}, \tag{1.3a}$$

$$K_k := \Sigma^0_k A^{\mathrm{T}}_k \Sigma^{-1}_k, \tag{1.3b}$$

$$\breve{x}_k := \phi_{k-1}\tilde{x}_{k-1}, \quad z_k := y_k - A_k\breve{x}_k \quad \text{('innovation')}, \tag{1.3c}$$

$$\tilde{x}_k := \breve{x}_k + K_k \cdot z_k, \quad E\{\tilde{x}_k - x_k\} = 0. \tag{1.3d}$$

Of course, from a theoretical point of view, the solutions of both algorithms should not differ. Any numerical differences that may occur nonetheless, may thus be attributed to accumulated rounding errors.

Now let us introduce the extended dynamic linear model that allows for a time-invariant, but *stochastic* perturbation of the observation equations, and thus for a better absorption of any unmodelled gravity disturbances:

$$y_k = [A_k \quad X_k] \cdot \begin{bmatrix} x_k \\ b_k \end{bmatrix} + e_k, \tag{1.4a}$$

$$\begin{bmatrix} x_k \\ b_k \end{bmatrix} = \begin{bmatrix} \phi_{k-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x_{k-1} \\ b_{k-1} \end{bmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} u_k, \tag{1.4b}$$

$$\begin{bmatrix} \tilde{x}_k \\ \tilde{b}^{(k-1)} \end{bmatrix} = \begin{bmatrix} x_{k-1} \\ b_{k-1} \end{bmatrix} + \begin{bmatrix} e^0_{k-1} \\ v^0_{k-1} \end{bmatrix}, \tag{1.4c}$$

with

$$\begin{bmatrix} e_k \\ u_k \\ e_{k-1}^0 \\ v_{k-1}^0 \end{bmatrix} \sim \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \Sigma_k & 0 & 0 & 0 \\ 0 & \Theta_k & 0 & 0 \\ 0 & 0 & \Sigma_{xx_{k-1}}^0 & \Sigma_{xb_{k-1}}^0 \\ 0 & 0 & \Sigma_{bx_{k-1}}^0 & \Sigma_{bb_{k-1}}^0 \end{bmatrix} \right). \tag{1.4d}$$

This model is indeed an *extension* of the previous *dynamic linear model* in so far as the respective inconsistencies that describe the discrepancy between the model $A_k x_k$ and the measurements in y_k were solely attributed to random noise in eqs (1.1a–c), whereas in eqs (1.4a–c) they may be caused by so-called perturbations as well. We call these $b_0, b_1, b_2, \ldots, b_{k-1}, b_k, \ldots$.

Note that, according to eq. (1.4b), we indeed have reduced their variability

$$b_0 = b_1 = b_2 = \cdots = b \quad (\text{'time-independent'}), \tag{1.5}$$

but with time-dependent estimates $\tilde{b}^{(k-1)}$ that enter the prior condition (1.4c). From a structural point of view, the above model resembles the original model as long as the variance-covariance matrix $\Sigma_{bb_{k-1}}^0 = D\{v_{k-1}^0\} = D\{\tilde{b}^{(k-1)} - b\}$ is a finite matrix for all $k \in N$ (with D denoting 'dispersion'), and can thus be handled in a similar fashion. It is namely a similar fashion eventually resulting in a properly modified filter. It is then straightforward to derive the formal least-squares solution in 'Kalman form':

$$\begin{bmatrix} K_{x_k} \\ K_{b_k} \end{bmatrix} := \begin{bmatrix} \Theta_k + \phi_{k-1}\Sigma_{xx_{k-1}}^0 \phi_{k-1}^{\mathrm{T}} & \vdots & \phi_{k-1}\Sigma_{xb_{k-1}}^0 \\ \Sigma_{bx_{k-1}}^0 \phi_{k-1}^{\mathrm{T}} & \vdots & \Sigma_{bb_{k-1}}^0 \end{bmatrix} \begin{bmatrix} A_k^{\mathrm{T}} \\ X_k^{\mathrm{T}} \end{bmatrix} \cdot \left(\Sigma_k + [A_k \quad X_k] \begin{bmatrix} \Theta_k + \phi_{k-1}\Sigma_{xx_{k-1}}^0 \phi_{k-1}^{\mathrm{T}} & \vdots & \phi_{k-1}\Sigma_{xb_{k-1}}^0 \\ \Sigma_{bx_{k-1}}^0 \phi_{k-1}^{\mathrm{T}} & \vdots & \Sigma_{bb_{k-1}}^0 \end{bmatrix} \begin{bmatrix} A_k^{\mathrm{T}} \\ X_k^{\mathrm{T}} \end{bmatrix} \right)^{-1}, \tag{1.6a}$$

$$\begin{bmatrix} \breve{x}_k \\ \breve{b}^{(k)} \end{bmatrix} := \begin{bmatrix} \phi_{k-1}\tilde{x}_{k-1} \\ \tilde{b}^{(k-1)} \end{bmatrix}, \quad z_k := y_k - [A_k \quad X_k] \begin{bmatrix} \breve{x}_k \\ \breve{b}^{(k)} \end{bmatrix} = y_k - A_k\breve{x}_k - X_k\tilde{b}^{(k-1)}, \tag{1.6b}$$

$$\begin{bmatrix} \tilde{x}_k \\ \tilde{b}^{(k)} \end{bmatrix} := \begin{bmatrix} \breve{x}_k \\ \tilde{b}^{(k-1)} \end{bmatrix} + \begin{bmatrix} K_{x_k} \\ K_{b_k} \end{bmatrix} \cdot z_k, \quad E\left\{ \begin{bmatrix} \tilde{x}_k - x_k \\ \tilde{b}^{(k)} - b \end{bmatrix} \right\} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{1.6c}$$

$$\begin{bmatrix} \Sigma_{xx_k}^0 & \Sigma_{xb_k}^0 \\ \Sigma_{bx_k}^0 & \Sigma_{bb_k}^0 \end{bmatrix} := MSPE\left\{ \begin{bmatrix} \tilde{x}_k \\ \tilde{b}^{(k)} \end{bmatrix} \right\} = D\left\{ \begin{bmatrix} \tilde{x}_k - x_k \\ \tilde{b}^{(k)} - b \end{bmatrix} \right\} = \left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} K_{x_k}A_k & \vdots & K_{x_k}X_k \\ K_{b_k}A_k & \vdots & K_{b_k}X_k \end{bmatrix} \right) \begin{bmatrix} \Theta_k + \phi_{k-1}\Sigma_{xx_{k-1}}^0 \phi_{k-1}^{\mathrm{T}} & \vdots & \phi_{k-1}\Sigma_{xb_{k-1}}^0 \\ \Sigma_{bx_{k-1}}^0 \phi_{k-1}^{\mathrm{T}} & \vdots & \Sigma_{bb_{k-1}}^0 \end{bmatrix}. \tag{1.6d}$$

Alternatively, its 'Bayes form' reads:

$$\begin{bmatrix} \Sigma_{xx_k}^0 & \Sigma_{xb_k}^0 \\ \Sigma_{bx_k}^0 & \Sigma_{bb_k}^0 \end{bmatrix} := \left(\begin{bmatrix} A_k^{\mathrm{T}}\Sigma_k^{-1}A_k & \vdots & A_k^{\mathrm{T}}\Sigma_k^{-1}X_k \\ X_k^{\mathrm{T}}\Sigma_k^{-1}A_k & \vdots & X_k^{\mathrm{T}}\Sigma_k^{-1}X_k \end{bmatrix} + \begin{bmatrix} \Theta_k + \phi_{k-1}\Sigma_{xx_{k-1}}^0 \phi_{k-1}^{\mathrm{T}} & \vdots & \phi_{k-1}\Sigma_{xb_{k-1}}^0 \\ \Sigma_{bx_{k-1}}^0 \phi_{k-1}^{\mathrm{T}} & \vdots & \Sigma_{bb_{k-1}}^0 \end{bmatrix}^{-1} \right)^{-1}, \tag{1.7a}$$

$$\begin{bmatrix} K_{x_k} \\ K_{b_k} \end{bmatrix} := \begin{bmatrix} \Sigma_{xx_k}^0 & \Sigma_{xb_k}^0 \\ \Sigma_{bx_k}^0 & \Sigma_{bb_k}^0 \end{bmatrix} \begin{bmatrix} A_k^{\mathrm{T}} \\ X_k^{\mathrm{T}} \end{bmatrix} \Sigma_k^{-1}, \tag{1.7b}$$

$$\begin{bmatrix} \breve{x}_k \\ \breve{b}^{(k)} \end{bmatrix} := \begin{bmatrix} \phi_{k-1}\tilde{x}_{k-1} \\ \tilde{b}^{(k-1)} \end{bmatrix}, \quad z_k := y_k - [A_k \quad X_k] \begin{bmatrix} \breve{x}_k \\ \breve{b}^{(k)} \end{bmatrix} = y_k - A_k\breve{x}_k - X_k\tilde{b}^{(k-1)}, \tag{1.7c}$$

$$\begin{bmatrix} \tilde{x}_k \\ \tilde{b}^{(k)} \end{bmatrix} := \begin{bmatrix} \breve{x}_k \\ \tilde{b}^{(k-1)} \end{bmatrix} + \begin{bmatrix} K_{x_k} \\ K_{b_k} \end{bmatrix} \cdot z_k, \quad E\left\{ \begin{bmatrix} \tilde{x}_k - x_k \\ \tilde{b}^{(k)} - b \end{bmatrix} \right\} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{1.7d}$$

In the next section, we shall modify these formulae in such a way that the results are produced as updates to the original solution (1.3a–d) that belongs to the simple dynamic linear model. The algorithm thus ought to be designed recursively so that the case of non-random perturbations β, where

$$b \to \beta = E\left\{\hat{\beta}^{(1)}\right\} = E\left\{\hat{\beta}^{(2)}\right\} = \cdots, \quad \left(\Sigma_{bb}^0\right)^{-1} =: Q_{bb}^0 \to Q_{\beta\beta}^0 = 0, \tag{1.8}$$

replaces the condition (1.5), is also covered. Here, the hat refers to *estimates* of the vector β whose components have received infinite variance now and will no longer impact the solution by possibly incorrect prior information on these perturbations (as would be the case with random perturbations b with finite variance).

2 AN ALGORITHM FOR THE BAYES FILTER IN FRIEDLAND FORM

We first show how, according to Friedland (1969), the modified formulae of the Kalman filter, eqs (1.6a–d), can be given an update form on the basis of the ordinary Kalman filter, eqs (1.2a–d), and why they may fail in the case of non-random perturbations as described through eq. (1.8).

For this, let us just consider the first interval with $k = 1$ in which case formula (1.6a) would, due to $\Sigma_{xb_0} = 0 = \Sigma_{bx_0}^{\mathrm{T}}$, read:

$$\begin{bmatrix} K_{x_1} \\ K_{b_1} \end{bmatrix} = \begin{bmatrix} \left(\Theta_1 + \phi_0 \Sigma_{xx_0}^0 \phi_0^{\mathrm{T}}\right) A_1^{\mathrm{T}} \\ \Sigma_{bb_0}^0 X_1^{\mathrm{T}} \end{bmatrix} \cdot \left[\Sigma_1 + A_1\left(\Theta_1 + \phi_0 \Sigma_{xx_0}^0 \phi_0^{\mathrm{T}}\right) A_1^{\mathrm{T}} + X_1 \Sigma_{bb_0}^0 X_1^{\mathrm{T}}\right]^{-1}, \tag{2.1}$$

thus requiring $\Sigma_{bb_0}^0$ to be a finite matrix.

Alternatively we may, however, use

$$K_{x_1} = K_1 \left\{ I - X_1 \Sigma_{bb_0}^0 X_1^{\mathrm{T}} \left[\Sigma_1 + A_1\left(\Theta_1 + \phi_0 \Sigma_{xx_0}^0 \phi_0^{\mathrm{T}}\right) A_1^{\mathrm{T}} + X_1 \Sigma_{bb_0}^0 X_1^{\mathrm{T}}\right]^{-1} \right\} = K_1\left(I - X_1 K_{b_1}\right) \tag{2.2a}$$

with K_1 as in eq. (1.2a) and

$$K_{b_1} = \left\{ Q_{bb}^0 + X_1^{\mathrm{T}} \left[\Sigma_1 + A_1\left(\Theta_1 + \phi_0 \Sigma_{xx_0}^0 \phi_0^{\mathrm{T}}\right) A_1^{\mathrm{T}}\right]^{-1} X_1 \right\}^{-1} X_1^{\mathrm{T}} \left[\Sigma_1 + A_1\left(\Theta_1 + \phi_0 \Sigma_{xx_0}^0 \phi_0^{\mathrm{T}}\right) A_1^{\mathrm{T}}\right]^{-1} \tag{2.2b}$$

instead, which would also be computable as K_{β_1} when $Q_{bb}^0 \to 0$ as in eq. (1.8) if we can only assume that X_1 is of full rank and, therefore, all perturbations are readily estimable.

In the next step, formulae (1.6b–c) would lead to

$$\tilde{b}^{(1)} = \tilde{b}^{(0)} + K_{b_1}\left(y_1 - A_1 \phi_0 \tilde{x}_0 - X_1 \tilde{b}^{(0)}\right), \tag{2.3a}$$

$$\begin{aligned} \tilde{x}_1 &= \phi_0 \tilde{x}_0 + K_{x_1}\left(y_1 - A_1 \phi_0 \tilde{x}_0 - X_1 \tilde{b}^{(0)}\right) = [\breve{x}_1 + K_1(y_1 - A_1 \breve{x}_1)] - K_1 X_1 \tilde{b}^{(0)} - K_1 X_1 \left[K_{b_1}\left(y_1 - A_1 \phi_0 \tilde{x}_0 - X_1 \tilde{b}^{(0)}\right)\right] \\ &= \bar{x}_1 - K_1 X_1 \tilde{b}^{(0)} - K_1 X_1 \left(\tilde{b}^{(1)} - \tilde{b}^{(0)}\right) = \bar{x}_1 - K_1 X_1 \tilde{b}^{(1)}, \end{aligned} \tag{2.3b}$$

in general, and to

$$\hat{\beta}^{(1)} = \hat{\beta}^{(0)} + K_{\beta_1}\left(y_1 - A_1 \phi_0 \tilde{x}_0 - X_1 \hat{\beta}^{(0)}\right) = K_{\beta_1}(y_1 - A_1 \breve{x}_1), \tag{2.4a}$$

$$\tilde{x}_1 = \bar{x}_1 - K_1 X_1 \hat{\beta}^{(1)}, \tag{2.4b}$$

in the case of $Q_{bb}^0 \to 0$, where

$$\bar{x}_1 := \breve{x}_1 + K_1(y_1 - A_1 \breve{x}_1) \tag{2.5}$$

represents the classical Kalman filter solution, without any perturbations, as in eq. (1.2c) or eq. (1.3d).

If we now, however, turn to formula (1.6c) we obtain

$$\begin{bmatrix} \Sigma_{xx_1}^0 & \Sigma_{xb_1}^0 \\ \Sigma_{bx_1}^0 & \Sigma_{bb_1}^0 \end{bmatrix} = \begin{bmatrix} I - K_{x_1} A_1 & \vdots & -K_{x_1} X_1 \\ -K_{b_1} A_1 & \vdots & I - K_{b_1} X_1 \end{bmatrix} \begin{bmatrix} \Theta_1 + \phi_0 \Sigma_{xx_0}^0 \phi_0^{\mathrm{T}} & \vdots & 0 \\ 0 & \vdots & \Sigma_{bb_0}^0 \end{bmatrix}, \tag{2.6}$$

and consequently, with $\bar{\Sigma}_1^0$ taken from eq. (1.2d), the representations

$$\Sigma_{xx_1}^0 = \left(I - K_1 A_1 + K_1 X_1 K_{b_1} A_1\right)\left(\Theta_1 + \phi_0 \Sigma_{xx_0}^0 \phi_0^{\mathrm{T}}\right) = \bar{\Sigma}_1^0 + K_1 X_1 \left\{ Q_{bb}^0 + X_1^{\mathrm{T}} \left[\Sigma_1 + A_1\left(\Theta_1 + \phi_0 \Sigma_{xx_0}^0 \phi_0^{\mathrm{T}}\right) A_1^{\mathrm{T}}\right]^{-1} X_1 \right\}^{-1} X_1^{\mathrm{T}} K_1^{\mathrm{T}}, \tag{2.7a}$$

$$\Sigma_{xb_1}^0 = -K_1 X_1 \left(I - K_{b_1} X_1\right) \Sigma_{bb_0}^0 = -\left(\Theta_1 + \phi_0 \Sigma_{xx_0}^0 \phi_0^{\mathrm{T}}\right) A_1^{\mathrm{T}} K_{b_1}^{\mathrm{T}} = \left(\Sigma_{bx_1}^0\right)^{\mathrm{T}}, \tag{2.7b}$$

$$\Sigma_{bb_1}^0 = \left(I - K_{b_1} X_1\right) \Sigma_{bb_0}^0 = \left\{ Q_{bb}^0 + X_1^{\mathrm{T}} \left[\Sigma_1 + A_1\left(\Theta_1 + \phi_0 \Sigma_{xx_0}^0 \phi_0^{\mathrm{T}}\right) A_1^{\mathrm{T}}\right]^{-1} X_1 \right\}^{-1}, \tag{2.7c}$$

where only the second identities can be used in the case that $Q_{bb}^0 \to 0$, i.e. the case of non-random perturbations. These identities readily result from the Bayes filter solution as given by eq. (1.7) which yields

$$\begin{bmatrix} \Sigma_{xx_1}^0 & \Sigma_{xb_1}^0 \\ \Sigma_{bx_1}^0 & \Sigma_{bb_1}^0 \end{bmatrix} = \begin{bmatrix} A_1^{\mathrm{T}} \Sigma_1^{-1} A_1 + \left(\Theta_1 + \phi_0 \Sigma_{xx_0}^0 \phi_0^{\mathrm{T}}\right)^{-1} & \vdots & A_1^{\mathrm{T}} \Sigma_1^{-1} X_1 \\ X_1^{\mathrm{T}} \Sigma_1^{-1} A_1 & \vdots & X_1^{\mathrm{T}} \Sigma_1^{-1} X_1 + Q_{bb}^0 \end{bmatrix}^{-1}, \tag{2.8}$$

and thus, with $\bar{\Sigma}_1^0$ now taken from eq. (1.3a) and K_1 from eq. (1.3b),

$$\begin{aligned} \Sigma_{xx_1}^0 &= \bar{\Sigma}_1^0 + \bar{\Sigma}_1^0 A_1^{\mathrm{T}} \Sigma_1^{-1} X_1 \left[Q_{bb}^0 + X_1^{\mathrm{T}} \left(\Sigma_1^{-1} - \Sigma_1^{-1} A_1 \bar{\Sigma}_1^0 A_1^{\mathrm{T}} \Sigma_1^{-1}\right) X_1\right]^{-1} X_1^{\mathrm{T}} \Sigma_1^{-1} A_1 \bar{\Sigma}_1^0 \\ &= \bar{\Sigma}_1^0 + K_1 X_1 \left[Q_{bb}^0 + X_1^{\mathrm{T}} \Sigma_1^{-1} (I - A_1 K_1) X_1\right]^{-1} X_1^{\mathrm{T}} K_1^{\mathrm{T}}, \end{aligned} \tag{2.9a}$$

$$\Sigma_{bx_1}^0 = -\bar{\Sigma}_1^0 A_1^{\mathrm{T}} \Sigma_1^{-1} X_1 \left[Q_{bb}^0 + X_1^{\mathrm{T}} \left(\Sigma_1^{-1} - \Sigma_1^{-1} A_1 \bar{\Sigma}_1^0 A_1^{\mathrm{T}} \Sigma_1^{-1}\right) X_1\right]^{-1} = -K_1 X_1 \left[Q_{bb}^0 + X_1^{\mathrm{T}} \Sigma_1^{-1} (I - A_1 K_1) X_1\right]^{-1} = \left(\Sigma_{bx_1}^0\right)^{\mathrm{T}}, \tag{2.9b}$$

$$\Sigma_{bb_1}^0 = \left[Q_{bb}^0 + X_1^{\mathrm{T}} \Sigma_1^{-1} (I - A_1 K_1) X_1\right]^{-1}. \tag{2.9c}$$

Note that the formulae (2.9a–c) become identical to eqs (2.7a–c) after some manipulations, using the relation

$$\Sigma_1^{-1}(I - A_1 K_1) = \left[\Sigma_1 + A_1\left(\Theta_1 + \phi_0 \Sigma_{xx_0}^0 \phi_0^{\mathrm{T}}\right) A_1^{\mathrm{T}}\right]^{-1}. \tag{2.10}$$

After exploiting formula (1.7b), we further obtain

$$K_{x_1} = \left(\Sigma^0_{xx_1} A_1^{\mathrm{T}} + \Sigma^0_{xb_1} X_1^{\mathrm{T}}\right) \Sigma_1^{-1}, \tag{2.11a}$$

$$K_{b_1} = \left(\Sigma^0_{bx_1} A_1^{\mathrm{T}} + \Sigma^0_{bb_1} X_1^{\mathrm{T}}\right) \Sigma_1^{-1}, \tag{2.11b}$$

which can be used in eqs (2.3a–b) in order to obtain the solutions $\tilde{x}_1$ and $\tilde{b}^{(1)}$.

All the above formulae remain valid even if $Q^0_{bb} \to 0$, assuming full column rank of X_1. We would then use

$$K_{x_1} = \left(\Sigma^0_{xx_1} A_1^{\mathrm{T}} + \Sigma^0_{x\beta_1} X_1^{\mathrm{T}}\right) \Sigma_1^{-1}, \tag{2.12a}$$

$$K_{\beta_1} = \left(\Sigma^0_{\beta x_1} A_1^{\mathrm{T}} + \Sigma^0_{\beta\beta_1} X_1^{\mathrm{T}}\right) \Sigma_1^{-1}, \tag{2.12b}$$

followed by (2.4a–b) to obtain $\tilde{x}_1$ and $\hat{\beta}^{(1)}$. Should X_1 not be of full rank, we may use a generalized inverse $[X_1^{\mathrm{T}} \Sigma_1^{-1}(I - A_1 K_1) X_1]^-$, e.g. the 'pseudo-inverse', instead. The matrices $\Sigma^0_{\beta\beta_1}$, $\Sigma^0_{x\beta_1} = (\Sigma^0_{\beta x_1})^{\mathrm{T}}$ and K_{β_1} will no longer be unique, and also $\hat{\beta}^{(1)}$ will depend on the chosen generalized-inverse that, however, will not affect $\tilde{x}_1$ and $\Sigma^0_{xx_1}$ owing to the invariance of $X_1 \Sigma^0_{\beta\beta_1} X_1^{\mathrm{T}}$, $X_1 \Sigma^0_{\beta x_1}$, and thus $X_1 K_{\beta_1}$.

For the next interval from t_1 to t_2, the formulae will obviously become more involved since we cannot assume that the off-diagonal block matrix $\Sigma^0_{x\beta_1} = (\Sigma^0_{bx_1})^{\mathrm{T}}$ still vanishes. Nevertheless, the above remarks are equally valid in suggesting the use of the '*Bayes filter*' eqs (1.7a–d) in order to come up with formulae in Friedland form that hold true even for the case of non-random perturbations when $Q^0_{bb} \to 0$. We shall, therefore, show the relevant derivations for this interval before designing the general algorithm.

Starting with eq. (1.7a), we now have to find the individual subblocks of the matrix

$$\begin{bmatrix} \Sigma^0_{xx_2} & \Sigma^0_{xb_2} \\ \Sigma^0_{bx_2} & \Sigma^0_{bb_2} \end{bmatrix} = \left(\begin{bmatrix} A_2^{\mathrm{T}} \\ X_2^{\mathrm{T}} \end{bmatrix} \Sigma_2^{-1} [A_2 \quad X_2] + \begin{bmatrix} \Theta_2 + \phi_1 \Sigma^0_{xx_1} \phi_1^{\mathrm{T}} & \vdots & \phi_1 \Sigma^0_{xb_1} \\ \Sigma^0_{bx_1} \phi_1^{\mathrm{T}} & \vdots & \Sigma^0_{bb_1} \end{bmatrix}^{-1} \right)^{-1}. \tag{2.13}$$

By using the relation

$$\Sigma^0_{xx_1} - \Sigma^0_{xb_1} \left(\Sigma^0_{bb_1}\right)^{-1} \Sigma^0_{bx_1} = \bar{\Sigma}^0_1 = \left[A_1^{\mathrm{T}} \Sigma_1^{-1} A_1 + \left(\Theta_1 + \phi_0 \Sigma^0_{xx_0} \phi_0^{\mathrm{T}}\right)^{-1} \right]^{-1} \tag{2.14}$$

from eqs (2.9a–c) and (1.3a), we can express the inner inverse of eq. (2.13) as

$$\begin{aligned} \begin{bmatrix} \Theta_2 + \phi_1 \Sigma^0_{xx_1} \phi_1^{\mathrm{T}} & \vdots & \phi_1 \Sigma^0_{xb_1} \\ \Sigma^0_{bx_1} \phi_1^{\mathrm{T}} & \vdots & \Sigma^0_{bb_1} \end{bmatrix}^{-1} &= \begin{bmatrix} 0 & 0 \\ 0 & \left(\Sigma^0_{bb_1}\right)^{-1} \end{bmatrix} + \begin{bmatrix} -I \\ \left(\Sigma^0_{bb_1}\right)^{-1} \Sigma^0_{bx_1} \end{bmatrix} \left(\Theta_2 + \phi_1 \bar{\Sigma}^0_1 \phi_1^{\mathrm{T}}\right)^{-1} \left[-I \quad \vdots \quad \Sigma^0_{xb_1} \left(\Sigma^0_{bb_1}\right)^{-1} \right] \\ &= \begin{bmatrix} \left(\Theta_2 + \phi_1 \bar{\Sigma}^0_1 \phi_1^{\mathrm{T}}\right)^{-1} & \vdots & -\left(\Theta_2 + \phi_1 \bar{\Sigma}^0_1 \phi_1^{\mathrm{T}}\right)^{-1} K_1 X_1 \\ -X_1^{\mathrm{T}} K_1^{\mathrm{T}} \left(\Theta_2 + \phi_1 \bar{\Sigma}^0_1 \phi_1^{\mathrm{T}}\right)^{-1} & \vdots & \left(\Sigma^0_{bb_1}\right)^{-1} + X_1^{\mathrm{T}} K_1^{\mathrm{T}} \left(\Theta_2 + \phi_1 \bar{\Sigma}^0_1 \phi_1^{\mathrm{T}}\right)^{-1} K_1 X_1 \end{bmatrix}. \end{aligned} \tag{2.15}$$

Alternatively, we can directly tackle the outer inverse and find

$$\begin{aligned} \begin{bmatrix} \Sigma^0_{xx_2} & \Sigma^0_{xb_2} \\ \Sigma^0_{bx_2} & \Sigma^0_{bb_2} \end{bmatrix} &= \begin{bmatrix} \Theta_2 + \phi_1 \Sigma^0_{xx_1} \phi_1^{\mathrm{T}} & \vdots & \phi_1 \Sigma^0_{xb_1} \\ \Sigma^0_{bx_1} \phi_1^{\mathrm{T}} & \vdots & \Sigma^0_{bb_1} \end{bmatrix} - \begin{bmatrix} \Theta_2 + \phi_1 \Sigma^0_{xx_1} \phi_1^{\mathrm{T}} & \vdots & \phi_1 \Sigma^0_{xb_1} \\ \Sigma^0_{bx_1} \phi_1^{\mathrm{T}} & \vdots & \Sigma^0_{bb_1} \end{bmatrix} \begin{bmatrix} A_2^{\mathrm{T}} \\ X_2^{\mathrm{T}} \end{bmatrix} \cdot \\ &\left(\Sigma_2 + [A_2 \quad X_2] \begin{bmatrix} \Theta_2 + \phi_1 \Sigma^0_{xx_1} \phi_1^{\mathrm{T}} & \vdots & \phi_1 \Sigma^0_{xb_1} \\ \Sigma^0_{bx_1} \phi_1^{\mathrm{T}} & \vdots & \Sigma^0_{bb_1} \end{bmatrix} \begin{bmatrix} A_2^{\mathrm{T}} \\ X_2^{\mathrm{T}} \end{bmatrix} \right)^{-1} \cdot [A_2 \quad X_2] \begin{bmatrix} \Theta_2 + \phi_1 \Sigma^0_{xx_1} \phi_1^{\mathrm{T}} & \vdots & \phi_1 \Sigma^0_{xb_1} \\ \Sigma^0_{bx_1} \phi_1^{\mathrm{T}} & \vdots & \Sigma^0_{bb_1} \end{bmatrix} \end{aligned} \tag{2.16}$$

with

$$\begin{aligned} \Sigma_2 + [A_2 \quad X_2] &\begin{bmatrix} \Theta_2 + \phi_1 \Sigma^0_{xx_1} \phi_1^{\mathrm{T}} & \vdots & \phi_1 \Sigma^0_{xb_1} \\ \Sigma^0_{bx_1} \phi_1^{\mathrm{T}} & \vdots & \Sigma^0_{bb_1} \end{bmatrix} \begin{bmatrix} A_2^{\mathrm{T}} \\ X_2^{\mathrm{T}} \end{bmatrix} \\ &= \Sigma_2 + [A_2 \quad X_2] \cdot \left(\begin{bmatrix} \Theta_2 + \phi_1 \bar{\Sigma}^0_1 \phi_1^{\mathrm{T}} & \vdots & 0 \\ 0 & \vdots & 0 \end{bmatrix} + \begin{bmatrix} \phi_1 K_1 X_1 \\ -I \end{bmatrix} \Sigma^0_{bb_1} \left[X_1^{\mathrm{T}} K_1^{\mathrm{T}} \phi_1^{\mathrm{T}} \quad \vdots \quad -I \right] \right) \begin{bmatrix} A_2^{\mathrm{T}} \\ X_2^{\mathrm{T}} \end{bmatrix} \\ &= \left[\Sigma_2 + A_2 \left(\Theta_2 + \phi_1 \bar{\Sigma}^0_1 \phi_1^{\mathrm{T}}\right) A_2^{\mathrm{T}} \right] + (X_2 - A_2 \phi_1 K_1 X_1) \Sigma^0_{bb_1} (X_2 - A_2 \phi_1 K_1 X_1)^{\mathrm{T}}. \end{aligned} \tag{2.17}$$

The respective subblocks are now readily obtained as

$$\begin{aligned} \Sigma^0_{bb_2} &= \Sigma^0_{bb_1} - \Sigma^0_{bb_1} (X_2 - A_2 \phi_1 K_1 X_1)^{\mathrm{T}} \cdot \left\{ \left[\Sigma_2 + A_2 \left(\Theta_2 + \phi_1 \bar{\Sigma}^0_1 \phi_1^{\mathrm{T}}\right) A_2^{\mathrm{T}} \right] + (X_2 - A_2 \phi_1 K_1 X_1) \Sigma^0_{bb_1} (X_2 - A_2 \phi_1 K_1 X_1)^{\mathrm{T}} \right\}^{-1} \\ &\quad \cdot (X_2 - A_2 \phi_1 K_1 X_1) \Sigma^0_{bb_1} \\ &= \left\{ \left(\Sigma^0_{bb_1}\right)^{-1} + (X_2 - A_2 \phi_1 K_1 X_1)^{\mathrm{T}} \left[\Sigma_2 + A_2 \left(\Theta_2 + \phi_1 \bar{\Sigma}^0_1 \phi_1^{\mathrm{T}}\right) A_2^{\mathrm{T}} \right]^{-1} (X_2 - A_2 \phi_1 K_1 X_1) \right\}^{-1}, \end{aligned} \tag{2.18a}$$

$$\begin{aligned} \Sigma^0_{xb_2} &= -\phi_1 K_1 X_1 \Sigma^0_{bb_1} - \left[\left(\Theta_2 + \phi_1 \bar{\Sigma}^0_1 \phi_1^{\mathrm{T}}\right) A_2^{\mathrm{T}} - \phi_1 K_1 X_1 \Sigma^0_{bb_1} (X_2 - A_2 \phi_1 K_1 X_1)^{\mathrm{T}} \right] \\ &\quad \cdot \left\{ \left[\Sigma_2 + A_2 \left(\Theta_2 + \phi_1 \bar{\Sigma}^0_1 \phi_1^{\mathrm{T}}\right) A_2^{\mathrm{T}} \right] + (X_2 - A_2 \phi_1 K_1 X_1) \Sigma^0_{bb_1} (X_2 - A_2 \phi_1 K_1 X_1)^{\mathrm{T}} \right\}^{-1} \cdot (X_2 - A_2 \phi_1 K_1 X_1) \Sigma^0_{bb_1} \end{aligned}$$

$$
\begin{aligned}
&= -\phi_1 K_1 X_1 \cdot \left\{ \left(\Sigma^0_{bb_1}\right)^{-1} + (X_2 - A_2\phi_1 K_1 X_1)^{\mathrm{T}} \left[\Sigma_2 + A_2\left(\Theta_2 + \phi_1 \bar{\Sigma}^0_1 \phi_1^{\mathrm{T}}\right) A_2^{\mathrm{T}}\right]^{-1} (X_2 - A_2\phi_1 K_1 X_1) \right\}^{-1} \\
&\quad - \left(\Theta_2 + \phi_1 \bar{\Sigma}^0_1 \phi_1^{\mathrm{T}}\right) A_2^{\mathrm{T}} \left[\Sigma_2 + A_2\left(\Theta_2 + \phi_1 \bar{\Sigma}^0_1 \phi_1^{\mathrm{T}}\right) A_2^{\mathrm{T}}\right]^{-1} (X_2 - A_2\phi_1 K_1 X_1) \\
&\quad \cdot \left\{ \left(\Sigma^0_{bb_1}\right)^{-1} + (X_2 - A_2\phi_1 K_1 X_1)^{\mathrm{T}} \left[\Sigma_2 + A_2\left(\Theta_2 + \phi_1 \bar{\Sigma}^0_1 \phi_1^{\mathrm{T}}\right) A_2^{\mathrm{T}}\right]^{-1} (X_2 - A_2\phi_1 K_1 X_1) \right\}^{-1} \\
&= -[\phi_1 K_1 X_1 + K_2(X_2 - A_2\phi_1 K_1 X_1)] \cdot \Sigma^0_{bb_2} = \left(\Sigma^0_{bx_2}\right)^{\mathrm{T}},
\end{aligned} \tag{2.18b}
$$

with

$$
K_2 := \left(\Theta_2 + \phi_1 \bar{\Sigma}^0_1 \phi_1^{\mathrm{T}}\right) A_2^{\mathrm{T}} \left[\Sigma_2 + A_2\left(\Theta_2 + \phi_1 \bar{\Sigma}^0_1 \phi_1^{\mathrm{T}}\right) A_2^{\mathrm{T}}\right]^{-1} = \bar{\Sigma}^0_2 A_2^{\mathrm{T}} \Sigma_2^{-1}, \tag{2.19a}
$$

$$
I - A_2 K_2 = \Sigma_2 \left[\Sigma_2 + A_2\left(\Theta_2 + \phi_1 \bar{\Sigma}^0_1 \phi_1^{\mathrm{T}}\right) A_2^{\mathrm{T}}\right]^{-1}, \tag{2.19b}
$$

and, after inverting both sides of eq. (2.13) and inserting eq. (2.15),

$$
\bar{\Sigma}^0_2 = \left[A_2^{\mathrm{T}} \Sigma_2^{-1} A_2 + \left(\Theta_2 + \phi_1 \bar{\Sigma}^0_1 \phi_1^{\mathrm{T}}\right)^{-1}\right]^{-1} = \Sigma^0_{xx_2} - \Sigma^0_{xb_2}\left(\Sigma^0_{bb_2}\right)^{-1} \Sigma^0_{bx_2} \tag{2.19c}
$$

so that we finally obtain

$$
\Sigma^0_{xx_2} = \bar{\Sigma}^0_2 + \Sigma^0_{xb_2}\left(\Sigma^0_{bb_2}\right)^{-1} \Sigma^0_{bx_2} = \bar{\Sigma}^0_2 - \Sigma^0_{xb_2}[\phi_1 K_1 X_1 + K_2(X_2 - A_2\phi_1 K_1 X_1)]^{\mathrm{T}}. \tag{2.18c}
$$

In the next step, we apply eq. (1.7b) to obtain

$$
K_{b_2} = \left(\Sigma^0_{bx_2} A_2^{\mathrm{T}} + \Sigma^0_{bb_2} X_2^{\mathrm{T}}\right)\Sigma_2^{-1} = \Sigma^0_{bb_2} \cdot \{X_2 - A_2[\phi_1 K_1 X_1 + K_2(X_2 - A_2\phi_1 K_1 X_1)]\}^{\mathrm{T}} \cdot \Sigma_2^{-1}, \tag{2.20a}
$$

$$
\begin{aligned}
K_{x_2} &= \left(\Sigma^0_{xx_2} A_2^{\mathrm{T}} + \Sigma^0_{xb_2} X_2^{\mathrm{T}}\right)\Sigma_2^{-1} = \bar{\Sigma}^0_2 A_2^{\mathrm{T}} \Sigma_2^{-1} + \Sigma^0_{xb_2}\{X_2 - A_2[\phi_1 K_1 X_1 + K_2(X_2 - A_2\phi_1 K_1 X_1)]\}^{\mathrm{T}} \cdot \Sigma_2^{-1} \\
&= K_2 - [\phi_1 K_1 X_1 + K_2(X_2 - A_2\phi_1 K_1 X_1)] \cdot K_{b_2}.
\end{aligned} \tag{2.20b}
$$

With $\breve{x}_2 := \phi_1 \tilde{x}_1$ and formulae (1.7c–d), we finally arrive at

$$
\tilde{b}^{(2)} = \tilde{b}^{(1)} + K_{b_2}\left(y_2 - A_2\breve{x}_2 - X_2\tilde{b}^{(1)}\right) = K_{b_2}(y_2 - A_2\phi_1\bar{x}_1) + \left[I - K_{b_2}(X_2 - A_2\phi_1 K_1 X_1)\right]\tilde{b}^{(1)}, \tag{2.21a}
$$

$$
\begin{aligned}
\tilde{x}_2 &= \breve{x}_2 + K_{x_2}\left(y_2 - A_2\breve{x}_2 - X_2\tilde{b}^{(1)}\right) = \breve{x}_2 + K_2(y_2 - A_2\breve{x}_2) - K_2 X_2 \tilde{b}^{(1)} - [\phi_1 K_1 X_1 + K_2(X_2 - A_2\phi_1 K_1 X_1)]\left(\tilde{b}^{(2)} - \tilde{b}^{(1)}\right) \\
&= \phi_1\left(\tilde{x}_1 + K_1 X_1 \tilde{b}^{(1)}\right) + K_2\left[y_2 - A_2\phi_1\left(\tilde{x}_1 + K_1 X_1 \tilde{b}^{(1)}\right)\right] - [K_2 X_2 + (I - K_2 A_2)\phi_1 K_1 X_1]\,\tilde{b}^{(2)} \\
&= [\phi_1 \bar{x}_1 + K_2(y_2 - A_2\phi_1\bar{x}_1)] - [\phi_1 K_1 X_1 + K_2(X_2 - A_2\phi_1 K_1 X_1)]\,\tilde{b}^{(2)}.
\end{aligned} \tag{2.21b}
$$

The formulae (2.18)–(2.21) comprise the 'Bayes filter in Friedland form' which updates the results from $t = t_1$ to $t = t_2$ for the extended dynamic linear model eqs (1.4a–d). Applying similar symbols as Friedland (1969), we may thus design the general update from $t = t_{k-1}$ to $t = t_k$ that works even when some of the matrices involved become singular (as in our application):

$$
\begin{aligned}
&\text{(i) } \bar{\Sigma}^0_k := \left[A_k^{\mathrm{T}} \Sigma_k^{-1} A_k + \left(\Theta_k + \phi_{k-1} \bar{\Sigma}^0_{k-1} \phi_{k-1}^{\mathrm{T}}\right)^{-1}\right]^{-1}, \\
&\text{(ii) } K_k := \bar{\Sigma}^0_k A_k^{\mathrm{T}} \Sigma_k^{-1}, \\
&\text{(iii) } U_k := \phi_{k-1} V_{k-1}, \\
&\text{(iv) } S_k := X_k + A_k U_k, \\
&\text{(v) } V_k := U_k - K_k S_k, \\
&\text{(vi) } N_{kk} := S_k^{\mathrm{T}} \Sigma_k^{-1} (I - A_k K_k) S_k, \\
&\text{(vii) } \begin{cases} \Sigma^0_{bb_k} := \left[\left(\Sigma^0_{bb_{k-1}}\right)^{+} + N_{kk}\right]^{+} \quad \text{(using pseudo-inverses)}, \\ \Sigma^0_{xb_k} := V_k \Sigma^0_{bb_k}, \quad \Sigma^0_{xx_k} := \bar{\Sigma}^0_k + \Sigma^0_{xb_k} V_k^{\mathrm{T}}, \end{cases} \\
&\text{(viii) } K_{b_k} := \Sigma^0_{bb_k}(A_k V_k + X_k)^{\mathrm{T}} \Sigma_k^{-1}, \\
&\text{(ix) } \bar{x}_k := \phi_{k-1}\bar{x}_{k-1} + K_k(y_k - A_k\phi_{k-1}\bar{x}_{k-1}), \\
&\text{(x) } \begin{cases} \tilde{b}^{(k)} := K_{b_k}(y_k - A_k\phi_{k-1}\bar{x}_{k-1}) + \left(I - K_{b_k} S_k\right)\tilde{b}^{(k-1)}, \\ \tilde{x}_k := \bar{x}_k + V_k \tilde{b}^{(k)}. \end{cases}
\end{aligned} \tag{2.22}
$$

With increasing numbers of iteration, the singularity of $\Sigma^0_{bb_k}$ should normally disappear even if the participating matrices $X_1, \ldots, X_k$ have all less than full column rank. The uniqueness of both $\tilde{x}_k$ and $\Sigma^0_{xx_k}$ will not be affected anyway.

Furthermore, the algorithm (2.22) is readily applicable to the case of non-random perturbations as in the application that will be presented in the following section. In this case, we should only change some of the notations, namely $\Sigma^0_{bb_k} \to \Sigma^0_{\beta\beta_k}$, $\Sigma^0_{xb_k} \to \Sigma^0_{x\beta_k}$, $K_{b_k} \to K_{\beta_k}$, $\tilde{b}^{(k)} \to \hat{\beta}^{(k)}$ to indicate that we started with $(\Sigma^0_{bb})^{-1} = Q^0_{bb} \to 0$.

3 AIRBORNE VECTOR GRAVIMETRY WITH AN INTEGRATED INS/GPS SYSTEM

In this section, the application of the Bayes filter derived in the previous section to the INS/GPS airborne vector gravimetry is described. For this purpose, the observation equation of the INS/GPS integrated system of the airborne gravimetry is derived first. Then, the problem in the derived observation equation, namely, modelling of the gravity is addressed. Finally, how the problem can be avoided using the derived algorithm will be explained.

The fundamental equation in airborne gravimetry, based on Newton's law of motion under the existence of the gravity field, is given in a non-rotating, freely falling coordinate frame (i-frame):

$$\ddot{\mathbf{x}}^i = \mathbf{a}^i + \mathbf{g}^i, \tag{3.1}$$

where the superscript i refers to the i-frame; $\ddot{\mathbf{x}}^i$ is the second time derivative of position, namely the kinematic acceleration; $\mathbf{a}^i$ is the acceleration due to an applied force, also known as the specific force; and $\mathbf{g}^i$ is the gravitation. In INS/GPS vector gravimetry, the kinematic acceleration $\ddot{\mathbf{x}}^i$ can be derived from GPS 3-D positions, and the specific force $\mathbf{a}^i$ can be measured by a triad of single-axis accelerometers.

Denoting the observed quantities with a tilde and the errors in the observations with δ, the fundamental equation can be expressed in terms of observations:

$$\tilde{\ddot{\mathbf{x}}}^i - \delta\ddot{\mathbf{x}}^i = \tilde{\mathbf{a}}^i - \delta\mathbf{a}^i + \mathbf{g}^i, \tag{3.2}$$

where $\tilde{\ddot{\mathbf{x}}}^i$ is the observed acceleration derived from GPS; $\tilde{\mathbf{a}}^i$ is the specific force obtained by the INS; and $\delta\ddot{\mathbf{x}}^i$, $\delta\mathbf{a}^i$ are the respective errors in GPS and INS observed accelerations. Since the accelerometer measurements refer to the body frame (b-frame), it is necessary to express the accelerometer error $\delta\mathbf{a}^i$ in terms of the sensor errors in the body frame and the orientation error according to the Coriolis law:

$$\delta\mathbf{a}^i = C_b^i\delta\mathbf{a}^b + \tilde{\mathbf{a}}^i \times \psi^i, \tag{3.3}$$

where the superscript b indicates the body (vehicle) frame, C_b^i is the transformation matrix from the body to the inertial frame, and ψ^i is the orientation error of the body in the inertial frame. Note that error eq. (3.3) is a linear approximation that neglects higher-order terms in $\delta\mathbf{a}^i$ and ψ^i.

For inertial measurement unit (IMU) errors, only some essential parameters such as biases, scale factor errors, and random noises are considered. GPS-observed accelerations are assumed to have no systematic errors. Although this is not strictly true owing to increased noise at high frequencies arising from the numerical differentiation, we assume that sufficient smoothing has already been applied to reduce the high-frequency errors. The models for the IMU errors are given as:

$$\delta\mathbf{a}^b = \mathbf{b}_\mathrm{a} + \mathrm{diag}\,(\mathbf{a}^b)\boldsymbol{\kappa}_\mathrm{a} + \boldsymbol{\varepsilon}_\mathrm{a}, \tag{3.4}$$

$$\delta\boldsymbol{\omega}_{ib}^b = \mathbf{b}_\mathrm{g} + \mathrm{diag}\left(\boldsymbol{\omega}_{ib}^b\right)\boldsymbol{\kappa}_\mathrm{g} + \boldsymbol{\varepsilon}_\mathrm{g}, \tag{3.5}$$

$$\delta\ddot{\mathbf{x}}^i = \boldsymbol{\varepsilon}_\mathrm{G}, \tag{3.6}$$

where $\boldsymbol{\varepsilon}_\mathrm{g} \sim N(\mathbf{0}, D_\mathrm{g})$, $\boldsymbol{\varepsilon}_\mathrm{a} \sim N(\mathbf{0}, D_\mathrm{a})$, $\boldsymbol{\varepsilon}_\mathrm{G} \sim N(\mathbf{0}, D_\mathrm{G})$ are zero-mean, Gaussian, white-noise processes with indicated dispersion matrices for gyro (D_g), accelerometer (D_a) and GPS observations (D_G). In eq. (3.5), $\boldsymbol{\omega}_{ib}^b$ is the angular rate of the body frame with respect to the i-frame, expressed in the b-frame. The dynamics for the biases, $\mathbf{b}_\mathrm{a}$, $\mathbf{b}_\mathrm{g}$ and scale factors $\boldsymbol{\kappa}_\mathrm{a}$, $\boldsymbol{\kappa}_\mathrm{g}$ are modelled as random constants over time:

$$\dot{\mathbf{b}}_\mathrm{a} = \mathbf{0}, \quad \dot{\boldsymbol{\kappa}}_\mathrm{a} = \mathbf{0}, \quad \dot{\mathbf{b}}_\mathrm{g} = \mathbf{0}, \quad \dot{\boldsymbol{\kappa}}_\mathrm{g} = \mathbf{0}. \tag{3.7}$$

With eq. (3.5), the dynamics of the orientation error are given by

$$\begin{aligned}\dot{\psi}^i &= -C_b^i\delta\boldsymbol{\omega}_{ib}^b \\ &= -C_b^i\mathbf{b}_\mathrm{g} - C_b^i\,\mathrm{diag}\left(\boldsymbol{\omega}_{ib}^b\right)\boldsymbol{\kappa}_\mathrm{g} - C_b^i\boldsymbol{\varepsilon}_\mathrm{g}.\end{aligned} \tag{3.8}$$

Combining eqs (3.7) and (3.8), the system model expressed by a set of linear, first-order, differential equations in terms of the INS system error parameters as well as orientation errors is obtained as:

$$\begin{bmatrix}\dot{\mathbf{b}}_\mathrm{a}\\ \dot{\mathbf{b}}_\mathrm{g}\\ \dot{\boldsymbol{\kappa}}_\mathrm{a}\\ \dot{\boldsymbol{\kappa}}_\mathrm{g}\\ \dot{\psi}^i\end{bmatrix} = \begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&-C_b^i&0&-C_b^i\,\mathrm{diag}\left(\boldsymbol{\omega}_{ib}^b\right)&0\end{bmatrix}\begin{bmatrix}\mathbf{b}_\mathrm{a}\\ \mathbf{b}_\mathrm{g}\\ \boldsymbol{\kappa}_\mathrm{a}\\ \boldsymbol{\kappa}_\mathrm{g}\\ \psi^i\end{bmatrix} + \begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&-C_b^i\end{bmatrix}\begin{bmatrix}0\\0\\0\\0\\ \boldsymbol{\varepsilon}_\mathrm{g}\end{bmatrix} = F\mathbf{x} + G\mathbf{w}, \tag{3.9}$$

which implicitly defines the state vector, $\mathbf{x}$, the noise vector, $\mathbf{w}$, and corresponding coefficient matrices, F and G.

The external observations are a combination of kinematic acceleration calculated from GPS positions and normal gravity. The corresponding update to the specific force is, therefore, given by

$$\mathbf{y} = \tilde{\ddot{\mathbf{x}}}^i - \tilde{\mathbf{a}}^i - \bar{\gamma}^i, \tag{3.10}$$

where $\bar{\gamma}^i$ denotes normal gravitation in the i-frame. From eqs (3.2)–(3.4), we also have

$$\mathbf{y} = -C_b^i \mathbf{b}_\mathrm{a} - C_b^i \,\mathrm{diag}(\tilde{\mathbf{a}}^b)\boldsymbol{\kappa}_\mathrm{a} - \tilde{\mathbf{a}}^i \times \psi^i + \delta\mathbf{g}^i - C_b^i \varepsilon_\mathrm{a} + \delta\ddot{\mathbf{x}}^i, \tag{3.11}$$

where

$$\delta\mathbf{g}^i = \mathbf{g}^i - \bar{\gamma}^i \tag{3.12}$$

is the gravity disturbance vector. Note that the first equation, (3.10), consists of actually observed (calculated) and sensed quantities (GPS and INS accelerations), while the second equation, (3.11), is a model of this update in terms of the error parameters of the system.

As seen in eq. (3.9), the gravity disturbance vector $\delta\mathbf{g}^i$ is not included as a parameter in the system because of its non-stochastic behaviour. To take care of this, Jekeli & Kwon (1999) intentionally excluded the gravity disturbance vector from the observation model and declared the residuals from the Kalman filter as gravity disturbance vector estimates. Although they obtained a good result, the approach still shows a lack of the theoretical completeness (no model for the gravity field). For this matter, an *ad hoc* deterministic model for the gravity disturbance vector is adopted in this study. It is the nth-order expansion of trigonometric functions, and no *a priori* information is assumed. In other words, the jth component of the gravity disturbance vector is modelled as

$$\begin{aligned}\delta\mathbf{g}_j \approx \sum_{k=0}^{n} a_{jk} \cos\frac{2\pi kt}{T} + \sum_{k=0}^{n} b_{jk} \sin\frac{2\pi kt}{T} = a_{j0} + a_{j1}\cos\frac{2\pi t}{T} + a_{j2}\cos\frac{2\pi\cdot 2t}{T} + \cdots + a_{jn}\cos\frac{2\pi\cdot nt}{T} + b_{j1}\sin\frac{2\pi t}{T} \\ + b_{j2}\sin\frac{2\pi\cdot 2t}{T} + \cdots + b_{jn}\sin\frac{2\pi\cdot nt}{T} = X_j \cdot \boldsymbol{\beta}_j,\end{aligned} \tag{3.13}$$

where T is the period of the trigonometric functions, k is the wavenumber, and a_k and b_k are constant coefficients representing the amplitude of the corresponding components of wavenumber k. The matrix X_j consists of trigonometric functions dependent on time, t. The vector $\boldsymbol{\beta}_j$ consists of the coefficients of the trigonometric functions. With all three components combined, we have

$$\delta\mathbf{g} = \mathrm{X}\cdot\boldsymbol{\beta}, \tag{3.14}$$

where

$$\underset{3\times 3(2n+1)}{\mathbf{X}} = \begin{bmatrix} \mathrm{X}_1 & 0 & 0 \\ 0 & \mathrm{X}_2 & 0 \\ 0 & 0 & \mathrm{X}_3 \end{bmatrix},$$

$$\underset{1\times(2n+1)}{X_1} = X_2 = X_3 = \begin{bmatrix} 1 & \cos\frac{2\pi t}{T} & \cos\frac{2\pi t\cdot 2}{T} & \cdots & \sin\frac{2\pi t}{T} & \sin\frac{2\pi t\cdot 2}{T} & \cdots & \sin\frac{2\pi t\cdot n}{T} \end{bmatrix},$$

$$\underset{1\times(6n+3)}{\boldsymbol{\beta}} = \begin{bmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \\ \boldsymbol{\beta}_3 \end{bmatrix}; \quad \boldsymbol{\beta}_1 = \begin{bmatrix} a_{10} \\ a_{11} \\ a_{12} \\ \vdots \\ a_{1n} \\ b_{11} \\ \vdots \\ b_{1n} \end{bmatrix}, \quad \boldsymbol{\beta}_2 = \begin{bmatrix} a_{20} \\ a_{21} \\ a_{22} \\ \vdots \\ a_{2n} \\ b_{21} \\ \vdots \\ b_{2n} \end{bmatrix}, \quad \boldsymbol{\beta}_3 = \begin{bmatrix} a_{30} \\ a_{31} \\ a_{32} \\ \vdots \\ a_{3n} \\ b_{31} \\ \vdots \\ b_{3n} \end{bmatrix}. \tag{3.15}$$

Combining eq. (3.11) with eqs (3.14) and (3.15), one can finally obtain the observation equations that will enter the Bayes filtering algorithm developed in the previous section:

$$\mathbf{y} = A\mathbf{x} + X\boldsymbol{\beta} + e, \quad e := -C_b^i \varepsilon_\mathrm{a} + \delta\ddot{\mathbf{x}}^i, \tag{3.16}$$

where the design matrix is $A_{3\times 15} := [-C_b^i \;\; 0 \;\; -\mathrm{diag}(\tilde{\mathbf{a}}^i)C_b^i \;\; 0 \;\; [\tilde{\mathbf{a}}\times]]$, and the last element $[\tilde{\mathbf{a}}\times]$ is a skew-symmetric matrix with elements arranged to emulate the cross-product.

With the initial information on the parameters that can usually be obtained from manufacturer's specification, with the system dynamics equation (3.9) and the observation equation (3.16), one can conduct the gravity estimation using the above dual Kalman, respectively, Bayes filtering algorithm. Clearly, one has to decide the maximum order n of the trigonometric expansion. Higher orders would generate more detail in the gravity disturbance signal, but would require much more calculation time.

4 NUMERICAL TEST RESULTS

In this section, the results from the developed dual Kalman (respectively, Bayes) filtering applied to real flight data are presented. The flight data are provided by the University of Calgary collected on 1995 June 1 and made available by the Special Study Group 3.164 of the International

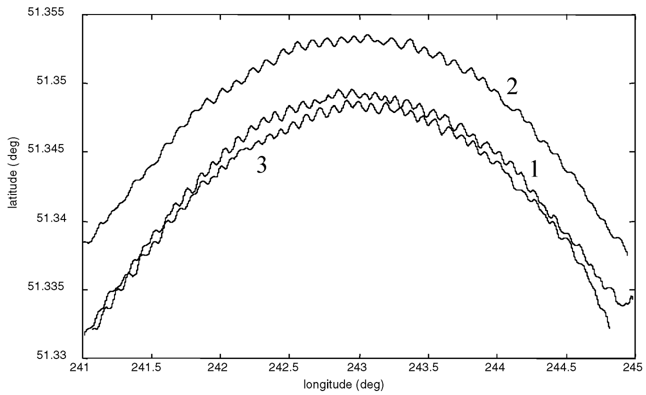

Figure 1. Three flight trajectories for the test data of 1995 June.

Association of Geodesy. They include the coordinates of an airborne GPS antenna at 0.5 s intervals, and raw accelerometer and gyro data at a data rate of 50 Hz. The total length of the profile was 250 km and the flying altitude was around 5.5 km above mean sea level. Average flying speed was about 430 km h^{-1}, so the corresponding spectral resolution for 60 s smoothing is about 3.5 km. There are three almost overlapping tracks available in this test flight (Fig. 1). This provides a useful method of internal consistency checking. For details on the data description, see Wei & Schwarz (1998).

To analyse the differences between the new algorithm developed and the 'conventional algorithm', the gravity disturbance estimates from these two algorithms are compared. In the (conventional) Kalman filter, the gravity disturbance is treated as a stochastic process and the amplitudes of the trigonometric functions are included as part of the state vector **x**. The initial means and variances for the amplitudes are set to zeros and 1 m^2 s^{-4}, respectively. All other state parameters related to the system errors are set identically in both filters. Note that the term 'conventional' is used here in order to separate our new filter algorithm from regular Kalman filtering. It should not indicate that Kalman filtering is the only (or even preferred) way to analyse INS/GPS gravity data.

Figs 2–4 show the gravity disturbance estimates for all three lines from those two filters. Obviously, the overall trends appear to be more or less the same in both cases. This is plausible because the new algorithm corresponds to the conventional Kalman filter with infinite *a priori* variances for the amplitude of gravity disturbances and the assigned *a priori* values of 1 m^2 s^{-4} are relatively large with respect to the magnitude of gravity disturbances. The main differences appear to be in the long-wavelength signal as shown in the horizontal components of all three lines. At the beginning of line 3 (longitude greater than 244° in Fig. 4), however, anomalies in the medium frequencies appear.

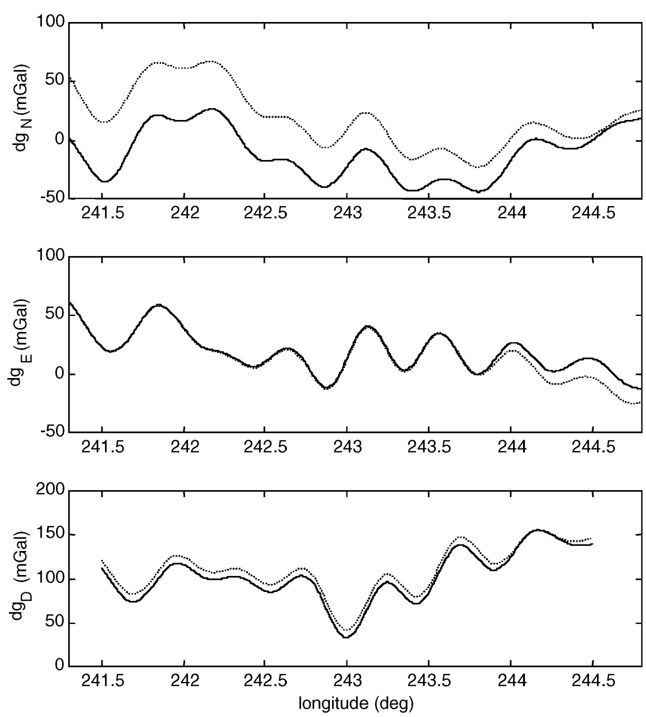

Figure 2. Estimated gravity disturbance vector for line 1; north (top), east (middle), down (bottom). The dashed line is from the conventional Kalman filtering and the solid line is from the new dual Kalman, respectively, Bayes filtering.

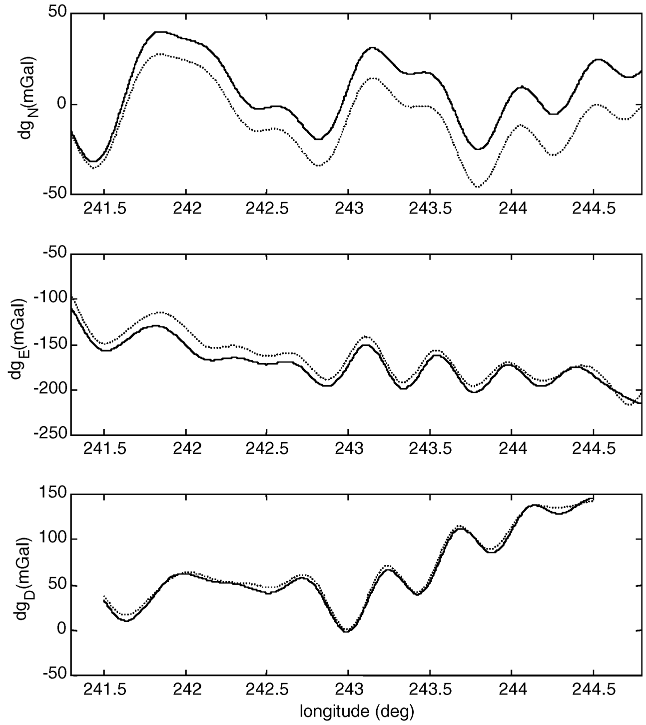

Figure 3. Estimated gravity disturbance vector for line 2; north (top), east (middle), down (bottom). The dashed line is from the conventional Kalman filtering and the solid line is from the new dual Kalman, respectively, Bayes filtering.

Apparently, this is considered as being the smeared effect of the turn of the vehicle, but detailed analysis is still under investigation. The standard deviations between two estimates range from ± 2 to ± 16 mGal (Table 1).

To show the relative superiority of the developed algorithm, the estimates are compared with control data provided by NIMA (National Imagery and Mapping Agency) for horizontal and by the University of Calgary for vertical components. Tables 2 and 3 show the standard deviations with respect to the control data from the conventional and the new filter, respectively. Overall, significant improvements of

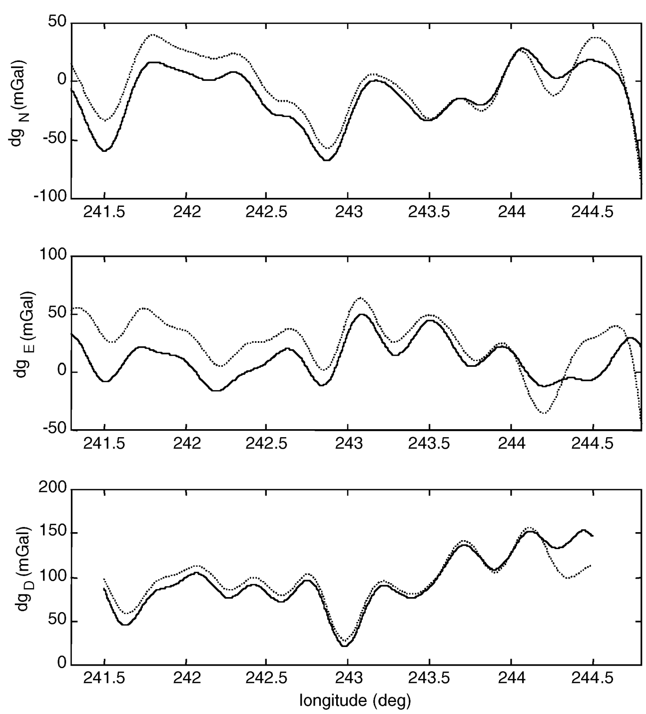

Figure 4. Estimated gravity disturbance vector for line 3; north (top), east (middle), down (bottom). The dashed line is from the conventional Kalman filtering and the solid line is from the new dual Kalman, respectively, Bayes filtering.

Table 1. Standard deviations of the differences between the gravity disturbances from the new and conventional Kalman filter (mGal).

	Line 1	Line 2	Line 3
North	±14.2	±5.7	±10.5
East	±5.2	±4.9	±15.8
Down	±2.6	±2.2	±13.0

Table 2. Standard deviations of the gravity disturbances from the conventional Kalman filter with respect to the control data (mGal).

	Line 1	Line 2	Line 3
North	±24.1	±12.5	±18.0
East	±16.6	±22.0	±21.7
Down	±12.6	±9.4	±16.4

Table 3. Standard deviations of the gravity disturbances from the new Bayes filter with respect to the control data (mGal).

	Line 1	Line 2	Line 3
North	±15.7	±10.3	±15.8
East	±13.1	±19.0	±17.1
Down	±11.3	±9.8	±6.2

10–60 per cent are achieved in the new filter except for the down component of line 2. Especially, the down component of line 3 (16.4 versus 6.2) and the north component of line 1 (24.1 versus 15.7) show a tremendous improvement. The former caused by improvements at the beginning of the filtering as mentioned previously, and the latter caused by the better long-wavelength component in the new filter.

The reason for better results in the new dual Kalman (respectively, Bayes) filter lies in the non-stochastic modelling of the gravity disturbance vector. In other words, the gravity disturbance has to be modelled as a stochastic process in the conventional Kalman filter. Then, prior information has to be assigned for the initial mean and variances of the gravity disturbances. This prior information could be quite misleading so that the filter estimates of the gravity disturbances end up in a wrong state space. As a matter of fact, the gravity field cannot simply be modelled stochastically. Without proper prior information, therefore, it is natural to model the gravity field in a non-stochastic way, which is done in this study.

It should be noted that the trigonometric expansion contains a singularity for the first ten epochs (see eq. 3.15). Therefore, one ought to apply the pseudo-inverses as in (2.22 vii) until the singularity has disappeared. As a matter of fact, one can develop other basis function for the gravity disturbances without such singularity. We chose, however, the trigonometric expansion for the gravity disturbance in this study because it is very simple and easy to implement.

5 CONCLUSIONS AND OUTLOOK

One of the motives for this study was to estimate gravity disturbances as non-random perturbations in the general structure of a dynamic linear model. Starting from a dual Kalman-type algorithm, an efficient Bayes filter in Friedland form for the estimation of these non-random perturbations has been developed and successfully applied to GPS/INS airborne vector gravimetry. The main advantage of the new algorithm consists in allowing us to retain the non-random characteristics of the gravity disturbances while keeping the system noise in the model. In addition, it is a one-step filter that does not create the correlation problem between the perturbation estimates and the system noise that is indicative for the two-step filters.

Our application results from the new algorithm showed 10–60 per cent improvement compared with those from the extended dynamic linear model that is based on a stochastic gravity field description. Therefore, using the new filter, we achieved theoretical soundness (non-stochastic modelling of the gravity field) as well as numerical improvements although the gravity field was only represented by a low-order trigonometric combination in this case.

While our emphasis here was on the development of the new filter algorithm, further improvements may be possible by using different basis functions, particularly if considering the size of the area and the roughness of the gravity field there. However, this is left for future investigations, such as studying the most suitable representations for a local or regional gravity field; spherical wavelets are certainly among those candidates. In the end, we may well achieve results that are similar in accuracy to those recently reported by Bruton (2000) in his dissertation.

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