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The method of self-determined probability weighted moments revisited

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Abstract

Haktanir originally introduced the method of self-determined probability weighted moments as an extension of the traditional method of probability weighted moments for parameter estimation. While this method possesses many advantages, his algorithms introduced certain mathematical manipulations for numerical convenience or based upon special knowledge of the behavior of a data sample. Also, some of these algorithms relied upon inputs from numerical tables that are not widely accessible. To improve the usefulness of this method, new algorithms have been developed that directly implement the relevant equations and do not rely upon external results. In this paper, we show that these features extend the applicability of self-determined probability weighted moments without loss of accuracy in the parameter estimates. Examples from flood peak analysis and extreme wind speed estimation are presented.

Keywords: Self-determined probability weighted moments; Parameter estimation; Numerical methods; Flood peak analysis; Extreme wind speed estimation

1. Introduction

The problem of estimating the parameters of a probability distribution from a sample is crucial to many fields of science and engineering, particularly for predicting future behavior of a phenomenon from previously observed behavior. A wide variety of methods have been developed to perform parameter estimation; see, e.g. Rao and Hamed, 2000 for a discussion of some commonly used methods. Despite the efforts of many researchers, there is an on-going need to create a method that is easily used, has the flexibility to accommodate many different distributions, and can produce accurate and robust parameter estimates from (frequently limited) sets of data.

The method of self-determined probability weighted moments (Haktanir, 1997) was originally introduced as an extension of the traditional method of probability weighted moments (Greenwood et al., 1979) for parameter estimation. One goal of the self-determined probability weighted moment (SD-PWM) method was to enhance the accuracy of the probability weighted moment (PWM) method by more fully utilizing the mathematical properties of the underlying probability distribution. (This could also provide an informal 'test' of the appropriateness of the assumed distribution in describing the data, as explained more fully in Section 2.) In addition, the SD-PWM method could more accurately account for

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variations in the data sample, thus improving the parameter estimation. Haktanir (1997) documented the improved performance of the SD-PWM method over the PWM method and the maximum likelihood method by estimating parameters from five distributions (generalized extreme-value (GEV), log-logistic (LL), three parameter lognormal (LN3), and Pearson type three (P3), and Gumbel) using annual flood data from a location in Turkey.

Because of the general complexity of the SD-PWM equations to be solved, Haktanir introduced numerical algorithms to determine the SD-PWM parameter estimates. While these algorithms are for the most part direct implementations of the SD-PWM equations, certain mathematical manipulations were introduced for numerical convenience or based upon special knowledge of the behavior of a data sample. Moreover, some of these algorithms relied upon multiple inputs from numerical tables that are published in journals that are not widely accessible. While not detracting from the accuracy of his results, all of these features make the algorithms harder to use and potentially susceptible to problems when applied to other types of data.

In an attempt to simplify and unify the enforcement of the definition of SD-PWMs across the various distributions, new SD-PWM algorithms have been developed for each distribution. These new algorithms are entirely self-contained and directly implement the theoretical equations for the SD-PWM method with no modifications. They rely upon powerful and efficient numerical techniques for integration and root finding implemented in the software package MATLAB® (1999), thus assuring accuracy and wide accessibility. These new algorithms have been tested both on Haktanir's original flood data and on extreme wind speed data, and their performance is assessed below. In general, the new algorithms perform as well as Haktanir's original ones, and they are shown to eliminate certain problems that were found in the Haktanir's implementation. Thus, these new algorithms should be more appropriate for general use in parameter estimation while not sacrificing the quality of Haktanir's approach.

The organization of the paper is as follows. In Section 2, we describe the theory of both probability weighted moments and self-determined probability weighted moments, highlighting the advantages that the latter method should have over the former. Section 3 gives implementation details for three of the distributions and points out differences with Haktanir's approach. In Section 4, results of testing of the new algorithms are reported. Finally, conclusions and future directions of the research are discussed in Section 5.

2. Theory

2.1. Probability weighted moments

For a given probability distribution, its probability weighted moments $M_{p,r,s}$ are defined as

$$M_{p,r,s} = E[X^p F^r (1-F)^s] = \int_0^1 [x(F)]^p F^r (1-F)^s \mathrm{d}F \tag{1}$$

where $F = F(x, \phi_1, \phi_2, \ldots, \phi_k) = P(X \le x)$ is the cumulative distribution function having $\phi_1, \phi_2, \ldots, \phi_k$ as parameters, $x(F)$ is the inverse cumulative distribution function, and p, r, and s are integers. Two particular sets of PWMs, α_s and β_r, are usually considered:

$$\alpha_s \equiv M_{1,0,s} = \int_0^1 x(F)(1-F)^s \mathrm{d}F = \int x(1-F)^s f(x)\mathrm{d}x \tag{2a}$$

and

$$\beta_r \equiv M_{1,r,0} = \int_0^1 x(F)F^r \mathrm{d}F = \int xF^r f(x)\mathrm{d}x \tag{2b}$$

where $f(x) = f(x, \phi_1, \phi_2, \ldots, \phi_k)$ is the probability density function and the upper and lower bounds on x in the second integrals are such that $F(x) = 1$ and $F(x) = 0$, respectively. In general, α_s and β_r are nonlinear functions of the distribution parameters $\phi_1, \phi_2, \ldots \phi_k$. It can be shown that the sets $\{\alpha_s : s = 0, 1, 2, \ldots, N\}$ and $\{\beta_r : r = 0, 1, 2, \ldots, N\}$ are linearly dependent, implying that either definition of PWM may be used for parameter estimation without loss of generality (Hosking, 1986).

To estimate the parameters of a distribution, PWM estimators of the ordered sample $x = \{x_1 \le x_2 \le \cdots \le x_{n-1} \le x_n\}$ are defined as follows, using

Haktanir's notation (1997):

$$\alpha'_s = \frac{1}{n}\sum_{i=1}^{n}(1 - P_{nex,i})^s x_i \quad \text{for } s = 0, 1, 2, \ldots, n-1 \tag{3a}$$

and

$$\beta'_r = \frac{1}{n}\sum_{i=1}^{n} P^r_{nex,i} x_i \quad \text{for } r = 0, 1, 2, \ldots, n-1 \tag{3b}$$

where $P_{nex,i}$ is an estimate for the non-exceedance probability of the ith event. Using the unbiased estimators suggested by Landwehr et al. (1979), $P_{nex,i}$ is taken as

$$P^j_{nex,i} = \frac{\binom{i-1}{j}}{\binom{n-1}{j}} \qquad j = 0, 1, 2, \ldots, n-1. \tag{4}$$

(Eq. (3a) requires that the polynomial in $P_{nex,i}$ be expanded prior to use of Eq. (4)). Various authors (e.g. Greenwood et al., 1979; Hosking, 1986) have suggested using a biased estimator employing a plotting position formula for $P_{nex,i}$, which sometimes yields better estimates for both distribution parameters and quantiles (Hosking, 1991). However, the unbiased PWM estimators obtained using Eq. (4) are employed here. The PWM parameter estimates $\phi'_1, \phi'_2, \ldots, \phi'_k$ are defined as those values that make the first k PWMs equal to the first k PWM estimators; i.e. $\phi'_1, \phi'_2, \ldots, \phi'_k$ are those values such that

$$\alpha'_s = \alpha_s(\phi'_1, \phi'_2, \ldots, \phi'_k) \quad \text{for } s = 0, 1, \ldots, k-1, \tag{5a}$$

or

$$\beta'_r = \beta_r(\phi'_1, \phi'_2, \ldots, \phi'_k) \quad \text{for } r = 0, 1, \ldots, k-1. \tag{5b}$$

Estimates that satisfy Eq. (5a) must also satisfy Eq. (5b), and vice versa (Savage, 2001).

2.2. Self-determined probability weighted moments

When estimating the parameters for a probability distribution, it is assumed that the observations in a sample X follow this distribution and thus that X displays some relevant behavior of the distribution. The relevant behavior of X used in the method of PWMs is simply that the PWM estimators obtained from X are equal to the PWMs of the given distribution; the assumed applicability of the probability distribution to the sample is not exploited any further. In particular, the non-exceedance probability $P_{nex,i}$ is not assigned to x_i according to the assumed distribution, but based solely on the position of x_i within the ordered sample X. The method of self-determined probability weighted moments attempts to improve estimation performance by using the assumed distribution more fully.

The method of SD-PWMs assumes that the non-exceedance probabilities of the observations can be assigned via the cumulative distribution function of the assumed distribution. Thus, the linearly related SD-PWM estimators α''_s and β''_r are

$$\alpha''_s = \frac{1}{n}\sum_{i=1}^{n}[1 - F(x_i, \phi_1, \phi_2, \ldots, \phi_k)]^s x_i \tag{6a}$$

for $s = 0, 1, 2, \ldots$

and

$$\beta''_r = \frac{1}{n}\sum_{i=1}^{n} F^r(x_i, \phi_1, \phi_2, \ldots, \phi_k) x_i \quad \text{for } r = 0, 1, 2, \ldots. \tag{6b}$$

Unlike the scalar PWM estimators, the SD-PWM estimators are nonlinear functions, in general, of the distribution parameters. For a given sample X and a given k-parameter probability distribution, the SD-PWM parameter estimates $\phi''_1, \phi''_2, \ldots, \phi''_k$ are defined as those values that the first k PWMs equal the first k SD-PWM estimators:

$$\alpha''_s(\phi''_1, \phi''_2, \ldots, \phi''_k) = \alpha_s(\phi''_1, \phi''_2, \ldots, \phi''_k) \tag{7a}$$

for $s = 0, 1, \ldots, k-1$,

or

$$\beta''_r(\phi''_1, \phi''_2, \ldots, \phi''_k) = \beta_r(\phi''_1, \phi''_2, \ldots, \phi''_k) \tag{7b}$$

for $r = 0, 1, \ldots, k-1$.

As with the PWM estimates from Eqs. (5a) and (5b), it can be shown that any set of SD-PWM estimates that satisfies Eq. (7a) also satisfies Eq. (7b), and vice versa (Savage, 2001).

Although the complexity of the resulting set of k equations to be solved increases when moving from

PWMs to SD-PWMs, Haktanir (1997) suggests two advantages that the assignment of non-exceedance probabilities directly from the cumulative distribution function has over the position-based assignment of traditional PWMs. First, because the method of PWMs assigns non-exceedance probabilities only according to rank order in the sample, the relative distance between two sample points does not affect these probabilities. Such a feature can be problematic if a sample contains one or more outliers, although methods do exist for circumventing this problem (e.g. using PWMs developed for censored samples or based on historical information). However, computing non-exceedance probabilities directly from the cumulative distribution function permits direct influence of the values by the sample points. Thus, a more informed estimate of the distribution parameters is possible via the method of SD-PWMs. Also, Haktanir suggests that calculating the non-exceedance probability via the assumed distribution permits the corresponding results to reflect the strength or weakness of the selected distribution in describing the sample. If a distribution is appropriate for describing a sample, errors generated by the method of SD-PWMs should be less than errors generated by other methods since more information about the distribution itself is used. Larger errors, conversely, should reflect an inappropriate choice of distribution.

3. SD-PWM algorithms

In this section, the basic ideas and equations for the new SD-PWM algorithms are presented. To simplify the discussions of the algorithm development, a uniform set of variables is used for the distribution parameters: a, the shape parameter; b, the scale parameter; and c, the location parameter. The only deviation from this standardized set is seen in the three-parameter log–normal distribution (LN3). For LN3, the location parameter c is maintained, but the traditional LN3 variables μ_y and σ_y are used to designate the mean and standard deviation of y, where $y = \ln(x - c)$.

For the first three distributions considered—the Gumbel, the general extreme value, and the log–logistic distributions—the SD-PWM equations and algorithms generally follow the equations and algorithms given by Haktanir (1997), although deviations from Haktanir's algorithms for the Gumbel distribution are explicitly noted. Alternatively, for the three-parameter log–normal and Pearson type three distributions, the algorithms presented deviate significantly from those presented by Haktanir. Unlike Haktanir's methods, the recommended algorithms presented here parallel the algorithms developed for the first three distributions.

Since the development process proceeds similarly for all five algorithms, we will provide details only for those algorithms that are dealt with in Section 4. A summary of the relevant SD-PWM equations to be solved, as well as the appropriate initial estimates for the parameters, is given for the GEV and LL distributions in Table 1. Note that, for all five distributions considered, $\alpha_0'' = \beta_0'' = \bar{x}$, the sample mean of the data. This fact is used to simplify the SD-PWM equations.

3.1. General algorithm development issues

Because of its robust numerical procedures and wide usage within the engineering and scientific communities, the algorithms have been developed into a set of MATLAB® scripts utilizing MATLAB's® iterative root-finding scheme *fsolve*. By default, *fsolve* employs its 'large scale' solution process to solve for the zeros of the relevant equations. This process is a subspace trust region method based upon a version of Newton's method. (See Coleman and Li, 1996 as well as the MATLAB® help menu for *fsolve*.) To improve performance, constraint settings on the solution process were determined and implemented to provide reasonable results for each script. In general, this entailed setting the bandwidth of the initial preconditioner to Inf (i.e. using full bandwidth) for the preconditioned conjugate gradient iterations as well as requiring a very tight tolerance on the termination value. See Savage (2001) for further information on these scripts.[1]

Since the SD-PWM parameter estimates are determined using iterative techniques, initial guesses on the parameters are required. Because the SD-PWM method follows from the method of PWMs, Haktanir (1997)

[1] The MATLAB® scripts can be obtained upon request from the authors.

Table 1
Summary of SD-PWM equations and initial estimates for GEV and LL distributions

Distribution and CDF	Parameter equations	Initial estimates
Generalized extreme value	$\frac{3\beta_2'' - \bar{x}}{2\beta_1'' - \bar{x}} = \frac{1 - 3^{-a}}{1 - 2^{-a}}$,	$a_0 = 7.8590r + 2.9554r^2$,
$F(x) = \exp\left\{-\left[1 - \frac{a(x-c)}{b}\right]^{1/a}\right\}$,	$b = \frac{a(2\beta_1'' - \bar{x})}{\Gamma(1+a)(1 - 2^{-a})}$	$b_0 = \frac{a_0(2\beta_1' - \bar{x})}{\Gamma(1+a_0)(1 - 2^{-a_0})}$,
$a \neq 0$.	$c = \bar{x} - \frac{b[1 - \Gamma(1+a)]}{a}$	$r = \frac{2\beta_1' - \bar{x}}{3\beta_2' - \bar{x}} - \frac{\ln 2}{\ln 3}$.
Log–Logistic	$a = 3 - \frac{2(\bar{x} - 3\alpha_2'')}{\bar{x} - 2\alpha_1''}$,	$a_0 = 3 - \frac{2(\bar{x} - 3\alpha_2')}{\bar{x} - 2\alpha_1'}$,
$F(x) = \left[1 + \left(\frac{x-c}{b}\right)^{1/-a}\right]^{-1}$	$b = \frac{\bar{x} - 2\alpha_1''}{a\,\Gamma(1+a)\,\Gamma(1-a)}$	$b_0 = \frac{x - 2\alpha_1'}{a_0\,\Gamma(1+a_0)\,\Gamma(1-a_0)}$
	$c = \bar{x} - b\,\Gamma(1+a)\,\Gamma(1-a)$	

proposed that PWM parameter estimates are reasonable initial guesses for the SD-PWM parameter estimates. However, for some distributions, the method of PWMs itself requires an iterative solution process. In such instances, approximations for the PWM parameter estimates are used to establish reasonable initial guesses for the SD-PWMs algorithms.

As mentioned previously, the SD-PWM parameter estimate equations are generally nonlinear functions of the parameters. However, it was discovered during the development process that it was always possible to solve one equation explicitly for a parameter in terms of the other parameters and the sample mean. This implies that the number of parameter equations to be solved iteratively could always be reduced by one—the remaining parameter is determined by a simple function evaluation. This reduction in dimension helps both the speed and accuracy of the algorithm, since higher dimensional root finding is generally a more demanding numerical procedure.

3.2. Gumbel distribution

For the Gumbel distribution, the cumulative distribution function is

$$F(x) = \exp\{-\exp[-b(x-c)]\} \tag{8}$$

where $-\infty < x < \infty$. Applying PWMs to the Gumbel distribution, Greenwood et al. (1979) considered the set α_s and found

$$\alpha_0 = c + \frac{\gamma}{b} \tag{9a}$$

and

$$\alpha_1 = \frac{c}{2} + \frac{\gamma - \ln 2}{2b} \tag{9b}$$

where Euler's constant $\gamma \approx 0.5772157$. Enforcing Eq. (7a) with the SD-PWM estimators and solving for b and c gives

$$b = \frac{\ln 2}{\bar{x} - 2\alpha_1''} \tag{10a}$$

and

$$c = \bar{x} - \frac{\gamma}{b} \tag{10b}$$

From Eq. (10b), the location parameter c is a function of the scale parameter b and the sample mean $\bar{x}$.

Substituting Eq. (10b) into the definition of α_1'' (Eq. (6a)) yields one equation to be solved for the scale parameter b:

$$b = \frac{\ln 2}{\bar{x} - \frac{2}{n}\left(\sum_{i=1}^{n}[1 - \exp\{-\exp[-b(x_i - \bar{x}) - \gamma]\}]x_i\right)} \quad (11)$$

The value of b which satisfies Eq. (11) is the SD-PWM scale parameter estimate for the Gumbel distribution and sample x. A reasonable initial guess b_0 is the PWM estimate for b:

$$b_0 = \frac{\ln 2}{\bar{x} - 2\alpha_1'} \quad (12)$$

Once b is found from Eq. (11), determination of c via Eq. (10b) is trivial.

To ensure that the scale parameter b is always non-negative, Haktanir (1997) suggests the following equation in place of Eq. (10a):

$$b = \frac{\ln 2}{|\bar{x} - 2\alpha_1''|} \quad (13)$$

In the most general case, however, the absolute value should not be included in the Gumbel PWM functions. Thus, the absolute value function has been excluded from the SD-PWMs algorithm presented above. The consequences and validity of Eq. (13) are discussed in Section 4.

3.3. Three-parameter log–normal distribution

Unlike the Gumbel distribution, the three-parameter log–normal distribution (LN3) must be considered explicitly for both positively and negatively skewed forms of the distribution. The governing SD-PWM equations for both cases are provided below. To determine which case is appropriate for a given sample x, Haktanir uses the skew measure R as defined by Ding et al. (1989b):

$$R = \frac{\beta_2' - \bar{x}/3}{\beta_1' - \bar{x}/2} \quad (14)$$

When $R > 1$, the parameters are estimated for the positively skewed LN3 distribution. Alternatively, the negatively skewed LN3 distribution is used when $R < 1$.

3.3.1. Positively skewed LN3

For the positively skewed LN3 distribution, the cumulative distribution function is

$$F(x) = \int_c^x f(t)\mathrm{d}t = \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{\ln(x - c) - \mu_y}{\sigma_y\sqrt{2}}\right)\right] \quad (15)$$

where $c \leq x < \infty$. Applying PWMs to the LN3 distribution, Song and Hou (1988) considered the set β_r. However, the resulting expressions obtained by Song and Hou contain integrals that cannot be evaluated analytically. Therefore, Song and Hou developed a numerical table using approximate integration useful for relating PWM estimators and functions of the distribution parameters.

For his LN3 SD-PWM algorithm, Haktanir (1997) follows Song and Hou and presents a complex algorithm dependent upon their numerical table. The resulting algorithm is undesirable for two reasons. First, user input is required at various intermediate steps within the algorithm based upon interpolation from Song and Hou's table, which may not be readily obtained by the user. Second, instead of requiring direct equality of the SD-PWM estimators and moments, Haktanir's algorithm requires the solving of polynomials based on the functions of distribution parameters for which Song and Hou developed the aforementioned table. Therefore, it is not immediately evident that the LN3 SD-PWMs algorithm suggested by Haktanir directly enforces the definition of SD-PWMs as set forth in Eqs. (7a) and (7b). Thus, to provide a closed algorithm (i.e. no intermediate input required) that both explicitly enforces the definition of SD-PWMs and eliminates the need for Song and Hou's numerical table, an algorithm substantially different from Haktanir's is presented below.

As was done by Song and Hou (1988), the set β_r of PWMs for the LN3 distribution is considered here. Additionally, Hosking (1986, 1991) considered the L-moments for the LN3 distribution (see Hosking (1991) for the definition of L-moments), which are linearly related to the PWMs of a distribution. Hosking's results were used to obtain the necessary PWMs:

$$\beta_0 = c + \exp(\mu_y + \sigma_y^2/2) \quad (16a)$$

$$\beta_1 = \frac{c}{2} + \tfrac{1}{2}\exp(\mu_y + \sigma_y^2/2)[1 + \mathrm{erf}(\sigma_y/2)] \tag{16b}$$

and

$$\beta_2 = \frac{c}{3} + \tfrac{1}{2}\exp(\mu_y + \sigma_y^2/2)\left[2S(\sigma_y) + \mathrm{erf}(\sigma_y/2) + \tfrac{2}{3}\right] \tag{16c}$$

where

$$S(\sigma_y) = \frac{1}{\sqrt{\pi}}\int_0^{\sigma_y/2} \mathrm{erf}(t/\sqrt{3})\exp(-t^2)\mathrm{d}t \tag{16d}$$

See Savage (2001) for the details of these derivations. For the method of SD-PWMs, substituting β_r'' for β_r and solving for μ_y, σ_y, and c gives

$$\frac{3\beta_2'' - 2\beta_1''}{2\beta_1'' - \bar{x}} = \frac{3S(\sigma_y)}{\mathrm{erf}(\sigma_y/2)} + \frac{1}{2} \tag{17a}$$

$$\mu_y = \ln\left(\frac{2\beta_1'' - \bar{x}}{\mathrm{erf}(\sigma_y/2)}\right) - \frac{\sigma_y^2}{2} \tag{17b}$$

and

$$c = \bar{x} - \exp(\mu_y + \sigma_y^2/2) \tag{17c}$$

Substitution of Eq. (17c) into the expressions for β_1'' and β_2'' yields two equations (Eqs. (17a) and (17b)) for the two unknowns μ_y and σ_y. Because Eq. (16d) must be evaluated numerically, $S(\sigma_y)$ is computed with MATLAB's® *quad8* numerical integration routine during the iterative solution process. Following the solution of μ_y and σ_y, the SD-PWM estimate for c is determined from Eq. (17c).

For initial parameter guesses, the PWM parameter estimates require an iterative solution. Therefore, an alternative is preferred. Making use of the PWM LN3 approximation given by Hosking (1986) and the approximation to the inverse cdf for the normal distribution given by Abramowitz and Stegun (1965), initial guesses μ_{y0} and σ_{y0} are computed as follows:

$$\sigma_{y0} = 0.999281z - 0.00618z^3 + 0.000127z^5 \tag{18a}$$

$$\mu_{y0} = \ln\left(\frac{2\beta_1' - \bar{x}}{\mathrm{erf}(\sigma_{y0}/2)}\right) - \frac{\sigma_{y0}^2}{2} \tag{18b}$$

where

$$z = \sqrt{\frac{8}{3}}\,\Phi^{-1}\left(\frac{3\beta_2' - 2\beta_1'}{2\beta_1' - \bar{x}}\right) \tag{18c}$$

$$\Phi^{-1}(F) = W - \frac{2.515517 + 0.802853W + 0.010328W^2}{1 + 1.432788W + 0.189269W^2 + 0.001308W^3} + \epsilon(F) \tag{18d}$$

$$W = \begin{cases} \sqrt{-2\ln(F)} & F \le 0.5 \\ \sqrt{-2\ln(1-F)} & F \ge 0.5 \end{cases} \tag{18e}$$

and the error $\epsilon(F)$ is less than 4.4×10^{-4}.

3.3.2. Negatively skewed LN3

For the negatively skewed LN3 distribution, the cumulative distribution function becomes

$$F(x) = \int_{-\infty}^{x} f(t)\mathrm{d}t = \frac{1}{2}\left[1 - \mathrm{erf}\left(\frac{\ln(c-x) - \mu_y}{\sigma_y\sqrt{2}}\right)\right] \tag{19}$$

where $-\infty \le x < c$. Considering the PWM set β_r, it can be shown that

$$\beta_0 = c - \exp(\mu_y + \sigma_y^2/2) \tag{20a}$$

$$\beta_1 = \frac{c}{2} - \tfrac{1}{2}\exp(\mu_y + \sigma_y^2/2)[1 - \mathrm{erf}(\sigma_y/2)] \tag{20b}$$

and

$$\beta_2 = \frac{c}{3} - \tfrac{1}{2}\exp(\mu_y + \sigma_y^2/2)\left[2S(\sigma_y) - \mathrm{erf}(\sigma_y/2) + \tfrac{2}{3}\right] \tag{20c}$$

For the method of SD-PWMs, substituting β_r'' for β_r and solving for μ_y, σ_y, and c gives

$$3 - \frac{3\beta_2'' - 2\beta_1''}{2\beta_1'' - \bar{x}} = \frac{3S(\sigma_y)}{\mathrm{erf}(\sigma_y/2)} + \frac{1}{2} \tag{21a}$$

$$\mu_y = \ln\left(\frac{2\beta_1'' - \bar{x}}{\mathrm{erf}(\sigma_y/2)}\right) - \frac{\sigma_y^2}{2} \tag{21b}$$

and

$$c = \bar{x} - \exp(\mu_y + \sigma_y^2/2) \tag{21c}$$

Solution of these three equations proceeds similarly to that of the positively skewed case. Because of their similar forms, the approximation given by Eqs. (18a)–(18e) can be used with slight modification to obtain the necessary initial guesses. Eq. (18c) must be replaced with

$$z = \sqrt{\frac{8}{3}}\Phi^{-1}\left(3 - \frac{3\beta_2' - 2\beta_1'}{2\beta_1' - \bar{x}}\right) \quad (22)$$

With this change, Eqs. (18a)–(18e) provides suitable initial parameter guesses μ_{y0} and σ_{y0} for the negatively skewed LN3 algorithm.

3.4. Pearson type three distribution

As with the LN3, the Pearson type three distribution (P3) must be considered for both positively skewed and negatively skewed cases explicitly. Again, the skewness measure R (Eq. 14) for a sample x is used to determined whether parameters are calculated for the positively skewed P3 or the negatively skewed P3.

3.4.1. Positively skewed P3

For the positively skewed P3, the cumulative distribution function is

$$F(x) = \int_c^x f(t)\mathrm{d}t = \Gamma(b(x-c), a) \quad (23)$$

where $c \le x < \infty$ and $\Gamma(z,k)$ is the normalized incomplete gamma function. Following the example of Song and Ding (1988) and Ding et al. (1989a,b), the set β_r was used for developing the SD-PWM equations. Substituting β_r'' for β_r and solving for a, b, and c gives

$$\frac{\beta_2'' - \bar{x}/3}{\beta_1'' - \bar{x}/2} = \frac{S_2^1(a) - a/3}{S_1^1(a) - a/2} \quad (24a)$$

$$b = \frac{S_1^1(a) - a/2}{\beta_1'' - \bar{x}/2} \quad (24b)$$

and

$$c = \bar{x} - \frac{a}{b} \quad (24c)$$

where

$$S_1^1(a) = \int_0^\infty \left[\int_0^t \frac{1}{\Gamma(a)} u^{a-1}\exp(-u)\mathrm{d}u\right] \frac{1}{\Gamma(a)} t^a \exp(-t)\mathrm{d}t \quad (24d)$$

and

$$S_2^1(a) = \int_0^\infty \left[\int_0^t \frac{1}{\Gamma(a)} u^{a-1}\exp(-u)\mathrm{d}u\right]^2 \frac{1}{\Gamma(a)} t^a \exp(-t)\mathrm{d}t \quad (24e)$$

No closed-form solution is known for either $S_1^1(a)$ or $S_2^1(a)$. Therefore, as with LN3, Song and Ding (1988) and Ding et al. (1989a) provide numerical tables based on approximate integration to relate PWM estimators and functions of the distribution parameters. As was the case for LN3, Haktanir presents a complex P3 SD-PWM algorithm that is heavily dependent upon the aforementioned numerical tables. Again, it is not immediately evident that the P3 SD-PWMs algorithm suggested by Haktanir strictly enforces the definition of SD-PWMs (Eqs. (7a) and (7b)). Therefore, to provide a closed algorithm that both explicitly follows the definition of SD-PWMs and eliminates the need for numerical tables, a new algorithm is created that deviates from the P3 SD-PWM algorithm suggested by Haktanir (1997).

Although no simple closed-form solutions are known to exist for either $S_1^1(a)$ or $S_2^1(a)$, Jakubowski (1992) presents alternative expressions in terms of the beta and incomplete beta functions that simplify the right-hand sides of Eqs. (24a) and (24b):

$$S_1^1(a) - a/2 = \frac{1}{2B(a, 1/2)} \quad (25a)$$

and

$$\frac{S_2^1(a) - a/3}{S_1^1(a) - a/2} = 2I_{1/3}(a, 2a) \quad (25b)$$

Substitution of these results into Eqs. (24a) and (24b) gives

$$\frac{\beta_2'' - \bar{x}/3}{\beta_1'' - \bar{x}/2} = 2I_{1/3}(a, 2a) \quad (26a)$$

and

$$b = \frac{1}{2B(a, 1/2)} \frac{1}{(\beta_1'' - \bar{x}/2)} \quad (26b)$$

Substitution of Eq. (24c) in the expressions for β_1'' and β_2'' yields two equations (Eqs. (26a) and (26b)) to be solved for a and b, with Eq. (24c) used to determine c.

Again for initial parameter guesses, the PWM parameter estimates require an iterative solution, thus creating a need for an alternative. Hosking (1986) considered the L-moments of the positively skewed P3 distribution, and Hosking (1991) suggests an approximation for these L-moment expressions. Due to the relationship between L-moments and PWMs, Savage (2001) shows that Hosking's approximation can be used to obtain the following initial guesses a_0 and b_0:

For $Q \geq 1/3$, let $t_m = 1 - Q$ and

$$a_0 = \frac{0.36067t_m - 0.5967t_m^2 + 0.25361t_m^3}{1 - 2.78861t_m + 2.56096t_m^2 - 0.77045t_m^3} \quad (27a)$$

for $Q < 1/3$, let $t_m = 3\pi Q^2$ and

$$a_0 = \frac{1 + 0.2906t_m}{t_m + 0.1882t_m^2 + 0.0442t_m^3} \quad (27b)$$

and for all Q

$$b_0 = \frac{1}{2B(a_0, 1/2)} \frac{1}{(\beta_1' - \bar{x}/2)} \quad (27c)$$

where

$$Q = 3\left(\frac{\beta_2' - x/3}{\beta_1' - \bar{x}/2} - 1\right) \quad (27d)$$

3.4.2. Negative skewness

For the negatively skewed P3, the cumulative distribution function is

$$F(x) = \int_{-\infty}^{x} f(t)\mathrm{d}t = 1 - \Gamma(b(c - x), a) \quad (28)$$

where $-\infty < x < c$. The first three PWMs are for the set β_r are

$$\beta_0 = c - \frac{a}{b} \quad (29a)$$

$$\beta_1 = \frac{c}{2} + \frac{S_1^1(a)}{b} - \frac{a}{b} \quad (29b)$$

and

$$\beta_2 = \frac{c}{3} + \frac{2S_1^1(a)}{b} - \frac{S_2^1(a)}{b} - \frac{a}{b} \quad (29c)$$

For the method of SD-PWMs, substituting β_r'' for β_r, solving for a, b, and c, and making use of Jabukowski's simplifications (Eqs. (25a) and (25b)) gives

$$\frac{\beta_2'' - \bar{x}/3}{\beta_1'' - \bar{x}/2} = 2 - 2I_{1/3}(a, 2a) \quad (30a)$$

$$b = \frac{1}{2B(a, 1/2)} \frac{1}{(\beta_1'' - \bar{x}/2)} \quad (30b)$$

and

$$c = \bar{x} + \frac{a}{b} \quad (30c)$$

Solution of these equations proceeds as before. As with the positively skewed form of the P3 distribution, use of the PWM parameter estimates for initial guesses requires an iterative solution. However with slight modification, Eqs. (27a)–(27d) can be used with the negatively skewed P3 as well (see Savage, 2001 for details). Specifically, if Eq. (27d) is replaced with

$$Q = 3\left(1 - \frac{\beta_2' - \bar{x}/3}{\beta_1' - \bar{x}/2}\right) \quad (31)$$

then Eqs. (27a)–(27d) provides suitable initial parameter guesses a_0 and b_0 for the negatively skewed P3 algorithm.

4. Algorithm analysis

As an example of parameter estimation by the method of SD-PWMs, Haktanir (1997) determined the SD-PWMs parameter estimates for all five distributions mentioned in Section 3 using a 51-element sample of annual flood peaks from the 902-Beskonak station on the Kopru creek in southern Turkey. The algorithms presented here were tested with the 902-Beskonak data, and the resulting estimates were compared to the estimates given by Haktanir (1997). For the Gumbel, GEV, and LL algorithms, the parameter estimates obtained with the present algorithms agree with the values determined by Haktanir, thus confirming that the present algorithms can perform at least as well as Haktanir's does for these situations. However, the absolute value function present in Eq. (13) can result in discrepancies between Haktanir's Gumbel algorithm and the Gumbel algorithm presented here. These discrepancies are explored below. Additionally, because the

LN3 and P3 SD-PWM algorithms presented here deviate significantly from the algorithms suggested by Haktanir (1997), the parameter estimates obtained for the 902-Beskonak data using the present algorithms differ from the parameter estimates obtained by Haktanir (1997). These variations are also discussed below.

In addition to testing the SD-PWMs algorithms using the 902-Beskonak data, daily fastest-mile wind speed recordings and the Gumbel, GEV, and LL SD-PWMs algorithms have been used to estimate distribution parameters for extreme wind speeds (see Savage, 2001; Whalen et al., 2002). The wind speed data proves useful for analyzing the aforementioned discrepancy in the Gumbel algorithm.

4.1. Gumbel distribution

For the Gumbel SD-PWM algorithm presented above, Eq. (11) must be satisfied for the Gumbel scale parameter estimate b. Rearranging slightly gives

$$0 = \ln 2 - b\left[\bar{x} - \frac{2}{n} \times \left(\sum_{i=1}^{n} [1 - \exp\{-\exp[-b(x_i - \bar{x}) - \gamma]\}]x_i\right)\right] \tag{32}$$

This equation defines a function $f_1(b)$ whose roots are, by definition, the SD-PWM Gumbel scale parameter estimates for the sample X. Because $f_1(b)$ is a non-linear function of b, multiple roots and thus multiple scale parameter estimates may exist for a given sample.

Alternatively, Haktanir (1997) suggests the SD-PWMs equation for the Gumbel scale parameter estimate b should take the form of Eq. (13). Expanding the SD-PWM estimator α_1'', substituting Eq. (10b) for the location parameter c, rearranging slightly, and defining a function $f_2(b)$ gives

$$f_2(b) = \ln 2 - b\left|\bar{x} - \frac{2}{n}\left(\sum_{i=1}^{n} [1 - \exp\{-\exp[-b(x_i - \bar{x}) - \gamma]\}]x_i\right)\right| = 0 \tag{33}$$

For Haktanir's (1997) algorithm, the roots of $f_2(b)$ are the SD-PWM scale parameter estimates. More specifically, Haktanir (1997) notes that the positive root to this equation is the SD-PWM estimate of b. It should be mentioned that the absolute value was excluded from Haktanir's equation (11) (the equivalent of Eq. (33) here). However, the equation for the scale parameter (Eq. (8) in Haktanir (1997) and the equivalent to Eq. (13) here) includes the absolute value function, and Haktanir makes no suggestion that it should be removed. Therefore, it is assumed that the absolute value function should be present in Haktanir's equation (11) as given in Eq. (33) above.

To explore the differences between the functions $f_1(b)$ and $f_2(b)$ and their impact on algorithm behavior, $f_1(b)$ and $f_2(b)$ are plotted in Figs. 1 and 2 for different data sets. For the 902-Beskonak sample (Fig. 1), observe that f_1 has two roots, one negative and one positive, while f_2 has only one positive root equal to the positive root of f_1. Because the scale parameter b is intended to be non-negative, the presence of the absolute function in f_2 seems advantageous, as it removes the negative root appearing in f_1 and yields only one possible solution. However, the absolute value function can present a problem when considering other sample sets. Fig. 2 gives f_1 and f_2 for a filtered wind recording from Albuquerque, NM with a threshold of 48 mph (see Savage, 2001 for details regarding the filtering and threshold). Comparison of Figs. 1 and 2 reveals that the general shape of the SD-PWM equations is extremely data dependent and that using f_2 can create an 'extra' root. Therefore, the absolute value function may not always produce the desired results, and its use should be avoided in a general-purpose algorithm.

To further explore the impact of data-dependency on the SD-PWM equations, consider Fig. 3, which shows examples of f_1 and f_2 for two wind speed samples: Albuquerque, NM/threshold = 48 mph and Greensboro, NC/threshold = 25 mph. For the Albuquerque sample, f_1 has three points of particular interest: two roots, b_1 and b_2, and a local minimum at b_v (at which the iterative solution could become 'stuck' and thus not generate a solution). The function f_2 has all of these features plus an additional root b_H (i.e. 'Haktanir's root'). However, because of the vertical position of the local minimum, b_H is the only root present in the Greensboro sample, and it exists

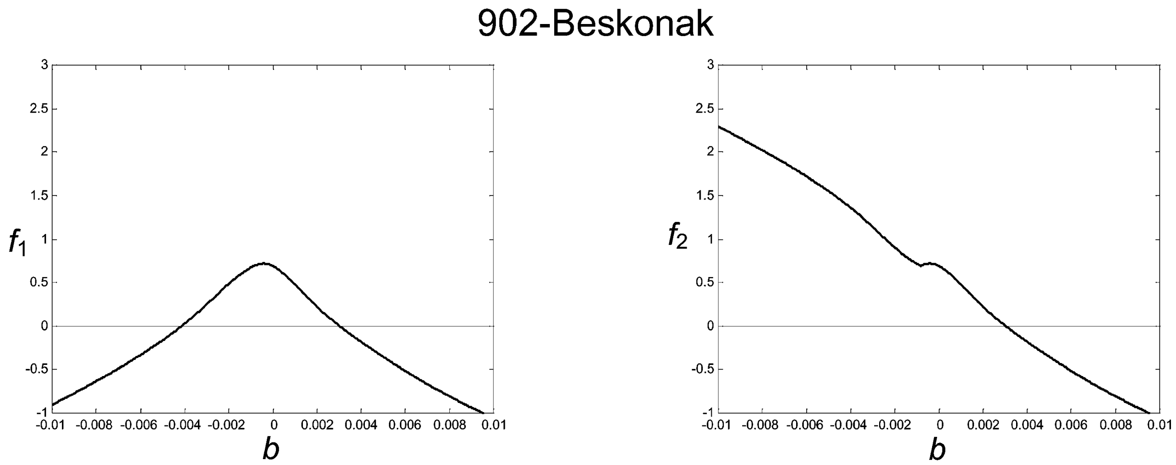

Fig. 1. SD-PWM Gumbel functions for the 902-Beskonak sample.

only for f_2. Contrary to b_1 and b_2, b_H is a mathematically appropriate SD-PWM scale parameter estimate only when the absolute value function is included. Along with b_1 and b_2, the adequacy of b_H, based on the resulting Gumbel cdf, is considered below.

Fig. 3 illustrates that at least two cases are possible for real datasets: $f_v < 0$ and $f_v > 0$. As observed in Fig. 3, when $f_v < 0$, both b_1 and b_2 exist for f_1, and all three roots (b_1, b_2, and b_H) exist for f_2. Alternatively, when $f_v > 0$, no solution exists for f_1, and b_H is the only possible solution for f_2. From the 12 aforementioned stations, a total of 132 wind speed samples (i.e. 11 thresholds per station for 12 stations) were created and analyzed. Of these 132 sets, 75 had $f_v < 0$, and 57 had $f_v > 0$. A summary of the algorithm behavior when $f_v > 0$ and when $f_v < 0$ is given in Table 2. Notice that, when using f_1, the algorithm always iterated to b_2 when $f_v < 0$, even though two solutions were possible. Using f_1 for the 57 cases in which $f_v > 0$, the algorithm always stopped in the region of b_v. In contrast, when using f_2, the algorithm iterated to b_H on a total of 20 occasions. Thus, we see that the presence of the absolute value in f_2 can change the results of the algorithm significantly.

To analyze the performance of the three parameter

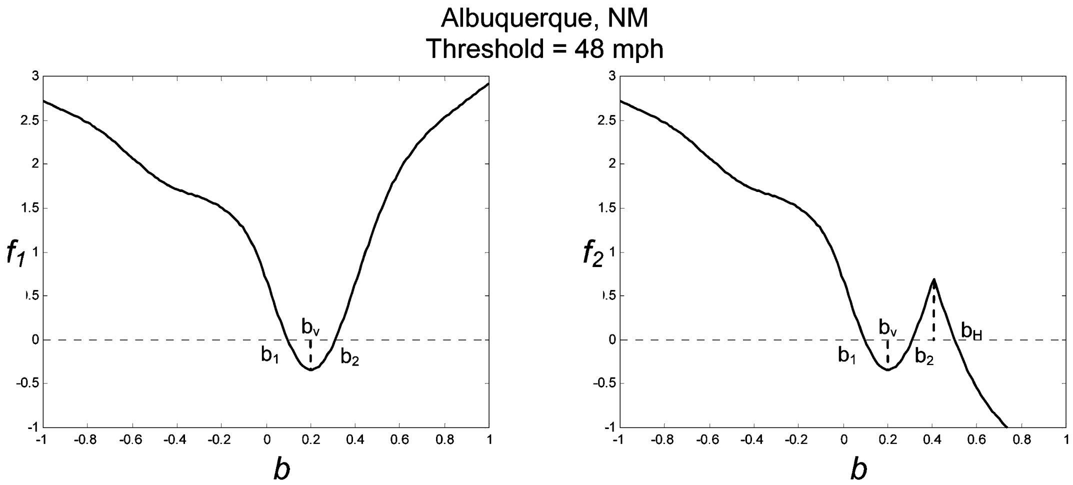

Fig. 2. SD-PWM Gumbel iterating functions for Albuquerque, NM data.

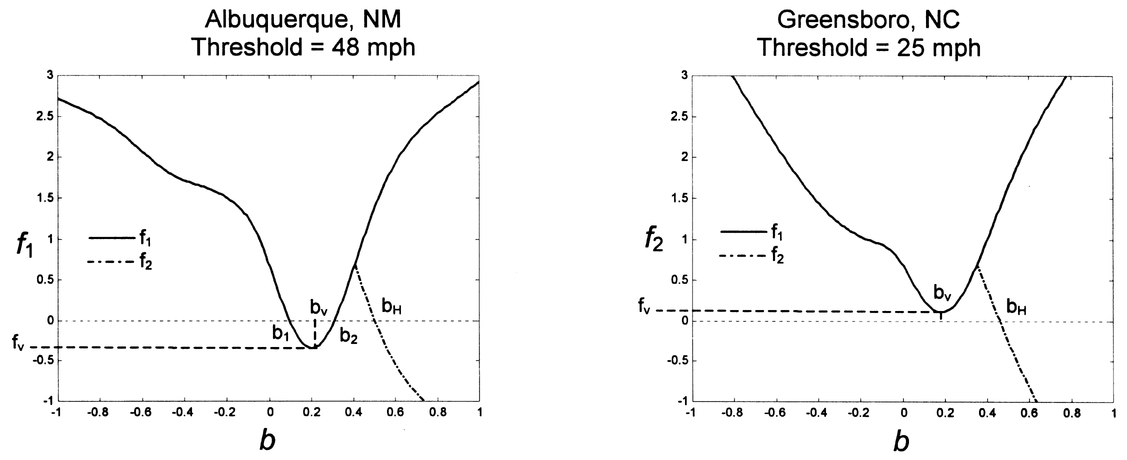

Fig. 3. Various Gumbel SD-PWM iterating functions and their corresponding roots.

estimates, b_1, b_2, and b_H, the resulting cdfs are now considered for the Albuquerque data. The three cdfs corresponding to the three roots are plotted in Fig. 4. For the data set, the probability of non-exceedance for each point was calculated using $P_{nex,i}$ ($j = 1$ in Eq. (4)). The notations cdf_1, cdf_2, and cdf_H correspond to the cdfs generated with b_1, b_2, and b_H, respectively. Based on visual inspection of Fig. 4, b_1 is clearly an inaccurate scale parameter estimate, due to the apparent lack of trend through the data points. Alternatively, both b_2 and b_H seem to generate reasonable fits. To quantify the performance of the various roots, total errors in the form of sums of squared errors are computed for each cdf. For a particular set of parameter estimates corresponding to a distribution with cdf $F(x)$, the sum of squared error e is the sum of the square of the difference between $P_{nex,i}$ and the cdf evaluated at x_i :

$$e = \sum_{i=1}^{n} \left[\frac{i-1}{n-1} - F(x_i) \right]^2 \qquad (34)$$

Table 2
Gumbel roots found for f_1 and f_2—132 cases

Equation	f_1			f_2			
Root found	b_1	b_2	b_v[a]	b_1	b_2	b_H	b_v[a]
Case							
$f_v < 0$	0	75	0	0	58	17	0
$f_v > 0$	0	0	57	0	0	3	54

[a] b_v indicates algorithm stopped in the region of b_v.

For the three estimates of b corresponding to the Albuquerque, NM/threshold = 48 mph sample, errors were computed based on Eq. (34). These errors were as follows: $e = 1.65$ for b_1, $e = 0.12$ for b_2, and $e = 0.55$ for b_H. For this sample, clearly b_2 produces the best fit to the data and is presumably the best parameter estimate of the three. This demonstrates that a direct implementation of the SD-PWM equations can be better suited for handling situations where sensitivity of the parameter estimates to the form of the parameter estimation equations exists. Since the presence such a condition is typically not known a priori, the direct implementation of the SD-PWM equations is recommended.

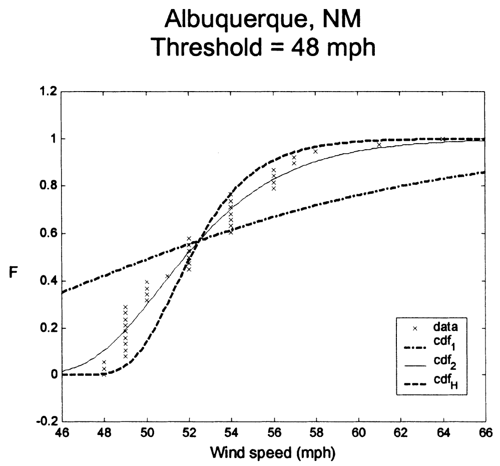

Fig. 4. Gumbel cumulative distribution functions.

Table 3
Algorithm results for the LN3 distribution and 902-Beskonak sample

	Parameter estimates		PWM equations (Eqs. (16a)–(16d)		SD-PWM estimators (Eq. (6b))	Sum of squared errors		PPCC value	
Present algorithm	σ_y	0.462313	β_0	888.615	888.618	e	0.02708	χ	0.99525
	μ_y	6.74910	β_1	565.971	565.974				
	c	−60.9146	β_2	426.922	426.924				
Haktanir's algorithm	σ_y	0.458436	β_0	889.549	888.618	e	0.02716	χ	0.99518
	μ_y	6.76085	β_1	566.662	565.253				
	c	−69.4897	β_2	427.399	426.006				

4.2. Three parameter log–normal distribution

For the 902-Beskonak sample, the LN3 SD-PWM parameter estimates are given in Table 3 for both the present algorithm and Haktanir's (1997) algorithm. Haktanir (1997) provides parameter estimates to six significant figures. Therefore in Table 3, the PWM equations β_0, β_1, and β_2 (Eqs. (16a)–(16d)) and the SD-PWM estimators β_0'', β_1'', and β_2'' (Eq. (6b)) are calculated based on the parameter estimates given to six significant figures for both algorithms.

While the PWM equations and the SD-PWM estimators are not identical for the present algorithm, they are equal to five significant figures. However, from Table 3, the same degree of accuracy is not obtained from Haktanir's SD-PWM parameter estimates. Moreover, it is not directly obvious that the LN3 SD-PWMs algorithm suggested by Haktanir (1997) generally requires the SD-PWM estimators and the PWM functions to be equal—only functions of the parameters are equated. Alternatively, the present algorithm is predicated upon the parameter estimates yielding equivalent PWM functions and SD-PWM estimators as set forth by the definition of SD-PWM estimates (Eqs. (7a) and (7b)).

To compare the accuracy of the parameter estimates provided by each algorithm, two measures of quality of fit were calculated. First, the sum of squared errors e for each algorithm was determined via Eq. (34). As shown in Table 3, these values are nearly the same for the two methods—the slightly smaller value for the present algorithm was deemed not to be statistically significant. As an additional test, each set of estimates was used to calculate a set of transformed data

$$z = \frac{x - c}{\sigma_y e^{\mu_y}} \tag{35}$$

where x represents the original 902-Beskonak flood data. According to the definition of the three-parameter lognormal distribution, the random variable z should have a normal distribution with a mean of zero and a standard deviation of one. This hypothesis was checked using a standard probability plot test in which the probability plot correlation coefficient (PPCC) χ was computed for the transformed data under the assumed normal distribution. The PPCC values are also reported in Table 3, and once again, the coefficient values for each algorithm are essentially equal. These facts lead to the conclusion that the present algorithm performs equally well in estimating the LN3 parameters of this data as Haktanir's algorithm does. Considering the simplicity and transparency in implementation that the present algorithm possesses by directly enforcing the definition of SD-PWM estimates without recourse to intermediate tables, we surmise that the present approach is better suited for general use without loss of accuracy in the estimates produced.

4.3. Pearson type three distribution

To analyze the log–Pearson type three distribution, Haktanir (1997) considers the natural logarithm of the 902-Beskonak sample and applies the P3 algorithm to the log-transformed data set. The same log-transformed data set is also considered for the present algorithm. The log transformed 902-Beskonak data has $R < 1$ (Eq. (14)), therefore the negatively skewed form of the P3 distribution is considered. The

Table 4
Algorithm results for P3 distribution and log-transformed 902-Beskonak sample

	Parameter estimates		PWM equations (Eqs. (29)–(29c))		SD-PWM estimators (Eq. (6b))	Sum of squared errors		PPCC	
Present Algorithm	a	929.241	β_0	6.66658	6.66660	e	0.02807	χ	0.99535
	b	60.7827	β_1	3.47475	3.47482				
	c	21.9545	β_2	2.36315	2.36322				
Haktanir's Algorithm	a	854.259	β_0	6.66662	6.66660	e	0.02786	χ	0.99534
	b	58.3786	β_1	3.47450	3.47445				
	c	21.2997	β_2	2.36312	2.36387				

SD-PWM parameter estimates are given in Table 4. Once again, Haktanir (1997) provides six significant figures for the parameter estimates. Therefore, the PWM equations β_0, β_1, and β_2 and the SD-PWM estimators β_0'', β_1'', and β_2'' are calculated based on the parameter estimates given to six significant figures for both algorithms. In this case, while Haktanir's (1997) P3 algorithm seems to give parameter estimates that yield equivalent PWM equations and SD-PWM estimators in most cases, some discrepancies exist, particularly for β_2.

As was done for the LN3 distribution, the sum of squared errors and probability plot correlation coefficients were computed for both P3 algorithms, with the results displayed in Table 4. In the case of the PPCC value, the flood data was transformed according to

$$z = b(c - \ln x) \quad (36)$$

where it can be shown that z should obey a gamma distribution with shape parameter a. Thus, a probability plot test for the gamma distribution, using an appropriate value of a, was performed for each set of estimates. As was found for the LN3 distribution, no significant difference between the two sets of estimates was indicated by either test, leading to the conclusion that the two algorithms performed equally well in determining the P3 parameter estimates for this data. From this, we reach the same conclusion that the present algorithm is expected to perform at least as accurately as Haktanir's algorithm while also being easier to implement and use. As a result, the present algorithm is a better candidate for general use.

5. Conclusions

The method of self-determined probability weighted moments holds much promise in enhancing the ability to estimate distribution parameters, particularly for 'extreme' phenomena where data is often limited. Haktanir's work has demonstrated notable improvements over the probability weighted moments method and other methods. In order for the method to become widely accepted, however, it should be implemented in a clear and easily understood way, rely as little as possible upon secondary sources of information, and avoid special cases that limit its applicability. We believe that the present algorithms achieve these goals without sacrificing the benefits of this method as proposed by Haktanir (1997).

In this paper, we have shown that certain features of Haktanir's algorithms can be eliminated without loss of accuracy in the parameter estimates. First, the absolute value function is not required for the SD-PWM equation for the Gumbel scale parameter b. Because Haktanir (1997) claims that the positive root is the appropriate SD-PWM Gumbel scale parameter estimate, from Fig. 1 it is evident that the absolute value function is included for convenience rather than mathematical necessity. However, this 'convenience' can create an inappropriate root for some samples. Also, the numerical tables from Song and Ding (1988) and Ding et al. (1989a) can be eliminated from Haktanir's algorithms for the LN3 and P3 distributions. By directly requiring the SD-PWM functions and estimators to be equivalent and utilizing very precise numerical integration methods, we have created a completely closed algorithm whose parameter estimates are at least as good as those of Haktanir (1997).

Future research directions are aimed at expanding the applicability of the SD-PWM method and testing its performance in various applications of extreme wind speed estimation. In a companion paper (Savage et al., 2001), we demonstrated that the self-determined probability weighted moments method offers some advantages over other techniques when applied to distributions of long return period wind speeds, but questions regarding the uniformity of the data prevented stronger conclusions from being drawn. Similar tests utilizing more appropriate data are being pursued. In addition, the robustness of the SD-PWM method versus the PWM method needs to be examined via Monte Carlo simulations of wind data generated from distributions other than those explored in this work (e.g. the Wakeby or Kappa distribution). A comparison of biases and mean square errors in estimated extreme winds will further expose the strengths and weaknesses of each approach. Also, efforts are being made to implement a version of this method for the generalized Pareto distribution, which is widely used in extreme wind speed analysis. Finally, finding the appropriate form of the SD-PWM equations from an iterative solution viewpoint is an open question needing investigation.

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