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Tidal groundwater level fluctuations in L-shaped leaky coastal aquifer system

Hailong Li[a,b,*], Jiu Jimmy Jiao[a]

[a]Department of Earth Sciences, The University of Hong Kong, Pokfulam Road, Hong Kong, People's Republic of China
[b]Institute of Biomathematics, Anshan Normal College, Anshan 114005, Liaoning, People's Republic of China

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Abstract

This paper presents an analytical solution to describe tidal groundwater level fluctuations in a coastal leaky aquifer system bounded by water–land boundaries that form a right angle (referred to as L-shaped coastlines). The system consists of an unconfined aquifer, a confined aquifer and a leaky layer between them. Previously published analytical solutions that discuss only single aquifer constitute a special case of the new solution when the permeability of leaky layer approaches zero. A simple approximate solution without integral is presented. Error analysis and hypothetical example show that the approximate solution has adequate accuracy for both groundwater level prediction and parameter estimation for an L-shaped leaky aquifer system.

Keywords: Periodic groundwater flow; Analytical solution; Coastal aquifer system; Sea tide; L-shaped coastline; Leaky aquifer

1. Introduction

Analytical studies of tidal effects play an important role in coastal hydrogeology. For example, Jacob (1950), Nielsen (1990), Li and Chen (1991), and Sun (1997) derived various solutions to describe the tidal groundwater fluctuations in a single coastal confined aquifer under the assumption that the coastline is straight. Jiao and Tang (1999), Li and Jiao (2001a,b), and Tang and Jiao (2001) derived analytical solutions for multi-layer coastal leaky aquifer systems. All these previous studies, however, assumed that the coastline is straight.

As the first work to address the impact of the coastline shape on tidal groundwater fluctuation in coastal aquifers, Li et al. (2000) derived a 2D analytical solution in an unconfined aquifer cut by coastlines which form a right angle (for simplicity, the coastlines are referred to as 'L-shaped coastlines' hereafter). Li and Jiao (2002b) improved their work by providing more simple analytical solutions to the same problem. Both of them, however, only consider a single aquifer.

In reality, coastal areas are not only bounded by very irregular coastlines full of inlets, bays, and headlands that cannot always be, even approximately, regarded as straight lines, but also composed of multi-layered aquifers separated by leaky layer(s) (Cheng and Chen, 2001; Carr and van der Kamp, 1969; Maas and De Lange, 1987; Liu, 1996). In this case, tidal wave propagation in the confined aquifer is affected

* Corresponding author. Address: Department of Earth Sciences, The University of Hong Kong, Pokfulam Road, Hong Kong, People's Republic of China. Tel.: +852-2546-8591; fax: +852-2517-6912.
E-mail address: hlica@hkusua.hku.hk (H. Li).

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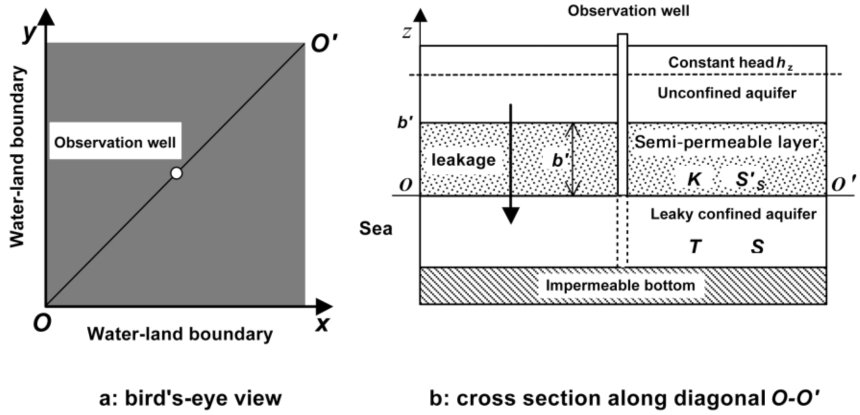

Fig. 1. An L-shaped leaky coastal aquifer system: (a) bird's eye view and (b) cross-section along the diagonal $O-O'$.

by three factors: the irregular water–land boundaries, the leaky layer and the tidal wave interference from the adjacent aquifer via the leaky layer. These will lead to a complicated problem. As an attempt to attack this problem, this paper considers an L-shaped coastal leaky aquifer system consisting of an unconfined aquifer, a confined aquifer and a leaky layer between them. An analytical solution to describe tidal groundwater head fluctuation in the semi-confined aquifer is derived. The assumptions underlying the solutions are: (a) negligible watertable variation in the upper unconfined aquifer, (b) negligible horizontal flow in the leaky layer, negligible vertical flow in the confined aquifer and (c) all formations have a clear-cut vertical boundary with seawater. Assumption (a) was proposed by Jiao and Tang (1999) and used by Li and Jiao (2001a,b). The discussions about assumption (a) by Volker and Zhang (2001), Jiao and Tang (2000), Li et al. (2001), Li and Jiao (2002a,c), and Jeng et al. (2002) show that the assumption is valid for realistic aquifer systems because the leakance of a realistic leaky layer is small and the specific yield of the unconfined aquifer is several orders of magnitude greater than the storativity of the confined aquifer. Assumption (b) was proposed by Hantush (1960) in the case of radial groundwater flow to a well. The validity of this assumption for coastal aquifer systems was examined using numerical solutions in Li and Jiao (2002a). Assumption (c) is used to simplify the geometry of the boundary so that analytical solutions can be derived. After the solution is derived, an attempt is made to compare this solution with previous solutions by various researchers for coastal leaky aquifer systems. Since the solution is complicated, an attempt is also made to derive an approximate solution. The accuracy of the approximate solution will be analyzed. Finally, a hypothetical inverse problem of aquifer parameter estimation is solved to examine the parameter estimation errors of the approximate solution.

2. Conceptual model and analytical solution

Consider an L-shaped subsurface system consisting of a leaky confined aquifer, an unconfined aquifer and a leaky layer between them. Assume that all the layers are homogeneous, horizontal, with constant thickness, and that the three assumptions in Section 1 applies. Choose an $x-y-z$ coordinate system so that the positive parts of both the x and y axes being the L-shaped coastlines, and the z-axis be vertical, positive upward with the $x-y$ plane coincides with the bottom plane of the leaky layer (Fig. 1). Then, according to the theory of Hantush (1960), similar to Li and Jiao (2001a), the groundwater head $h(x, y, z, t)$ in the leaky layer satisfies the following differential equation and boundary conditions

$$S'_S \frac{\partial h}{\partial t} = K' \frac{\partial^2 h}{\partial z^2}, \qquad -\infty < t < \infty, \ \ 0 < z < b', \quad (1)$$

$$h(x, y, b', t) = h_z = 0, \quad (2)$$

$$h(x, y, 0, t) = H(x, y, t), \quad (3)$$

where S_S', K' and b' are the specific storativity [L^{-1}], the vertical permeability [LT^{-1}] and thickness [L] of the leaky layer, respectively, $h_z = 0$ is the mean sea level, $H(x,y,t)$ is the hydraulic head [L] in the confined aquifer and satisfies the following differential equation

$$S\frac{\partial H}{\partial t} = T\left(\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2}\right) + K'\frac{\partial H}{\partial z}(x,y,0,t), \tag{4}$$

$$-\infty < t < \infty, \quad x, y > 0,$$

where S and T are the storativity (dimensionless) and the transmissivity [L^2T^{-1}] of the confined aquifer, respectively (Hantush, 1960). On one side ($y = 0$, $x > 0$), which represents the ocean–land boundary, the spatially constant tidal boundary condition

$$H(x,0,t) = A\cos(\omega t + c) = A\,\mathrm{Re}\{\exp[\mathrm{i}(\omega t + c)]\},$$

$$x > 0, \tag{5}$$

is used, where Re denotes the real part of the complex expression, $\mathrm{i} = \sqrt{-1}$, A and ω are the tidal amplitude [L] and frequency [T^{-1}], respectively, and c is the phase shift (dimensionless). On the other side ($y > 0$, $x = 0$), which represents the boundary in the estuary, the tidal attenuation is considered by the spatially variable tidal boundary condition (Li et al., 2000; Sun, 1997)

$$H(0,y,t) = A\,e^{-\kappa_{er}y}\cos(\omega t - \kappa_{ei}y + c)$$

$$= A\,\mathrm{Re}(\exp(-\kappa_e y + \mathrm{i}(\omega t + c))), \quad y > 0, \tag{6}$$

where y denotes the distance along the estuary from the entry; $\kappa_{er} \geq 0$ and $\kappa_{ei} \geq 0$ are the amplitude damping coefficient [L^{-1}] and wave number [L^{-1}] of the tidal wave in the estuary, respectively; $\kappa_e = \kappa_{er} + \mathrm{i}\kappa_{ei}$. The datum of the hydraulic head of the aquifer is set to be the mean sea level. In inland places far from the origin, no-flow boundary condition is used, i.e.

$$\lim_{x\to\infty}\frac{\partial H}{\partial x}(x,y,t) = \lim_{y\to\infty}\frac{\partial H}{\partial y}(x,y,t) = 0. \tag{7}$$

The derivation of the solution $h(x,y,z,t)$ and $H(x,y,t)$ to the boundary value problem (1)–(7) is presented in Appendix A. The analysis will focus on the groundwater head $H(x,y,t)$ in the confined aquifer because, in reality, it is much more useful than the groundwater head $h(x,y,z,t)$ in the leaky layer. Hence, only the expression of $H(x,y,t)$ will be given here. Details about $h(x,y,z,t)$ are presented in Appendix A. For the sake of clarity, three aquifer parameters are introduced. They are the aquifer's tidal propagation parameter a [L^{-1}], the leaky layer's buffer capacity [dimensionless] θ and dimensionless leakage u

$$a = \sqrt{\frac{\omega S}{2T}}, \tag{8}$$

$$\theta = b'\sqrt{\frac{\omega S_S'}{2K'}} = \sqrt{\frac{\omega S'}{2L}}, \tag{9}$$

$$u = \frac{L}{\omega S} = \frac{K'}{\omega S b'}, \tag{10}$$

where $S' = S_S'b'$, and $L = K'/b'$ [T^{-1}] is the specific leakage of the leaky layer (Hantush, 1960). The solution reads

$$H(x,y,t) = A\,\mathrm{Re}[(I(apx, apy; 1+\mathrm{i}q, 1+\mathrm{i}q) + I(apy, apx; m+\mathrm{i}n, 1+\mathrm{i}q) + e^{-(1+\mathrm{i}q)apy} + e^{-\kappa_e y-(m+\mathrm{i}n)apx})e^{\mathrm{i}(\omega t + c)}], \tag{11}$$

where

$$I(\xi,\eta;\mu,\lambda) = -\frac{\lambda\xi}{\pi}\int_0^\infty e^{-\mu\tau}\left(\frac{K_1(\lambda\rho(\xi,\eta-\tau))}{\rho(\xi,\eta-\tau)} - \frac{K_1(\lambda\rho(\xi,\eta+\tau))}{\rho(\xi,\eta+\tau)}\right)d\tau \tag{12}$$

with $K_n(z_c)$ being the modified second kind Bessel function of nth order and

$$\rho(\xi,\eta) = \sqrt{\xi^2 + \eta^2}; \tag{13}$$

the four dimensionless constants $p = p(u,\theta)$, $q = q(u,\theta)$, $m = m(a,u,\theta,\kappa_{er},\kappa_{ei})$ and $n = n(a,u,\theta,\kappa_{er},\kappa_{ei})$ are defined as

$$p = \sqrt{\sqrt{(1+L_i)^2 + L_r^2} + L_r}, \tag{14a}$$

$$q = \frac{1}{p}\sqrt{\sqrt{(1+L_i)^2 + L_r^2} - L_r} = \frac{1+L_i}{p^2}, \tag{14b}$$

$$m = \frac{1}{p}\sqrt{\sqrt{\left(1 + L_i - \frac{\kappa_{er}\kappa_{ei}}{a^2}\right)^2 + \left(L_r - \frac{\kappa_{er}^2 - \kappa_{ei}^2}{2a^2}\right)^2} + L_r - \frac{\kappa_{er}^2 - \kappa_{ei}^2}{2a^2}}, \tag{14d}$$

$$n = \frac{1}{p}\sqrt{\sqrt{\left(1 + L_i - \frac{\kappa_{er}\kappa_{ei}}{a^2}\right)^2 + \left(L_r - \frac{\kappa_{er}^2 - \kappa_{ei}^2}{2a^2}\right)^2} - L_r + \frac{\kappa_{er}^2 - \kappa_{ei}^2}{2a^2}}, \tag{14e}$$

in which the functions $L_i = L_i(u, \theta)$ and $L_r = L_r(u, \theta)$ are given by

$$L_i = u\theta \frac{1 - 2\,e^{-2\theta}\sin 2\theta - e^{-4\theta}}{1 - 2\,e^{-2\theta}\cos(2\theta) + e^{-4\theta}}, \tag{15a}$$

$$L_r = u\theta \frac{1 + 2\,e^{-2\theta}\sin 2\theta - e^{-4\theta}}{1 - 2\,e^{-2\theta}\cos(2\theta) + e^{-4\theta}}. \tag{15b}$$

3. Discussion of solution

If the leaky layer is very thin, then according to Eq. (9), one has $\theta \approx 0$. Let $\theta \to 0$ in Eqs. (15a) and (15b), it follows that

$$\lim_{\theta\to+0} L_i = 0, \qquad \lim_{\theta\to+0} L_r = u. \tag{16}$$

Therefore, for thin leaky layer which satisfies $\theta \approx 0$, the parameters p, q, m and n defined in Eqs. (14a)–(14d) can be significantly simplified.

3.1. Analytical solution for single aquifer

If the middle layer becomes completely impermeable, i.e. $K' = 0$, then one has

$$\lim_{K'\to+0} L_i = \lim_{K'\to+0} L_r = 0. \tag{17}$$

In fact, in the case of $S'_S > 0$, according to Eqs. (9) and (10), it follows that

$$\lim_{K'\to+0} \theta\Big|_{S'_S>0} = \infty, \quad \lim_{K'\to+0} u = 0, \quad \lim_{K'\to+0} u\theta = 0. \tag{18}$$

Based on Eqs. (18), (15a) and (15b), one immediately obtains Eq. (17). In the case that $S'_S = 0$, using Eq. (16), one will find that Eq. (17) still holds. Hence Eq. (17) applies for both $S'_S > 0$ and $S'_S = 0$. Substituting Eq. (17) into Eqs. (14a)–(14d), one finds that

$$\lim_{K'\to+0} p(s,u) = \lim_{K'\to+0} q(s,u) = 1, \tag{19}$$

and m and n become the same as Eqs. (8) and (9) of Li and Jiao (2002b), which are for single confined aquifer. Therefore, solution (11) becomes the Li and Jiao (2002b) solution if the leaky layer becomes completely impermeable.

3.2. Approximate simplification of solution (11)

Since solution (11) is very complicated, an approximate simplification will be helpful. It will be shown that the following simple expression

$$H_{approx}(x,y,t) = A\,\mathrm{Re}[(-e^{-(1+iq)apy-(m+in)apx} + e^{-(1+iq)apy} + e^{-\kappa_e y-(m+in)apx})e^{i\omega t+c}] \tag{20}$$

is an adequate approximation to solution (11), although solution (20) does not exactly satisfy the differential equation (4). In order to show this, let

$$R_r = \mathrm{Re}[I(apx, apy; 1+iq, 1+iq) + I(apy, apx; m+in, 1+iq) + e^{-(1+iq)apy-(m+in)apx}], \tag{21a}$$

$$R_i = \mathrm{Im}[I(apx, apy; 1+iq, 1+iq) + I(apy, apx; m+in, 1+iq) + e^{-(1+iq)apy-(m+in)apx}], \tag{21b}$$

then the spatial maximum error distribution of the approximate solution (20) relative to the tidal

Table 1
Maximum values of $R(apx, apy; q, 1 + iq)$ for different values of q

q	apx	apy	$R(apx, apy; q, 1 + iq)$
0.01	0.75	0.75	0.0503
0.2	0.75	0.75	0.0518
0.4	0.74	0.74	0.0559
0.6	0.74	0.74	0.0624
0.8	0.73	0.73	0.0710
1.0	0.72	0.72	0.0812

amplitude A is given by

$$R(apx, apy; q, m + in) \overset{\text{def}}{=} \max_t |H(x, y, t) - H_{\text{appox}}(x, y, t)|/A = \sqrt{R_r^2 + R_i^2}. \quad (22)$$

To discuss the error distribution $R(apx, apy; q, m + in)$ in the apx–apy space for all possible values of the parameter q and $m + in$, it is necessary to find the ranges of q and $m + in$ values. According to Eqs. (14a) and (14b), it follows that

$$0 < q \le 1, \quad (23)$$

and that $q = 1$ if and only if $p = 1$. From Eqs. (14c) and (14d), the range of $m + in$ value depends on κ_{er} and κ_{ei}. The field values of κ_{er} and κ_{ei} are less than $10^{-5}\ \text{m}^{-1}$ (Li et al., 2000; Sun, 1997). They are always several orders of magnitude smaller than the aquifer's tidal propagation parameter a, which is usually greater than $10^{-3}\ \text{m}^{-1}$. Therefore, the inequalities

$$\kappa_{er} < 0.1a, \qquad \kappa_{ei} < 0.1a \quad (24)$$

hold for all kinds of field data. Using Eqs. (24) and (14a)–(14d), one can show that (see Appendix B for the proof)

$$|m + in - (1 + iq)| < 0.0072, \quad (25)$$

namely, $m + in \approx 1 + iq$. For example, when $p = q = 1$, $\kappa_{er} = \kappa_{ei} = 0.1a$, from Eqs. (14c) and (14d) one has $m = n = 0.995$. Due to this reason, it is enough to discuss the error distribution $R(apx, apy; q, m + in)$ when $m + in = 1 + iq$ (or equivalently, $\kappa_{er} = \kappa_{ei} = 0$). In this case, the error becomes $R(apx, apy; q, 1 + iq)$, which has only one parameter q. Table 1 shows the maximum values of $R(apx, apy; q, 1 + iq)$ corresponding to six values of q ranging from 0.01 to 1. One can see that the maximum of $R(apx, apy; q, 1 + iq)$ ranges from 5.03% to 8.12% when q ranges from 0.01 to 1, and that all the maximum are on the diagonal line $apx = apy$. Fig. 2 shows how the contours $R(apx, apy; 0.01, 1 + 0.01i) = \varepsilon$ change with ε (when $\kappa_{er} = \kappa_{ei} = 0$ and $q = 0.01$). At inland point $(apx, apy) = (0.75, 0.75)$, the dimensionless error $R(apx, apy; 0.01, 1 + 0.01i)$ reaches its maximum of 5.03%. Li and Jiao (2002b) shows how the contours $R(ax, ay; 1, 1 + i) = \varepsilon$ change with ε when $\kappa_{er} = \kappa_{ei} = 0$ and $q = 1$ (hence $p = 1$). At inland point $(ax, ay) = (0.715, 0.715)$, the dimensionless error $R(ax, ay; 1, 1 + i)$ (defined as $R(ax, ay)$ in Li and Jiao (2002b)) reaches its maximum of 8.12% (see Fig. 2 of Li and Jiao (2002b)).

The physical importance of the problem (1)–(7) is that both the interference of the tidal groundwater waves induced by the sea tides at the two sides of the L-shaped coastlines and the impact of the leakance of the semi-permeable layer are considered comprehensively. Due to the simplicity of solution (20), it clearly describes both the interference and the impact. The interference is described by the superposition of three sinusoidal fluctuations in Eq. (20). The impact of the leakance is described by the parameters p and q given by Eqs. (14a)–(14d).

3.3. Asymptotic solutions for large x and y

From the expression of the approximate solution (20), it can be easily seen that for large x solution (11) becomes the straight-coastline solution of Li and Jiao (2001a), i.e.

$$H(x, y, t)|_{apx \gg 1} \approx H_{\text{approx}}(x, y, t)|_{apx \gg 1} \approx A\, e^{-apy} \cos(\omega t - apqy + c), \quad (26)$$

and for large y it becomes

$$H(x, y, t)|_{apy \gg 1} \approx H_{\text{approx}}(x, y, t)|_{apy \gg 1} \approx A \exp(-\kappa_{er} y - apmx) \cos(\omega t - \kappa_{ei} y - apnx + c). \quad (27)$$

If the leaky layer's storage is negligible, i.e. $\theta = 0$, then Eq. (16) applies and the parameters p, q, m and n

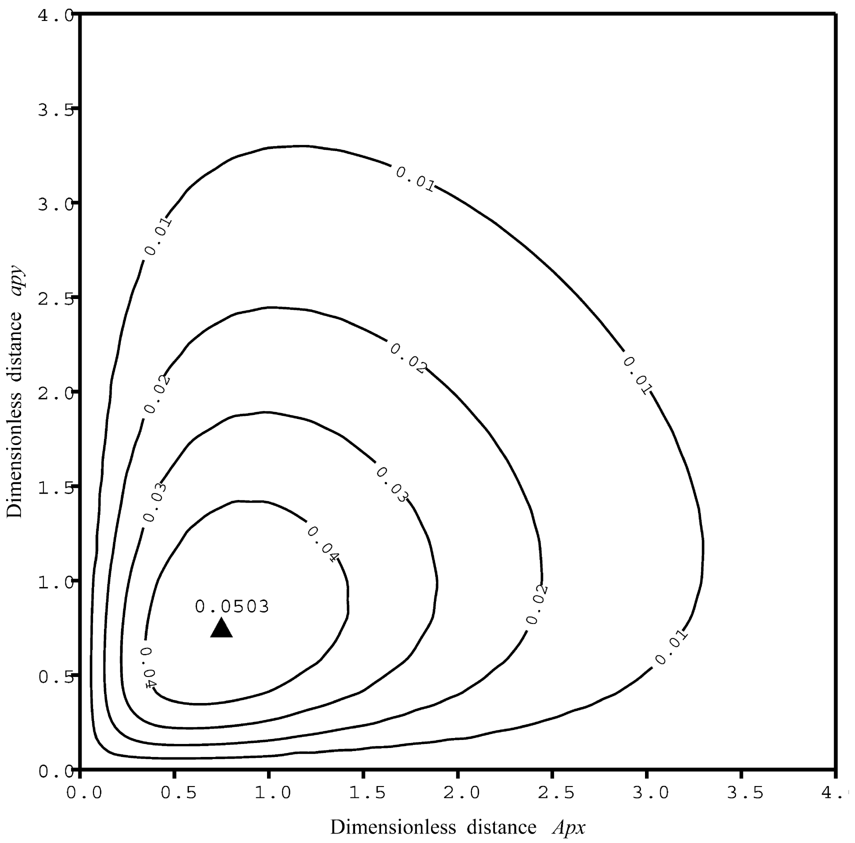

Fig. 2. Spatial error distribution $R(apx, apy; q, 1 + iq)$ of the simple approximate solution (20) when both $\kappa_{er} = \kappa_{ei} = 0$, $q = 0.01$.

in solution (27) can be simplified into the solution of Tang and Jiao (2001) (see equation (4) of their paper) by substituting Eq. (16) into Eqs. (14a)–(14d).

4. Hypothetical example of aquifer parameter estimation

A hypothetical example is designed to understand how much error can be introduced in estimating aquifer's parameters if there is observation error in the groundwater head data, and if the approximate solution (20) or Li and Jiao's (2001a) straight-coastline solution (26) are used in an L-shaped coastal leaky aquifer system. The approach used is as follows. A given set of aquifer parameters are used to generate the 'true values' of the parameters a, u and θ. Then based on the exact solution (11), the true values of the parameters a, u and θ are used to generate exact groundwater head fluctuation data forced by a given sinusoidal sea tide. These data are rounded into 'observed' groundwater head fluctuation data within an error of ± 0.5 cm. Then assume that the parameters a and θ are unknown, the sea tide, the dimensionless leakage u and the observed groundwater head fluctuation data are known. Inverse problems are solved to estimate the two unknown parameters a and θ based on the exact solution (11), the approximate solution (20) and the straight-coastline solution (26), respectively. By comparing the 'estimated' and the 'true' values of parameters a and θ, the aquifer-parameter-estimating applicabilities of the exact solution (11), the approximate solution (20) and straight-coastline solution (26) in L-shaped aquifer are examined.

Assume that the sea tide is semi-diurnal with the angular velocity $\omega = 0.506\ \text{h}^{-1}$, amplitude $A = 1.0$ m, phase shift $c = 0$, $\kappa_{er} = \kappa_{ei} = 0$. An

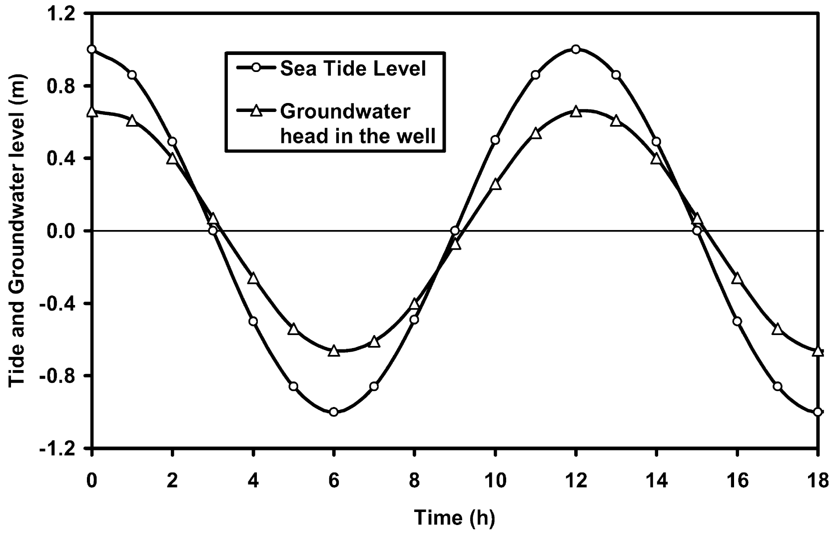

Fig. 3. Hypothetical semi-diurnal tide level and observed groundwater head fluctuation data at the observation well (the least-squares fittings based on different solutions (11), (20) and (26) coincide with the observed groundwater head data).

observation well is screened in an L-shaped leaky aquifer system at an inland point $(x_0, y_0) = (102.4, 102.4\ \text{m})$. The aquifer parameters are leaky layer's thickness $b' = 5$ m, vertical permeability $K' = 1.0\ \text{m d}^{-1}$, specific storage $S'_S = 0.0036\ \text{m}^{-1}$ (according to Raghunath (1987), this is reasonable for clay layer), the semi-confined aquifer's transmissivity $T = 2000\ \text{m}^2\ \text{d}^{-1}$, storativity $S = 0.001$. The true values of the three basic parameters are $a = 0.00177\ \text{m}^{-1}$, $u = 15.92$ and $\theta = 0.752$. These data leads to a value of $q = 0.20$ and a dimensionless 'observation well' position of $(apx_0, apy_0) = (0.75, 0.75)$, where the error $R(apx, apy; 0.2\text{i}, 1 + 0.2\text{i})$ of the approximate solution (20) reaches the maximum of 5.18%.

In order to estimate the parameters a and θ, the following least-squares problem

$$\min_{a,\theta} \sum_{j=1}^{12} [h_{\text{chosen}}(x_0, y_0, t_j; a, u, \theta)|_{u=15.92} - h_j^*]^2 \qquad (28)$$

is solved, where h_j^* $(j = 1, \ldots, 12)$ are 12 hourly observed groundwater head data at the observation well (x_0, y_0), as shown in Fig. 3, $h_{\text{chosen}}(x_0, y_0, t_j; a, u, \theta)$ is either the exact solution (11), or the approximation solution (20), or the straight-coastline solution (26). The results are listed in Table 2. As indicated by the least-squares residuals, the fit between the analytical solutions and the observed data is the same for the three different solutions and very good.

It can be seen from Table 2 that the values a and θ estimated by both the exact solution (11) and the approximate solution (20) are very close to their true values, while the straight-coastline solution (26) leads to significant errors. Although the estimated parameters a and θ based on the straight-coastline solution (26) have significant errors, the least-squares residual remains as small as those produced by the exact solution (11). This implies that, for the inverse problem, a satisfactory least-squares fitting does not necessarily mean that the parameter estimation is reliable.

5. Conclusions

An analytical solution is derived to describe tidal groundwater level fluctuations in an L-shaped leaky coastal aquifer system consisting of an unconfined aquifer, a semi-confined aquifer and a leaky layer between them. The watertable variation in the unconfined aquifer is neglected. The tidal attenuation in the estuary is taken into account on the one side of the L-shaped water–land boundaries.

Previous solutions of Jacob (1950), Sun (1997), Jiao and Tang (1999), Li and Jiao (2001a), and Tang

Table 2
Aquifer parameters estimated by least-squares fitting based on different solutions

	a (m^{-1})	θ	Least-squares residual (m^2)
True value	0.00177 (0%)	0.752 (0%)	0
Estimation of exact solution (11)	0.00180 (1.7%)	0.745 (−0.9%)	2.0×10^{-4}
Estimation of approx. solution (20)	0.00207 (17%)	0.685 (8.9%)	2.0×10^{-4}
Estimation of straight-coastline solution (26)	0.000923 (−48%)	0.946 (26%)	2.0×10^{-4}

Note: the percentages in the parentheses are the relative errors defined as (estimated value − true value)/(true value).

and Jiao (2001), which use the straight-coastline assumption, are special cases of the new solution when the distance from the coastline-bending point approaches infinity. Previous solutions of Li et al. (2000) and Li and Jiao (2002b) that discuss only single aquifer are special cases of the new solution when the vertical permeability of leaky layer approaches zero. A simple approximate solution without integral is presented. Error analysis and a hypothetical example show that the approximate solution has adequate accuracy for both groundwater level prediction and parameter estimation for an L-shaped leaky aquifer system.

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Appendix A. Derivation of the solution

Assume that

$$H(x,y,t) = A\,\mathrm{Re}[U(x,y)\exp(\mathrm{i}(\omega t + c))], \tag{A1}$$

$$h(x,y,z,t) = A\,\mathrm{Re}[Z(z)U(x,y)\exp(\mathrm{i}(\omega t + c))], \tag{A2}$$

where $U(x,y)$ and $Z(z)$ are complex functions. Substituting Eqs. (A1) and (A2) into Eqs. (1)–(3), and extending the three resultant real equations into complex ones with respect to the unknown complex functions, yield

$$\mathrm{i}\omega S'_S Z = K'Z'', \quad 0 < z < b', \tag{A3}$$

$$Z(b') = h_z = 0, \tag{A4}$$

$$Z(0) = 1. \tag{A5}$$

The solution of Eqs. (A3)–(A5) is

$$Z(z) = \frac{\exp(-(1+\mathrm{i})(z/b' - 1)\theta) - \exp((1+\mathrm{i})(z/b' - 1)\theta)}{\exp((1+\mathrm{i})\theta) - \exp(-(1+\mathrm{i})\theta)}. \tag{A6}$$

Using Eq. (A6), one obtains

$$K'Z'(0) = -\omega S(L_r + \mathrm{i}L_i), \tag{A7}$$

where L_i and L_r are defined in Eqs. (15a) and (15b). Now substituting Eqs. (A1), (A2) and (A7) into Eq. (4), and Eq. (A1) into Eqs. (5)–(7), and extending the four resulting real equations into complex ones with respect to the unknown functions, yield

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = \frac{\omega S}{T}[\mathrm{i}(1+L_i) + L_r]U, \tag{A8}$$

$$0 < x, y < +\infty,$$

$$U(x,0) = 1, \quad x > 0, \tag{A9}$$

$$U(0,y) = \exp(-\kappa_e y), \quad y > 0, \tag{A10}$$

$$\lim_{x\to\infty} \frac{\partial U}{\partial x}(x,y) = \lim_{y\to\infty} \frac{\partial U}{\partial y}(x,y) = 0. \tag{A11}$$

Because $\mathrm{e}^{-(1+\mathrm{i}q)apy}$ satisfies Eqs. (A8), (A9) and (A11), $\mathrm{e}^{-\kappa_e y-(m+\mathrm{i}n)apx}$ satisfies Eqs. (A8), (A10) and (A11), let

$$V(x,y) = U(x,y) - \mathrm{e}^{-(1+\mathrm{i}q)apy} - \mathrm{e}^{-\kappa_e y-(m+\mathrm{i}n)apx}, \tag{A12}$$

then $V(x,y)$ satisfies

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = a^2p^2(1+\mathrm{i}q)^2 V, \quad 0 < x, y < +\infty, \tag{A13}$$

$$V(x,0) = -e^{-(m+in)apx}, \quad x > 0, \tag{A14}$$

$$V(0,y) = -e^{-(1+iq)apy}, \quad y > 0, \tag{A15}$$

$$\lim_{x\to\infty} \frac{\partial V}{\partial x}(x,y) = \lim_{y\to\infty} \frac{\partial V}{\partial y}(x,y) = 0. \tag{A16}$$

Using the Green's function (Shimakura, 1992)

$$G = E(x,y;x_0,y_0) - E(x,y;-x_0,y_0) + E(x,y;-x_0,-y_0) - E(x,y;x_0,-y_0), \tag{A17}$$

where $E(x,y;x_0,y_0) = (1/2\pi)K_0((1+iq)\rho(ap(x-x_0), ap(y-y_0)))$, $K_n(z_c)$ denotes the modified second kind Bessel function of nth order, implementing the standard procedure to solve boundary value problem (A13)–(A16) (Shimakura, 1992, p. 43), one obtains

$$V(x,y) = \int_0^\infty V(x_0,0)\frac{\partial G}{\partial y_0}\bigg|_{y_0=0} dx_0 + \int_0^\infty V(0,y_0)\frac{\partial G}{\partial x_0}\bigg|_{y_0=0} dx_0. \tag{A18}$$

Using the formula (McLachlan, 1961; Shimakura, 1992, pp. 25, 26)

$$\frac{dK_0(z_c)}{dz_c} = -K_1(z_c), \tag{A19}$$

one finds

$$\int_0^\infty V(0,y_0)\frac{\partial G}{\partial x_0}\bigg|_{x_0=0} dy_0 = I(apy, apx; 1+iq, 1+iq), \tag{A20a}$$

$$\int_0^\infty V(x_0,0)\frac{\partial G}{\partial y_0}\bigg|_{y_0=0} dx_0 = I(apy, apx; m+in, 1+iq). \tag{A20b}$$

Substituting Eqs. (A20a) and (A20b) into Eq. (A18), then Eq. (A18) into Eq. (A12), yields

$$U(x,y) = I(apx, apy; 1+iq, 1+iq) + I(apy, apx; m+in, 1+iq) + e^{-(1+iq)apy} + e^{-\kappa_e y-(m+in)apx}. \tag{A21}$$

Finally, substituting Eq. (A21) into Eq. (A1) yields solution (11). The groundwater head $h(x,y,z,t)$ in the leaky layer is given in terms of Eqs. (A2), (A6) and (A21).

Appendix B. Proof of $|m + in - (1 + iq)| < 0.0072$

Because both $e^{-(1+iq)apy}$ and $e^{-\kappa_e y-(m+in)apx}$ satisfy Eq. (A8), one has

$$a^2p^2(1+iq)^2 = \frac{\omega S}{T}[i(1+L_i)+L_r], \tag{A22}$$

$$\kappa_e^2 + a^2p^2(m+in)^2 = \frac{\omega S}{T}[i(1+L_i)+L_r]. \tag{A23}$$

Subtracting Eq. (A23) from Eq. (A22), yields

$$a^2p^2(1+iq)^2 - a^2p^2(m+in)^2 = \kappa_e^2. \tag{A24}$$

Because $\kappa_e = \kappa_{er} + i\kappa_{ei}$, using Eqs. (24) and (A24), it follows that

$$|(1+iq)^2 - (m+in)^2| = \frac{1}{p^2}\left|\frac{\kappa_e}{a}\right|^2 < \frac{0.02}{p^2}. \tag{A25}$$

Using Eq. (14b) and $L_i \geq 0$ one finds that $1/p^2 = q/(1+L_i) \leq q$. Hence, from Eq. (A25) one obtains

$$|(1+iq)-(m+in)| < \frac{0.02}{p^2|1+iq+m+in|} \leq \frac{0.02q}{\sqrt{(1+m)^2+(q+n)^2}}. \tag{A26}$$

Because $m > 0$, $n > 0$, $1 \geq q > 0$, hence

$$q/\sqrt{(1+m)^2+(q+n)^2} < q/\sqrt{1+q^2} \leq \sqrt{2}/2.$$

Substituting this inequality into Eq. (A26), yields $|(1+iq)-(m+in)| < 0.01\sqrt{2}$. Therefore,

$$|(1+iq)+(m+in)| = |2(1+iq)+(m+in) - (1+iq)| \geq 2|1+iq| - |(m+in)-(1+iq)| > 2|1+iq| - 0.01\sqrt{2}. \tag{A27}$$

Substituting Eq. (A27) into Eq. (A26), it follows that

$$|(1+\mathrm{i}q)-(m+\mathrm{i}n)| < \frac{0.02q}{2|1+\mathrm{i}q|-0.01\sqrt{2}}$$

$$\leq \frac{0.02}{2|1+\mathrm{i}|-0.01\sqrt{2}} < 0.0072.$$

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