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Parametric representation of the elastic wave in anisotropic media[☆]

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Translation by K. Helbig, Commentary by K. Helbig and M.A. Slawinski

Abstract

Shortly after his appointment to the first geophysical professorship (1895 at the Jagiellonian University of Cracow), Rudzki had published two papers in which he made a strong case for anisotropy of crustal rocks [Beitr. Geophys. 2 (1898); Bull. Acad. Sci. Crac. (1899)]. He had solved the Christoffel equation for transversely isotropic media in terms of what we today would call the "slowness surface". Rudzki regarded this as the representation of the wave surface in *line coordinates*. The conversion to point coordinates lead to an equation of 12th degree. Rudzki had determined a few points, but this was not sufficient to obtain an impression of the wavefront. Costanzi [Boll. Soc. Sismol. Ital. 7 (1901)] had suggested to simplify the coordinate conversion by expressing the solution of the Christoffel equation in a parameter form. The first part of the current paper describes the implementation of this idea. For the first time, the cusps in the wave surface became visible. The results of this first part have been discussed and expanded by Helbig [Beitr. Geophys. 67 (1958) 177; Bull. Seismol. Soc. Am. 56 (1966) 527; Helbig, K., 1994. Foundations of Anisotropy for Exploration Geophysics. Pergamon] and Khatkevich [Isv. Akad. Nauk. SSSR, Ser. Geofiz. 9 (1964) 788]. In a second part, Rudzki applied the ideas to orthorhombic media. The process is straightforward: the elements of the characteristic determinant are of order 2 in the three line coordinates (the three slowness components), with squares of coordinates in the diagonal elements and products of two coordinates in the off-diagonal elements. The elements are easily manipulated so that they are expressed in terms of squares only. Next, the determinant is expanded in terms of rows. This leads to three (equivalent) expressions. The vanishing of any of the three expressions means that the characteristic determinant vanishes, i.e., it corresponds to a solution of the Christoffel equation. Each of the equations can be used to determine one of the sheets of the line coordinates of the wave surface (point coordinates of the slowness surface). To this end, it is expressed in terms of two parameters, which have been chosen strictly for mathematical convenience. After conversion of the line coordinates to point coordinates (formation of the envelopes), one obtains a parameter expression for the wave surface. Until today, the second part of the Rudzki's paper has not been closely studied. However, a blind test of the equations showed that they indeed describe the wave surface of orthorhombic media. The final sections discuss a few interesting aspects, among them the stability conditions for orthorhombic media and the condition under which a transversely isotropic medium transmits pure P- and S-waves.

[☆] In the text, additions by the translator are given in brackets.
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1. Introduction

I have discussed the shape of the elastic wave [front] in a transversely isotropic medium already in 1898 and 1899 (Rudzki, 1898, 1899), but could not finish the investigation. I came across an equation of degree 12 of such difficulty that I had to be content with approximate conclusions. In the subsequent text, these older papers will be called the "first paper" and the "second paper".

In a "résumé" to my investigation on the propagation of earthquake waves, Costanzi (1901) has shown that it should be relatively simple to obtain a parametric representation of the curve of degree 12. I give this representation in the present paper. It will be seen that the curve of degree 12 is more complicated than Costanzi assumed: with the numerical values I used in my earlier papers, one of the three branches of the curve is certainly not an oval. Thus I had been close to the truth; but with the cumbersome approximations, I could not determine the characteristic properties of the curve exactly.

Obviously, some of the earlier material has to be repeated, but more than three quarters of the present paper are completely new. New is, e.g., the investigation of the wave surface of a medium with three planes of symmetry in Sections 6–10.

2. Elastic waves in a transversely isotropic medium

A medium is called "transversely isotropic" (Love, 1906), if all directions parallel to a given plane are equivalent. If this plane is chosen as the xy plane, the elastic potential W has the following form (I use Love's notation throughout. In the earlier papers, I had used a different notation):

$$2W = c_{11}(e_{xx}+e_{yy})^2 + c_{33}e_{zz}^2 + 2c_{13}e_{zz}(e_{xx}+e_{yy}) - 4c_{66}e_{xx}e_{yy} + c_{44}(e_{xz}^2+e_{yz}^2) + c_{44}e_{xy}^2, \tag{1}$$

where

$$e_{xx} = \frac{\partial u}{\partial x},$$

$$e_{yz} = \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right),$$

$$e_{yy} = \frac{\partial v}{\partial y},$$

$$e_{xz} = \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right),$$

$$e_{zz} = \frac{\partial w}{\partial z},$$

$$e_{xy} = \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right).$$

Of course, u, v, w are displacements and c_{11}, c_{33}, ... are elastic constants. We see that a transversely isotropic medium is characterized by five independent elastic constants.

The stress components are

$$\begin{aligned}
X_x &= c_{11}e_{xx} + (c_{11}-2c_{66})e_{yy} + c_{13}e_{zz},\\
Y_y &= (c_{11}-2c_{66})e_{xx} + c_{11}e_{yy} + c_{13}e_{zz},\\
Z_z &= c_{13}e_{xx} + c_{13}e_{yy} + c_{33}e_{zz},\\
Z_y &= c_{44}e_{yz},\\
X_z &= c_{44}e_{xz},\\
Y_x &= c_{66}e_{xy}.
\end{aligned} \tag{2}$$

Now we write the differential equation for oscillations of small amplitude:

$$\rho\frac{d^2u}{dt^2} = \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z},$$

$$\rho\frac{d^2v}{dt^2} = \frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z},$$

$$\rho\frac{d^2w}{dt^2} = \frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z}, \tag{3}$$

divide everywhere through the density ρ, write for simplicity c_{11} instead of c_{11}/ρ, c_{33} instead of c_{33}/ρ, etc., and substitute the particular integral

$$u = A\lambda\sigma, \quad v = A\mu\sigma, \quad w = A\nu\sigma,$$

with

$$\sigma = e^{i(lx+my+nz)},$$

where A is an arbitrary constant (magnitude of the displacement); λ, μ, v the direction cosines of the displacement; l, m, n the direction cosines of the wave normal; and V the velocity of propagation of the oscillations.

In this way, we obtain the equations:

$$(H_{11} - V^2)\lambda + H_{12}\mu + H_{13}v = 0,$$

$$H_{12}\lambda + (H_{22} - V^2)\mu + H_{23}v = 0,$$

$$H_{13}\lambda + H_{23}\mu + (H_{33} - V^2)v = 0, \tag{4}$$

where

$$H_{11} = c_{11}l^2 + c_{66}m^2 + c_{44}n^2,$$

$$H_{22} = c_{66}l^2 + c_{11}m^2 + c_{44}n^2,$$

$$H_{33} = c_{44}l^2 + c_{44}m^2 + c_{33}n^2,$$

$$H_{12} = (c_{11} - c_{66})lm,$$

$$H_{12} = (c_{13} + c_{44})ln,$$

$$H_{12} = (c_{13} + c_{44})mn. \tag{5}$$

Now we follow Costanzi (1901); we divide Eq. (4) by V^2 and set

$$\frac{l}{V} = \xi, \quad \frac{m}{V} = \eta, \quad \frac{n}{V} = \zeta.$$

(Costanzi uses a substitution that makes ξ, η, ζ "Plücker" coordinates, i.e., $l/V=-\xi$, etc. I find it more convenient to set $l/V=\xi$). [In this form, ξ, η, ζ are the components of the slowness vector]. This transforms Eq. (4) into

$$(h_{11} - 1)\lambda + h_{12}\mu + h_{13}v = 0,$$

$$h_{12}\lambda + (h_{22} - 1)\mu + h_{23}v = 0,$$

$$h_{13}\lambda + h_{23}\mu + (h_{33} - 1)v = 0, \tag{4bis}$$

where

$$h_{11} = c_{11}\xi^2 + c_{66}\eta^2 + c_{44}\zeta^2,$$

$$h_{22} = c_{66}\xi^2 + c_{11}\eta^2 + c_{44}\zeta^2,$$

$$h_{33} = c_{44}\xi^2 + c_{44}\eta^2 + c_{33}\zeta^2,$$

$$h_{12} = (c_{11} - c_{66})\xi\eta,$$

$$h_{13} = (c_{13} + c_{44})\xi\zeta,$$

$$h_{23} = (c_{13} + c_{44})\eta\zeta. \tag{5bis}$$

Eq. (4 bis) can be satisfied only if the determinant of the coefficients of λ, μ, v vanishes. The equation thus obtained, i.e.,

$$\begin{vmatrix} h_{11} - 1 & h_{12} & h_{13} \\ h_{12} & h_{22} - 1 & h_{23} \\ h_{13} & h_{23} & h_{33} - 1 \end{vmatrix} = 0. \tag{6}$$

is the equation of the wave surface [If the ξ, η, ζ are considered as *point coordinates*, Eq. (6) is the equation of the slowness surface. Rudzki regards ξ, η, ζ as *line coordinates*. In this form, they describe all tangent planes of the wave surface]. Eq. (6) can be simplified significantly, because obviously the wave surface is a surface of revolution about the z-axis. It is sufficient to investigate a meridional curve, e.g., in the xz plane. We set

$$m = 0, \quad \eta = 0,$$

and obtain immediately

$$h_{12} = h_{23} = 0,$$

which lets Eq. (6) decouple into two equation. The first has the form:

$$h_{22} - 1 = 0,$$

i.e.,

$$c_{66}\xi^2 + c_{44}\zeta^2 - 1 = 0, \tag{7}$$

The second has the form

$$\begin{vmatrix} h_{11} - 1 & h_{13} \\ h_{13} & h_{33} - 1 \end{vmatrix} = 0,$$

i.e.,

$$(c_{11}\xi^2 + c_{44}\zeta^2 - 1)(c_{44}\xi^2 + c_{33}\zeta^2 - 1) - b^2\xi^2\zeta^2 = 0, \tag{8}$$

where for simplicity's sake we have set

$$c_{13} + c_{44} = b.$$

Eq. (7) represents a curve of class 2 (a conical section), Eq. (8) a curve of class 4. Curve (8) is of degree 12, as I have found in my second paper [the *degree* of a curve is given by the number of intersections of a straight line; the *class* of a curve is given by the number of tangents from a point]. From the well-known Plücker relations, it follows that curve (8) has neither points of inflection nor double tangents, but [up to] 28 double points and 24 cusps. The curve is of genus 3 and not rational. One sees immediately that both curves (7) and (8) are symmetric with respect to the x- and z-axes.

3. Parametric representation of curve (7)

We now follow Costanzi (1901) to obtain the parametric representation of the curves (7) and (8). The wave surface is nothing but the envelope of all planes

$$lx + my + nz = V(t - t_0),$$

where

$$l^2 + m^2 + n^2 = 1$$

and the time interval $(t-t_0)$ has a fixed value. If one sets $(t-t_0)=1$ and divides the equation of the planes by V, one can write:

$$x\xi + y\eta + z\zeta = 1.$$

[x, y, z are the coordinates of the wave surface, i.e., the components of the wave velocity vector]. In this particular case, we only need the equation of the meridional curve in the plane $y=0$ ($\eta=0$) that is the envelope to the lines $x\xi+z\zeta=1$.

One knows from the theory of curves that the envelope is determined by the system

$$\left.\begin{aligned} & x\xi + z\zeta = 1 \\ & x\frac{\partial\xi}{\partial u} + z\frac{\partial\zeta}{\partial u} = 0, \end{aligned}\right\} \tag{9}$$

where u is a parameter. If one solves the system (9) for x and z, one obtains:

$$x = \frac{\dfrac{\partial\zeta}{\partial u}}{\xi\dfrac{\partial\zeta}{\partial u} - \zeta\dfrac{\partial\xi}{\partial u}}, \quad z = \frac{-\dfrac{\partial\xi}{\partial u}}{\xi\dfrac{\partial\zeta}{\partial u} - \zeta\dfrac{\partial\xi}{\partial u}}, \tag{10}$$

and it only remains to represent ξ and ζ in terms of a single parameter (up to here this is a recapitulation of my older papers and of Costanzi (1901). What follows is new).

Let us first look at Eq. (7). One can write it as

$$c_{66}\xi^2 = u^2, \quad c_{44}\zeta^2 = 1 - u^2.$$

It follows that

$$\xi = \frac{u}{\sqrt{c_{66}}}, \qquad \zeta = \frac{\sqrt{1-u^2}}{\sqrt{c_{66}}},$$
$$\frac{\partial\xi}{\partial u} = \frac{1}{\sqrt{c_{66}}}, \quad \frac{\partial\zeta}{\partial u} = \frac{-1}{\sqrt{c_{66}}}\frac{u}{\sqrt{1-u^2}},$$

and

$$x = \sqrt{c_{66}}\,u, \quad z = \sqrt{c_{44}}\sqrt{1-u^2}.$$

That is the parametric representation of the ellipse

$$\frac{x^2}{c_{66}} + \frac{z^2}{c_{44}} = 1. \tag{11}$$

Thus, the first sheet of the wave surface is an ellipsoid of rotation with the semiaxes $\sqrt{c_{66}}$ and $\sqrt{c_{44}}$, as I had shown in a different way already in my first paper.

4. Parametric representation of curve (8)

Eq. (8) is identical with the equation

$$\begin{vmatrix} h_{11} - 1 & b\xi^2 \\ b\zeta^2 & h_{33} - 1 \end{vmatrix} = 0, \tag{12}$$

which can be separated in two ways. First, we write

$$\frac{h_{11} - 1}{b\zeta^2} = \frac{b\xi^2}{h_{33} - 1} = u, \tag{13}$$

and second

$$\frac{h_{11} - 1}{b\xi^2} = \frac{b\zeta^2}{h_{33} - 1} = -\frac{1}{u_1}. \tag{14}$$

The system (13) gives the second and the system (14) the third branch of the meridional curve. If one substitutes h_{11} and h_{22} from Eq. (5 bis) and solves Eqs. (8) and (9), respectively, for ξ^2 and ζ^2, one obtains from Eq. (8)

$$\left.\begin{aligned} \xi^2 &= \frac{(c_{33} - c_{44})u + bu^2}{bc_{44} + (c_{11}c_{33} - c_{44}^2 - b^2)u + bc_{44}u^2} \\ \zeta^2 &= \frac{b + (c_{11} - c_{44})u}{bc_{44} + (c_{11}c_{33} - c_{44}^2 - b^2)u + bc_{44}u^2} \end{aligned}\right\} \tag{15}$$

and from Eq. (14)

$$\left.\begin{aligned} \xi^2 &= \frac{(c_{33} - c_{44})u_1 + bu_1^2}{bc_{33} + (c_{11}c_{33} - c_{44}^2 + b^2)u_1 + bc_{11}u_1^2} \\ \zeta^2 &= \frac{b + (c_{11} - c_{44})u_1}{bc_{33} + (c_{11}c_{33} - c_{44}^2 + b^2)u_1 + bc_{11}u_1^2} \end{aligned}\right\} \tag{16}$$

First, we insert into Eq. (10) the values for ξ and ζ that follow from Eq. (15). For brevity, we write

$$c_{11} - c_{44} = e; \quad c_{33} - c_{44} = g; \quad c_{11}c_{33} - b^2 = R^2$$

and obtain

$$\left.\begin{aligned} x &= \sqrt{\frac{gu + bu^2}{bc_{44} + (R^2 - c_{44}^2)u + bc_{44}u^2}}\left(c_{44} + \frac{R^2}{g + 2bu + eu^2}\right) \\ z &= \sqrt{\frac{b + eu}{bc_{44} + (R^2 - c_{44}^2)u + bc_{44}u^2}}\left(c_{44} + \frac{R^2u^2}{g + 2bu + eu^2}\right) \end{aligned}\right\} \tag{17}$$

This is the required parametric representation of the second branch of the wave surface. [There seems to be a slip of the pen: it follows from Eq. (15) that the second term in the denominator of the radicand is $c_{11}c_{33}-c_{44}^2-b^2 \neq R^2-b^2$. The error does not invalidate any conclusion, but affects the detailed shape of the wave surface and the numerical values in Table 1 (in Rudzki's version, the cusps are exaggerated). In Helbig (1958), the calculations were repeated with the correct expressions.]

So little is known about the elastic constants of rocks (moreover, one should presumably distinguish between static and dynamic constants) that we are forced to take an arbitrary example. As a guideline, we take the elastic constants of beryl. According to W. Voigt (in his units), beryl has the constants c_{11}=27460, c_{33}=24090, c_{44}=6660, c_{13}=6740, c_{66}=8830. We assume (in different units) round values that stand in relations close to those of the constants of beryl. Thus, we deal with an arbitrary ideal medium that is somewhat similar to beryl and that has the following constants: c_{11}=10, c_{33}=8, c_{44}=c_{13}=2. We have b=c_{13}+c_{44}=4, e=c_{11}−c_{44}=8, g=c_{33}−c_{44}=6, R^2=eg−b^2=32. c_{66} does not occur in Eq. (17).

With these numerical values, one finds after a few reductions

$$\left.\begin{aligned} x &= \sqrt{\frac{6u + 4u^2}{2 + 15u + 2u^2}}\left(1 + \frac{8}{3 + 4u + 4u^2}\right) \\ z &= \sqrt{\frac{4 + 8u}{2 + 15u + 2u^2}}\left(1 + \frac{8u^2}{3 + 4u + 4u^2}\right) \end{aligned}\right\} \tag{18}$$

To obtain the curve in a quadrant, it is sufficient to let u vary between 0 and ∞. For u, one cannot use negative values, since then either x or z or both would be imaginary. Coordinates in other quadrants are obtained by mere changes of the algebraic sign.

Table 1

u	x	z
0.000	0.0000	1.4142
0.001	0.1998	1.4104
0.01	0.9646	1.3777
0.1	1.4180	1.1949
0.2	1.5627	1.1348
0.25	1.5731	1.1295
0.3	1.5663	1.1334
0.4	1.5274	1.1575
0.5	1.4757	1.1926
1.0	1.2531	1.3727
2.0	1.0846	1.5445
3.0	1.0544	1.5829
3.5	1.0554	1.5815
4.0	1.0608	1.5732
4.5	1.0686	1.5608
5.0	1.0775	1.5457
6.0	1.0966	1.5114
10.0	1.1638	1.3707
20.0	1.2553	1.1193
100.0	1.3744	0.5763
∞	1.4142	0.0000

We give the coordinates of a few points in the first quadrant together with the corresponding values of the parameter u.

From these values and Fig. 1, one sees that the curve has four double points and eight cusps. For a cusp to exist, simultaneously

$$\frac{\mathrm{d}x}{\mathrm{d}u}=0 \text{ and } \frac{\mathrm{d}z}{\mathrm{d}u}=0$$

must hold. If one combines this condition with Eq. (10), one finds that both condition reduce to the single equation

$$\frac{\mathrm{d}\zeta}{\mathrm{d}u}\frac{\mathrm{d}^2\xi}{\mathrm{d}u^2}-\frac{\mathrm{d}\xi}{\mathrm{d}u}\frac{d^2\zeta}{\mathrm{d}u^2}=0 \tag{19}$$

Substitution into Eq. (19) of ξ and ζ from Eq. (14), one obtains after a few obvious operations

$$a_0u^6+a_1u^5+a_2u^4+a_3u^3+a_4u^2+a_5u^1+a_6=0, \tag{20}$$

where

$$a_0=e^3+e^2\frac{R}{c_{44}},$$

$$a_1=6be^2,$$

$$a_2=3e\left(eg+4b^2-\frac{R^2}{c_{44}}(e+g)-\frac{R^4}{c_{44}^2}\right)-3eg\frac{R^2}{c_{44}},$$

$$a_3=12egb+8b^3-\frac{4R^2}{c_{44}}(e+g)\frac{eg+b^2}{b}-2R^2\frac{2eg+b^2}{b},$$

$$a_4=3g\left(eg+4b^2-\frac{R^2}{c_{44}}(e+g)-\frac{R^4}{c_{44}^2}\right)-3eg\frac{R^2}{c_{44}},$$

$$a_5=6bg^2,$$

$$a_6=g^3+g^2\frac{R}{c_{44}}.$$

[The factor 2 in the last term of the expression for a_3 should be 1].

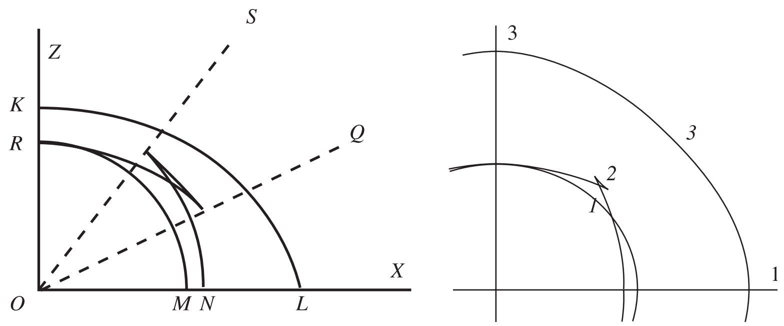

Fig. 1. Left, from Rudzki (1911). Right, from Helbig (1958).

In these equations, the former abbreviations have been kept: $b{=}c_{13}{+}c_{44}$, $e{=}c_{11}{-}c_{44}$, $g{=}c_{33}{-}c_{44}$, $R^2{=}eg{-}b^2$.

Eq. (20) is of degree 6 in u; thus, it can have [at most] six distinct roots. In view of the symmetry, each of the roots corresponds to four cusps. Thus, the curve has [not more than] 24 cusps, as we found earlier from Plücker's equations. With the assumed numbers and after division by 8, one finds $a_0{=}a_1{=}192$, $a_2{=}1392$, $a_3{=}3232$, $a_4{=}{-}1116$, $a_5{=}108$, $a_6{=}99$. Further, one finds that the equation has only two positive real roots, viz, $u_1{=}0.2505$ and $u_2{=}3.159$. To these two real positive roots of Eq. (20) thus correspond eight real cusps of the curve. The remaining roots correspond to imaginary cusps, since the other branch has no cusps (compare Section 5).

These are the conditions for the numerical example. But another case is possible: If Eq. (20) has no real positive roots, then not only has the branch no cusps, but also no double points if Eq. (20) has no real roots between $u{=}0$ and $u{=}\infty$, then $\mathrm{d}x/\mathrm{d}u$ and $\mathrm{d}z/\mathrm{d}u$ cannot change sign. x increases monotonously and z decreases monotonously, so that neither x nor z can assume the same value a second time.

The case $R{=}eg{-}b^2{=}0$ is worth noting. In this case, Eq. (20) reduces to

$$(\sqrt{e}u+\sqrt{g})^6=0,$$

which certainly has not real positive root. The branch of the curve degenerates to a circle:

$$x^2+z^2=C_{44},$$

i.e., the corresponding wave surface becomes a sphere. The propagation velocity has in all directions the same value: $\sqrt{c_{44}}$. In this case, Eq. (8) decouples into two equations, viz.,

$$c_{44}(\xi^2+\zeta^2)-1=0 \text{ and } c_{11}\xi^2+\zeta_{33}^2-1=0.$$

$R{=}0$ occurs, e.g., if the medium propagates dilatational oscillations separately from the torsional oscillations [pure longitudinal and transverse waves in all directions]. For a transversely isotropic medium, that happens if

$$c_{33}=c_{11} \text{ and } c_{13}=c_{11}-2c_{44}.$$

With these conditions, one has simultaneously $g{=}e{=}b$, and thus $R{=}0$. In this case, the number of independent elastic constants is reduced from five to three; however, in general, $R{=}0$ implies only the reduction of the number of constants from five to four.

Of course, it is possible that $R{\neq}0$ and all five constants are independent, but that nevertheless Eq. (20) has *no* real positive root. In this case, the second branch of the curves is a smooth oval.

5. The third branch of the meridian curve. Algebraic signs of the elastic constants

We take the values of and ζ^2 from Eq. (16), substitute them into Eq. (10), and obtain

$$\left.\begin{aligned} x&=\sqrt{\frac{gu_1+bu_1^2}{bc_{33}+(c_{11}c_{33}+b^2-c_{44}^2)u_1+bc_{11}u_1^2}\left(c_{11}-\frac{R^2}{g+2bu_1+eu_1^2}\right)}\\ z&=\sqrt{\frac{b+eu_1}{bc_{33}+(c_{11}c_{33}+b^2-c_{44}^2)u_1+bc_{11}u_1^2}\left(c_{33}-\frac{R^2u_1^2}{g+2bu_1+eu_1^2}\right)}\end{aligned}\right\} \tag{21}$$

As in the previous section, one needs only the curve in one quadrant. If one uses the same values for u as in Section 4, one obtains the following triplets (Table 2).

Thus, this third branch of the curve is an oval without cusps and double points, looking somewhat like an ellipse. In the same manner, as for the second branch, we can form the equation

$$\zeta\frac{\mathrm{d}^2\xi}{\mathrm{d}u^2}-\xi\frac{\mathrm{d}^2\zeta}{\mathrm{d}u^2}=0$$

Of course, it is of degree 6 in u_1

$$a_0u_1^6+a_1u_1^5+a_2u_1^4+a_3u_1^3+a_4u_1^2+a_5u_1^1+a_6=0, \tag{22}$$

Table 2

u	x	z
0	0.0000	2.8284
1/100	0.2029	2.1860
1/10	0.6634	2.7017
1/4	1.0878	2.5082
1/3	1.2694	2.4041
1/2	1.5635	2.2111
2/3	1.7903	2.0418
10/11	2.0366	1.8365
1	2.1102	1.7706
10/9	2.1892	1.6970
2	2.5684	1.2984
4	2.8528	0.9146
10	3.0369	0.5715
∞	3.1623	0.0000

but the coefficients a_0, etc. have now another meaning, viz.:

$$a_0 = e^3 - e^2\frac{R}{c_{33}},$$

$$a_1 = 6be^2,$$

$$a_2 = 3e\left(eg + 4b^2 + R^2\left(\frac{e}{c_{11}} + \frac{g}{c_{33}}\right) - \frac{R^4}{c_{11}c_{33}}\right) + \frac{3R^2eg}{c_{33}},$$

$$a_3 = 12egb + 8b^3 + 4R^2\left(\frac{e}{c_{11}} + \frac{g}{c_{33}}\right)\frac{eg + b^2}{b} - \frac{2R^4}{c_{11}c_{33}}\,\frac{2eg + b^2}{b},$$

$$a_4 = 3g\left(eg + 4b^2 + R^2\left(\frac{e}{c_{11}} + \frac{g}{c_{33}}\right) - \frac{R^4}{c_{11}c_{33}}\right) + \frac{3R^2eg}{c_{11}},$$

$$a_5 = 6bg^2,$$

$$a_6 = g^3 - g^2\frac{R}{c_{11}}.$$

The roots of Eq. (12) must be closely related to the roots of Eq. (20). Each equation must be a transform of the other, since the entire curve (8), i.e., the second and the third branch together, have 24 imaginary and real cusps. But both Eqs. (20) and (22) have six roots, and in view of the symmetry each root corresponds to four cusps. Thus, a single equation is sufficient to determine all real and imaginary cusps (in our example, the two positive roots of Eq. (20) correspond to two negative roots of Eq. (22)). By the way, for R=0, Eq. (22) reduces to the same equation

$$(\sqrt{e}u_1 + \sqrt{g})^6 = 0,$$

as Eq. (20). At the same time, the third branch reduces to an ellipse with the semiaxes $\sqrt{c_{11}}$ and $\sqrt{c_{33}}$.

With the numerical values for the constants used earlier, one obtains—after removal of a common factor:

$$a_0 = 160, \quad a_1 = 960, \quad a_2 = 2592, \quad a_3 = 3296,$$

$$a_4 = 1962, \quad a_5 = 540, \quad a_6 = 63.$$

Thus Eq. (22) has in this case certainly no positive real root, and thus the third branch of the curve has certainly neither cusps nor double points, a result that confirms the statement made in connection with Table 2. But the coefficients of Eq. (22) are all positive not only in this special case, but one can state generally that this occurs frequently. We shall give the reasons for this.

First, the elastic constants must satisfy certain inequalities that follow from the condition that the elastic potential must be always positive. Thus, the coefficients of the squares

$$e_{yz}^2, \quad e_{zx}^2, \quad e_{xy}^2$$

must be positive, i.e., one must have

$$c_{44} > 0, \quad c_{66} > 0.$$

Further, the quadratic form

$$c_{11}(e_{xx}+e_{yy})^2+c_{33}e_{zz}^2+2c_{13}e_{zz}(e_{xx}+e_{yy}) - 4c_{66}e_{xx}e_{yy}$$

must be always positive. It is known from the theory of orthogonal substitutions that this form can be converted to the sum of squares

$$\lambda_1 e_{x_1x_1}^2+\lambda_2 e_{y_1y_1}^2+\lambda_3 e_{z_1z_1}^2$$

Thus, the three coefficients λ_1, λ_2, λ_3 must be real and positive. These coefficients are the roots of the equation

$$\begin{vmatrix} c_{11}-\lambda & c_{11}-2c_{66} & c_{13} \\ c_{11}-2c_{66} & c_{11}-\lambda & c_{13} \\ c_{13} & c_{13} & c_{33}-\lambda \end{vmatrix}=0$$

After expanding and solving of this equation, one finds

$$\lambda_1=2c_{66},$$

$$\lambda_2=c_{11}-c_{66}+\frac{1}{2}c_{33} + \sqrt{\left(c_{11}-c_{66}+\frac{1}{2}c_{33}\right)^2-2(c_{33}(c_{11}-c_{66})-c_{13}^2)},$$

$$\lambda_3=c_{11}-c_{66}+\frac{1}{2}c_{33} - \sqrt{\left(c_{11}-c_{66}+\frac{1}{2}c_{33}\right)^2-2(c_{33}(c_{11}-c_{66})-c_{13}^2)}.$$

We have found already that c_{66} is positive; thus we need only determine the conditions under which λ_2 and λ_3 are positive. Since the function under the square root is essentially positive- it can be written in the form

$$\left(c_{11}-c_{66}-\frac{1}{2}c_{33}\right)^2+2c_{13}^2,$$

-there only remain the conditions

$$c_{11}-c_{66}+\frac{1}{2}c_{33}>0 \text{ and } c_{33}(c_{11}-c_{66})-c_{13}^2>0.$$

These can be satisfied if and only if

$$c_{33}>0 \text{ and } c_{11}-c_{66}>0.$$

But since c_{66}>0, there also must be c_{11}>0.

Finally, one obtains the following conditions:

$$c_{44}>0,\quad c_{66}>0\quad c_{11}>0,\quad c_{33}>0,$$
$$c_{11}-c_{66}>0,\quad c_{33}(c_{11}-c_{66})-c_{13}^2>0.$$

Only the constant c_{13} and therefore also b=c_{13}+c_{44} can be negative.

Further information is based on experience. Generally, one has not only

$$c_{11}-c_{66}>0,$$

but also

$$c_{33}-c_{66}>0,\quad c_{11}-c_{44}>0,\quad c_{33}-c_{44}>0.$$

Thus, also e and g are generally positive. Further, experience shows that b and R^2 are frequently positive. But if e, g, b, R^2, c_{44}, c_{33}, and c_{11} are positive, all coefficients of Eq. (22) are positive. This is easily seen if one converts the coefficients that contain negative terms into sums of positive terms. One can write

$$a_0=\frac{e^2}{c_{44}}(ec_{44}+b^2),$$

$$a_2 = 3e\left(eg + 4b^2 + \frac{R^2}{c_{11}c_{33}}(eg + c_{44}(e+g) + b^2)\right) + \frac{3R^2 eg}{c_{33}},$$

$$a_3 = 12egb + 8b^3 + \frac{2R^2}{c_{11}c_{33}b} \times (2e^2g^2 + 5egb^2 + b^4 + 2c_{44}(e+g) \times (eg + b^2)),$$

$$a_4 = 3g\left(eg + 4b^2 + \frac{R^2}{c_{11}c_{33}}(eg + c_{44}(e+g) + b^2)\right) + \frac{3R^2 eg}{c_{11}},$$

$$a_6 = \frac{g^2}{c_{11}}(gc_{44} + b^2).$$

[The above argument is not conclusive. It is correct that all terms *appear* to contain only squares, but $R^2=(c_{11}-c_{44})(c_{33}-c_{44})-(c_{13}+c_{44})^2$ is a square only for reasons of dimension. Nothing general can be said about its algebraic sign].[1]

[1] Translator's note on Section 5: Costanzi's (1901) suggestion to express the solution of the Christoffel equation in a parametric form was a significant advance. At a time when all numerical calculation had to be done by hand, the standard expressions were too complicated for a determination of the wave surface. When Rudzki followed the suggestion, he made considerable progress in the investigation of seismic anisotropy.

In hindsight, it is a pity that the parameter was established only formally (see Eqs. (12) and (13)), because in this way, the physical meaning of the parameter u got lost. It is easy to see that Rudzki's parameters are u=tan β tan α and u_1=tan β cot α, where β is the angle between wave normal and axis of symmetry and α is the angle between polarization direction and the axis of symmetry. For u=u_1, the propagation direction is the same and the polarization direction differs by 90°, as it should for qP- and qS waves (see Helbig, 1966). Moreover, as Khatkevich (1964) pointed out, the restriction to positive values of u is arbitrary. Some negative parameter values make indeed one of the slowness components imaginary (and thus tan β imaginary), but these values refer to evanescent waves. There is always a range of negative parameters that yield qS waves if inserted into the qP equation, and qP waves if inserted into the qS equation. While it is more convenient to use different equations for the two wave types, the joint formulation highlights the close relationship between the two wave types with in-plane polarization. For a thorough discussion of these aspects, see Chapter 6 of Helbig (1994).

6. The wave surface in a medium with three planes of symmetry

The elastic potential of such a medium contains nine (elastic) constants. It has the form

$$\begin{aligned} 2W = c_{11}e_{xx}^2 \quad &+2c_{12}e_{xx}e_{yy} \quad +2c_{13}e_{xx}e_{zz} \\ &+c_{22}e_{yy}^2 \quad +2c_{23}e_{yy}e_{zz} \\ &+c_{33}e_{zz}^2 \quad +c_{44}e_{yz}^2 + c_{55}e_{xz}^2 + c_{22}e_{xy}^2 \end{aligned} \tag{23}$$

The stress components are

$$\begin{aligned} X_x &= c_{11}e_{xx} + c_{12}e_{yy} + c_{13}e_{zz}, \\ Y_y &= c_{12}e_{xx} + c_{22}e_{yy} + c_{23}e_{zz}, \\ Z_z &= c_{13}e_{xx} + c_{23}e_{yy} + c_{33}e_{zz}, \\ Z_y &= Y_z = c_{44}e_{yz}, \\ X_z &= Z_x = c_{55}e_{xz}, \\ Y_x &= X_y = c_{66}e_{xy}. \end{aligned} \tag{24}$$

The meaning of the symbols e_{xz}, etc. was already given in Section 2. With operations as in Section 2, one successively obtains equations that are similar to Eqs. (3), (4), and (4 bis), but with a different meaning of the symbols H_{11} ... and h_{11} ... H_{11} need not be given, but the new meaning of h_{11} ... must be listed:

$$\begin{aligned} h_{11} &= c_{11}\xi^2 + c_{66}\eta^2 + c_{55}\zeta^2, \\ h_{22} &= c_{66}\xi^2 + c_{22}\eta^2 + c_{44}\zeta^2, \\ h_{33} &= c_{55}\xi^2 + c_{44}\eta^2 + c_{33}\zeta^2, \\ h_{12} &= (c_{12} + c_{66})\xi\eta = c\xi\eta, \\ h_{13} &= (c_{13} + c_{55})\xi\zeta = b\xi\zeta, \\ h_{23} &= (c_{23} + c_{44})\eta\zeta = a\eta\zeta. \end{aligned} \tag{25}$$

Further, we obtain the wave surface (6) in the same way as in Section 2. However, we write these equations in slightly different form by substituting for h_{12}, h_{13}, and h_{23} the expressions from Eq. (25). In this way, we obtain the equation of the wave surface in the form

$$\begin{vmatrix} h_{11}-1 & c\xi\eta & b\xi\zeta \\ c\xi\eta & h_{22}-1 & a\eta\zeta \\ b\xi\zeta & a\eta\zeta & h_{33}-1 \end{vmatrix} = 0, \tag{26}$$

or equivalently

$$\begin{vmatrix} h_{11}-1 & c\eta^2 & b\zeta^2 \\ c\xi^2 & h_{22}-1 & a\zeta^2 \\ b\xi^2 & a\eta^2 & h_{33}-1 \end{vmatrix} = 0, \tag{27}$$

The entire analysis in the following rests on the identity of Eqs. (26) and (27), since Eq. (27) leads to much simpler expressions. Eq. (27) can be written in three ways

$$\left.\begin{aligned} &(h_{11}-1)M_{11} + c\eta^2 M_{12} + b\zeta^2 M_{13} = 0 \\ &c\xi^2 M_{21} + (h_{22}-1)M_{22} + a\zeta^2 M_{23} = 0 \\ &b\xi^2 M_{31} + a\eta^2 M_{23} + (h_{33}-1)M_{33} = 0 \end{aligned}\right\} \tag{28}$$

where M_{11} ... are the minors of the determinant in Eq. (27). Each of the three equations in Eq. (28) can be used to determine another sheet of the wave surface. Take, e.g., the first equation. We replace it by the system of three equations

$$\left.\begin{aligned} &M_{11} + M_{12}u + M_{13}v = 0 \\ &c\eta^2 - (h_{11}-1)u = 0 \\ &b\zeta^2 - (h_{11}-1)v = 0 \end{aligned}\right\} \tag{29}$$

which give the parametric representation of the first sheet. Similarly, the system replacing Eq. (28.2) is the parametric representations of the second sheet and the system replacing Eq. (28.3) is the parametric representations of the third sheet.

By substitution of h_{11} from the first Eq. (25), one obtains from the second and third Eq. (29)

$$\eta^2 = \frac{c_{11}\xi^2 - 1}{F} bu, \quad \zeta^2 = \frac{c_{11}\xi^2 - 1}{F} cv, \tag{30}$$

where

$$F = bc - c_{66}bu - c_{55}cv.$$

On the other hand, the first Eq. (29) can be written as

$$(h_{22}-1)(h_{33}-1) - a^2\eta^2\zeta^2 + \xi^2((ab\zeta^2 - c(h_{33}-1))u + (ac\eta^2 - b(h_{22}-1))v) = 0.$$

On substituting and h_{33} from Eq. (25) and eliminating ξ^2 and η^2 with the help of Eq. (30), one obtains

$$A\xi^4 - 2B\xi^2 + C = 0, \tag{31}$$

where

$$\left.\begin{aligned} &A = F^2\alpha + c_{11}FL + c_{11}^2 M, \\ &2B = F^2\beta + c_{11}F(L + c_{11}(R+S)) + 2c_{11}M, \\ &C = F^2 + F(R+S) + M. \end{aligned}\right\} \tag{32}$$

In these expressions, the following abbreviations have been used:

$$\left.\begin{aligned} &F = bc - c_{66}bu - c_{55}cv, \\ &\alpha = c_{55}c_{66} - c_{55}cu - c_{66}bv, \\ &\beta = c_{66} - cu + c_{55} - bv, \\ &R = c_{22}bu + c_{44}cv, \\ &S = c_{44}bu + c_{33}cv, \\ &L = R(c_{55} - bv) + S(c_{66} - cu) + 2abcuv, \\ &M = RS - a^2bcuv. \end{aligned}\right\} \tag{33}$$

From Eq. (31), it follows that

$$\xi^2 = \frac{B \pm \sqrt{B^2 - AC}}{A}$$

It is easily shown that in $B^2 - AC$ all terms without the factor F and all terms that contain F only in the first power cancel. Thus, one can write

$$\xi^2 = \frac{B \pm F\sqrt{Q}}{A}, \tag{34}$$

where

$$Q = \left(\frac{1}{4}\beta^2 - \alpha\right)F^2 + \left(\left(\frac{1}{2}\beta - c_{11}\right)L + \left(\frac{1}{2}\beta c_{11} - \alpha\right)(R+S)\right)F + \frac{1}{4}(L - c_{11}(R+S))^2 - (\alpha - c_{11}\beta + c_{11}^2)M. \tag{35}$$

A and B are polynomials of degree 3 in u and v, Q is a polynomial of degree 4 in u and v, and F is linear in u and v.

If one now substitutes ξ^2 from Eq. (34) into Eq. (30), one obtains

$$\eta^2 = \frac{bu}{F}\,\frac{c_{11}B - A \pm c_{11}F\sqrt{Q}}{A},$$
$$\zeta^2 = \frac{cv}{F}\,\frac{c_{11}B - A \pm c_{11}F\sqrt{Q}}{A}$$

One sees immediately that terms without the factor F in the difference $c_{11}B - A$ cancel, so that $c_{11}B - A$ is divisible by F. If we set

$$\frac{c_{11}B - A}{F} = T = \left(\frac{1}{2}c_{11}\beta - \alpha\right)F - \frac{1}{2}c_{11}(L - c_{11}(R+S)),$$

we can write

$$\eta^2 = \frac{T \pm c_{11}\sqrt{Q}}{A}bu,$$
$$\zeta^2 = \frac{T \pm c_{11}\sqrt{Q}}{A}cv.$$

However, it follows from Eq. (35) and the first Eq. (32) that

$$c_{11}^2 Q = T^2 - A\gamma, \tag{37}$$

where

$$\gamma = \alpha - c_{11}\beta + c_{11}^2 = (c_{11} - c_{55})(c_{11} - c_{66}) + (c_{11} - c_{55})cu + (c_{11} - c_{66})bv. \tag{38}$$

Thus, we finally get the following expressions for ξ^2, η^2, ζ^2 in terms of the two parameters u, v:

$$\left.\begin{aligned}
\xi^2 &= \frac{B \pm F\sqrt{Q}}{A} = \frac{A + TF \pm \sqrt{T^2 - A\gamma}}{c_{11}A}, \\
\eta^2 &= \frac{T \pm \sqrt{T^2 - A\gamma}}{A}bu, \\
\zeta^2 &= \frac{T \pm \sqrt{T^2 - A\gamma}}{A}cv;
\end{aligned}\right\} \tag{39}$$

T is a polynomial of second degree in u and v.

7. Parametric representation of the first sheet

The wave surface is the envelope of all planes

$$lx + my + nz - V(t - t_0) = 0,$$

where $t - t_0$ has a fixed value and V the velocity of propagation. If one divides the equation of the plane by the propagation distance $V(t - t_0)$ and sets

$$\xi = \frac{l}{V(t - t_0)}, \quad \eta = \frac{m}{V(t - t_0)}, \quad \zeta = \frac{n}{V(t - t_0)},$$

it takes the form

$$x\xi + y\eta + z\zeta - 1 = 0,$$

and the envelope is determined by the system of equations:

$$\left.\begin{aligned} &x\xi + y\eta + z\zeta = 1,\\ &x\frac{\partial \xi}{\partial u} + y\frac{\partial \eta}{\partial u} + z\frac{\partial \zeta}{\partial u} = 0,\\ &x\frac{\partial \xi}{\partial v} + y\frac{\partial \eta}{\partial v} + z\frac{\partial \zeta}{\partial v} = 0,\end{aligned}\right\} \tag{40}$$

where u and v are two independent parameters. From Eq. (40), we get for x, y, z the expressions

$$x = \frac{D_\xi}{D}, \quad y = \frac{D_\eta}{D}, \quad z = \frac{D_\zeta}{D}, \tag{41}$$

where

$$\left.\begin{aligned} &D_\xi = \frac{\partial \eta}{\partial u}\frac{\partial \zeta}{\partial v} - \frac{\partial \zeta}{\partial u}\frac{\partial \eta}{\partial v}\\ &D_\eta = \frac{\partial \zeta}{\partial u}\frac{\partial \xi}{\partial v} - \frac{\partial \xi}{\partial u}\frac{\partial \zeta}{\partial v}\\ &D_\zeta = \frac{\partial \xi}{\partial u}\frac{\partial \eta}{\partial v} - \frac{\partial \eta}{\partial u}\frac{\partial \xi}{\partial v}\\ &D = \xi D_\xi + \eta D_\eta + \zeta D_\zeta.\end{aligned}\right\} \tag{42}$$

In view of Eq. (39), which contains only squares of ξ, η, ζ, the coordinates x, y, z should also be expressed in terms of squares of ξ, η, ζ and their derivatives. To this end, one multiplies in Eq. (41) the numerators and the denominators with ξ, η, ζ, respectively, and obtains

$$x = \frac{\xi P_\xi}{P}, \quad y = \frac{\eta P_\eta}{P}, \quad z = \frac{\zeta P_\zeta}{P}, \tag{43}$$

where

$$\left.\begin{aligned} &P_\xi = \eta\frac{\partial \eta}{\partial u}\zeta\frac{\partial \zeta}{\partial v} - \zeta\frac{\partial \zeta}{\partial u}\eta\frac{\partial \eta}{\partial v}\\ &P_\eta = \zeta\frac{\partial \zeta}{\partial u}\xi\frac{\partial \xi}{\partial v} - \xi\frac{\partial \xi}{\partial u}\zeta\frac{\partial \zeta}{\partial v}\\ &P_\zeta = \xi\frac{\partial \xi}{\partial u}\eta\frac{\partial \eta}{\partial v} - \eta\frac{\partial \eta}{\partial u}\xi\frac{\partial \xi}{\partial v}\\ &P = \xi^2 P_\xi + \eta^2 P_\eta + \zeta^2 P_\zeta.\end{aligned}\right\} \tag{44}$$

Now one has to substitute the values of ξ^2, η^2, ζ^2 from Eq. (39) into Eq. (44). If one uses the abbreviations

$$q = \frac{T \pm \sqrt{T^2 - A\gamma}}{A}, \tag{45}$$

writes Eq. (39) in the form

$$\xi^2 = \frac{1 + qF}{c_{11}}, \quad \eta^2 = buq, \quad \zeta^2 = cvq, \tag{46}$$

and substitutes these expressions in Eq. (44), one obtains

$$\left.\begin{aligned} &P_\xi = \frac{1}{4}bcq\left(q + u\frac{\partial q}{\partial u} + v\frac{\partial q}{\partial v}\right)\\ &P_\eta = \frac{1}{4}\frac{bcq}{c_{11}}\left(c_{66}\left(q + u\frac{\partial q}{\partial u} + v\frac{\partial q}{\partial v}\right) - c\frac{\partial q}{\partial u}\right)\\ &P_\zeta = \frac{1}{4}\frac{bcq}{c_{11}}\left(c_{55}\left(q + u\frac{\partial q}{\partial u} + v\frac{\partial q}{\partial v}\right) - b\frac{\partial q}{\partial v}\right)\\ &P = \frac{1}{4}\frac{bcq}{c_{11}}\left(q + u\frac{\partial q}{\partial u} + v\frac{\partial q}{\partial v} + bcq^2\right).\end{aligned}\right\} \tag{47}$$

The right-hand sides of the Eq. (47) contain only u and v; if one introduces these expressions into Eq. (43) and replaces ξ, η, ζ by the square roots of the right-hand sides of Eq. (46), one obtains

$$\left.\begin{aligned} &x = \sqrt{1 + qF}\,\frac{\sqrt{c_{11}}\left(q + u\frac{\partial q}{\partial u} + v\frac{\partial q}{\partial v}\right)}{q + u\frac{\partial q}{\partial u} + v\frac{\partial q}{\partial v} + bcq^2}\\ &y = \sqrt{buq}\,\frac{c_{66}\left(q + u\frac{\partial q}{\partial u} + v\frac{\partial q}{\partial v}\right) - c\frac{\partial q}{\partial u}}{q + u\frac{\partial q}{\partial u} + v\frac{\partial q}{\partial v} + bcq^2}\\ &z = \sqrt{cvq}\,\frac{c_{55}\left(q + u\frac{\partial q}{\partial u} + v\frac{\partial q}{\partial v}\right) - b\frac{\partial q}{\partial n}}{q + u\frac{\partial q}{\partial u} + v\frac{\partial q}{\partial v} + bcq^2}\end{aligned}\right\} \tag{48}$$

However, q is itself a fraction; if one introduces its value

$$q=\frac{T\pm\sqrt{T^2-A\gamma}}{A},$$

removes the denominators and uses the abbreviations

$$\left.\begin{aligned}
&M_1=A\frac{\partial T}{\partial u}-T\frac{\partial A}{\partial u},\\
&M_2=A\frac{\partial T}{\partial v}-T\frac{\partial A}{\partial v},\\
&M_3=2T^2-A\gamma,\\
&M=AT+uM_1+vM_2,\\
&N_1=AT\frac{\partial T}{\partial u}-\left(T^2-\frac{1}{2}A\gamma\right)\frac{\partial A}{\partial u}-\frac{1}{2}A^2\frac{\partial\gamma}{\partial u},\\
&N_2=AT\frac{\partial T}{\partial v}-\left(T^2-\frac{1}{2}A\gamma\right)\frac{\partial A}{\partial v}-\frac{1}{2}A^2\frac{\partial\gamma}{\partial v},\\
&N_3=2T(T^2-A\gamma),\\
&N=A(T^2-A\gamma)+uN_1+vN_2,\\
&\text{finally } H=M\sqrt{T^2-A\gamma}\pm N+bc\left(M_3\sqrt{T^2-A\gamma}\pm N_3\right);
\end{aligned}\right\} \tag{49}$$

one can write the Eq. 49 [Eq. (48)] in the following form:

$$\left.\begin{aligned}
&x=\sqrt{\frac{A+TF\pm F\sqrt{T^2-A\gamma}}{A}}\sqrt{c_{11}}\frac{M\sqrt{T^2-A\gamma}\pm N}{H}\\
&y=\sqrt{\frac{bu}{A}(T\pm\sqrt{T^2-A\gamma})}\frac{c_{66}(M\sqrt{T^2-A\gamma}\pm N)-c(M_1\sqrt{T^2-A\gamma}\pm N_1)}{H}\\
&z=\sqrt{\frac{av}{A}(T\pm\sqrt{T^2-A\gamma})}\frac{c_{55}(M\sqrt{T^2-A\gamma}\pm N)-b(M_2\sqrt{T^2-A\gamma}\pm N_2)}{H}.
\end{aligned}\right\} \tag{50}$$

Here γ and F are linear in u and v, T is of second degree, A of third degree, M_1, M_2, M_3 and $T^2-A\gamma$ of fourth degree, M of fifth degree, N_1, N_2, N_3 of sixth degree, and N and H of seventh degree. Thus, under the square root, we have fractions where numerator and denominator are of degree 3, and outside the square roots fractions with numerator and denominator are of degree 7.

With the help of the expressions of the last paragraph, one could replace the symbols M, N, T, etc., by polynomials in u and v. However, the resulting expressions would be too complicated.

By cyclic substitution, one can obtain from the expressions for the first sheet the corresponding expressions for the second and the third sheet.

8. Algebraic signs of the elastic constants. Condition for separate propagation of dilatational and torsional waves

Obviously, the expressions under the square root must be positive to make x, y, and z real; but the algebraic signs of these expressions depend in turn on the signs of the coefficients and ultimately on the signs of the elastic constants.
The condition the potential W (see Eq. (23)) must be positive requires that

$$c_{44}>0,\quad c_{55}>0,\quad c_{66}>0$$

In addition, the quadratic form

$$\begin{aligned}
c_{11}e_{xx}^2&+2c_{12}e_{xx}e_{xx}+2c_{13}e_{xx}e_{zz}\\
&+c_{22}e_{yy}^2+2c_{23}e_{yy}e_{zz}\\
&\quad+c_{33}e_{zz}^2
\end{aligned}$$

must be always positive. It is known that this quadratic form can, by an orthogonal substitution, be converted into the sum of squares

$$\lambda_1e_{x_1x_1}^2+\lambda_2e_{x_2x_2}^2+\lambda_3e_{x_3x_3}^2$$

in which all three coefficients λ_1, λ_2, λ_3 obviously must be positive to make the form always positive. The coefficients λ_1, λ_2, λ_3 are the roots of the equation

$$\begin{vmatrix} c_{11}-\lambda & c_{12} & c_{13}\\ c_{12} & c_{22}-\lambda & c_{23}\\ c_{13} & c_{23} & c_{33}-\lambda \end{vmatrix}=0, \tag{51}$$

of which it is known that he roots are real if the constants c_{11}, etc. are real. There is no doubt that these constants are real; thus, λ_1, λ_2, λ_3 are certainly real; however, they must, in addition, be positive.

One can show easily the real λ_1, λ_2, λ_3 are positive if

$$\lambda_1 + \lambda_2 + \lambda_3 > 0,$$

$$\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 > 0,$$

$$\lambda_1\lambda_2\lambda_3 > 0.$$

But the left members of these inequalities can be expressed in terms of symmetric functions of the coefficients of Eq. (51). Thus, the conditions for positive λ_1, λ_2, λ_3 can be written as

$$c_{11} + c_{22} + c_{33} > 0,$$

$$c_{11}c_{22} + c_{22}c_{33} + c_{33}c_{11} - (c_{12}^2 + c_{23}^2 + c_{13}^2) > 0,$$

$$c_{11}c_{22}c_{33} + 2c_{12}c_{23}c_{13} - (c_{11}c_{23}^2 + c_{22}c_{13}^2 + c_{33}c_{12}^2) > 0. \tag{52}$$

The inequalities (52) can be satisfied if one of the three constants c_{11}, c_{22}, c_{33} is negative; but for physical reasons, all three are positive. The constants c_{12}, c_{23}, c_{13} can be positive or negative, but their absolute values must be smaller than the absolute values of the constants c_{11}, c_{22}, c_{33}. From experience, one should assume that the numerical values of the constants c_{44}, c_{55}, c_{66} are smaller than the numerical values of the positive constants c_{11}, c_{22}, c_{33}. In consequence, the differences $c_{11}-c_{55}$, $c_{11}-c_{66}$, etc. should be regarded as positive. On the other hand, the sums a=c_{23}+c_{44}, b=c_{13}+c_{55}, c=c_{23}+c_{44}, c_{12}+c_{66} can individually or collectively be negative.

In this occasion, we answer the question, in which relation the nine constants must stand if the medium is to propagate dilatational [longitudinal] oscillations separately from torsional [transverse] oscillations. One can find the relations, e.g., in the following way. One differentiates the Differential Equation (3) (valid for oscillations of small amplitude), the first with respect to x, the second with respect to y, and the third with respect to z. Then, one adds the three equation and finds the relation between the constants under which the differential equation thus obtained contains only the dilatation

$$\delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

and its derivatives. One finds that the following relations are necessary and sufficient:

$$c_{11} = c_{22} = c_{33} = c_{12} + 2c_{66} = c_{13} + 2c_{55}$$
$$= c_{23} + 2c_{44}.$$

We call the common value of these quantities k and write

$$c_{11} = c_{22} = c_{33} = k, \quad c_{12} = k - 2c_{66},$$
$$c_{13} = k - 2c_{55}, \quad c_{23} = k - 2c_{44}.$$

This reduces the differential equation in δ to

$$\frac{d^2\delta}{dt^2} = k\nabla^2\delta. \tag{53}$$

The simplification is significant: instead of nine, there are only four independent constants. Eq. (53) shows that the dilatational wave propagates in all direction with the common velocity

$$\sqrt{k}$$

i.e., that the wavefront is a sphere (one should not forget that we discuss waves that emanate from a point).

The shape of the torsional wave is less simple; it is the well-known Fresnel's wave surface of optically triaxial media. I do not think it likely that an elastic material, say a rock, would be isotropic with respect to dilatational deformations *and* anisotropic with respect to torsional deformations. Nevertheless, I shall give the equations corresponding to Fresnel's wave surface. They are inherently interesting and provide the occasion to test the method on a well-known and thoroughly investigated case.

9. Fresnel's wave surface

Since the waves are purely torsional, the dilatation δ must vanish. The condition δ=0 is written as

$$\lambda l + \mu m + \nu n = 0$$

or, in terms of the variables ξ, η, ζ,

$$\lambda\xi + \mu\eta + \nu\zeta = 0. \tag{54}$$

Let us write Eq. (4 bis) in extenso and use the relations determined in Section 8:

$$c_{11} = c_{22} = c_{33} = k,$$

$$c_{12} = k - 2c_{66}, \quad c_{13} = k - 2c_{55} = c_{23} = k - 2c_{44}.$$

We obtain

$$\begin{aligned}
&(k\xi^2 + c_{66}\eta^2 + c_{55}\zeta^2 - 1)\lambda + (k - c_{66})\xi\eta\mu + (k - c_{55})\xi\zeta\nu &= 0\\
&(k - c_{66})\xi\eta\lambda + (c_{66}\xi^2 + k\eta^2 + c_{44}\zeta^2 - 1)\mu + (k - c_{44})\eta\zeta\nu &= 0\\
&(k - c_{55})\xi\zeta\lambda + (k - c_{44})\eta\zeta\mu + (c_{55}\xi^2 + c_{44}\eta^2 + k\zeta^2 - 1)\nu &= 0
\end{aligned}$$

or, in view of Eq. (54),

$$\begin{aligned}
&(c_{66}\eta^2 + c_{55}\zeta^2 - 1)\lambda - c_{66}\xi\eta\mu - c_{55}\xi\zeta\nu = 0\\
&-c_{66}\xi\eta\lambda + (c_{66}\xi^2 + c_{44}\zeta^2 - 1)\mu + (-c_{44}\eta\zeta\nu) = 0\\
&-c_{55}\xi\zeta\lambda - c_{44}\eta\zeta\mu + (c_{55}\xi^2 + c_{44}\eta^2 - 1)\nu = 0.
\end{aligned} \tag{55}$$

These equations can exist together if the determinant vanishes:

$$\begin{vmatrix}
c_{66}\eta^2 + c_{55}\zeta^2 - 1 & -c_{66}\xi\eta & -c_{55}\xi\zeta\\
-c_{66}\xi\eta & c_{66}\xi^2 + c_{44}\zeta^2 - 1 & -c_{44}\xi\eta\\
-c_{55}\xi\zeta & -c_{44}\eta\zeta & c_{55}\xi^2 + c_{44}\eta^2 - 1
\end{vmatrix} = 0. \tag{56}$$

We replace this last equation by the equivalent equation

$$\begin{vmatrix}
c_{66}\eta^2 + c_{55}\zeta^2 - 1 & -c_{66}\eta^2 & -c_{55}\zeta^2\\
-c_{66}\xi^2 & c_{66}\xi^2 + c_{44}\zeta^2 - 1 & -c_{44}\xi^2\\
-c_{55}\xi^2 & -c_{44}\eta^2 & c_{55}\xi^2 + c_{44}\eta^2 - 1
\end{vmatrix} = 0. \tag{57}$$

Eqs. (56) and (57) are only apparently of degree 6, since all sixth-degree terms cancel. An equation of degree 4 remains; the equation of Fresnel's wave surface in line [plane] coordinates. It is

$$\begin{aligned}
&c_{55}c_{66}\xi^4 + c_{44}c_{66}\eta^4 + c_{44}c_{55}\zeta^4 + c_{44}(c_{55} + c_{66})\eta^2\zeta^2\\
&\quad + c_{55}(c_{44} + c_{66})\xi^2\zeta^2 + c_{66}(c_{44} + c_{55})\xi^2\eta^2\\
&\quad - (c_{55} + c_{66})\xi^2 - (c_{44} + c_{66})\eta^2 - (c_{44} + c_{55})\zeta^2\\
&\quad + 1 = 0.
\end{aligned} \tag{58}$$

With the methods of Section 6, one finds immediately

$$\left.\begin{aligned}
&\eta^2 = \frac{1}{c_{66}}\frac{u}{F}, \quad \zeta^2 = \frac{1}{c_{55}}\frac{v}{F},\\
&\text{and finally} \quad \xi^2 = \frac{1}{2c_{55}c_{66}F}(M \pm \sqrt{R}),
\end{aligned}\right. \tag{59}$$

where

$$\left.\begin{aligned}
F &= u + v + 1,\\
M &= (c_{66} - c_{44})u + (c_{55} - c_{44})v + c_{55} + c_{66},\\
R &= ((c_{66} - c_{44})u + (c_{55} - c_{44})v)^2\\
&\quad -2(c_{55} - c_{66})((c_{66} - c_{44})u\\
&\quad +(c_{55} - c_{44})v) + (c_{55} - c_{66})^2,
\end{aligned}\right\} \tag{60}$$

u and v are the two independent parameters.

With the methods of Section 7, one finds the following parametric representation of Fresnel's wave surface:

$$\left.\begin{aligned}
x &= \xi\frac{\pm 2c_{55}c_{66}\sqrt{R}}{(c_{55} - c_{66})L \pm (c_{55} + c_{66})\sqrt{R}}\\
y &= \eta c_{66}\frac{(c_{55} - c_{44})P \mp (c_{55} + c_{44})\sqrt{R}}{(c_{55} - c_{66})L \pm (c_{55} + c_{66})\sqrt{R}}\\
z &= \zeta c_{55}\frac{(c_{66} - c_{44})Q \mp (c_{66} + c_{44})\sqrt{R}}{(c_{55} - c_{66})L \pm (c_{55} + c_{66})\sqrt{R}}
\end{aligned}\right\} \tag{61}$$

where

$$\left.\begin{aligned}
L &= -(c_{66} - c_{44})u + (c_{55} - c_{44})v + c_{55} - c_{66},\\
P &= (c_{66} - c_{44})u + (2c_{66} - c_{44} - c_{55})v - c_{55} + c_{66},\\
Q &= (2c_{55} - c_{44} - c_{66})u + (c_{55} - c_{44})v + c_{55} - c_{66}.
\end{aligned}\right\} \tag{62}$$

The meaning of the other symbols was given in connection with Eqs. (59) and (60).

One can verify that the expressions in Eq. (61) satisfy the equation of Fresnel's wave surface:

$$(x^2+y^2+z^2)(c_{44}x^2+c_{55}y^2+c_{66}z^2)$$
$$-c_{44}(c_{55}+c_{66})x^2-c_{55}(c_{44}+c_{66})y^2$$
$$-c_{66}(c_{44}+c_{55})z^2+c_{44}c_{55}c_{66}=0. \tag{63}$$

The necessary calculations are too lengthy to be given here. Obviously, it is sufficient to let u and v vary between zero and infinity, since the factors ξ, η, ζ in the expressions in Eq. (61) are square roots and can assume both signs "+" and "−".

10. The line element of the wave surface

We will indicate briefly how the square of the line element of the wave surface can be determined. It is known that

$$ds^2 = Mdu^2+2Ndudv+Pdv^2, \tag{64}$$

where

$$\left.\begin{aligned} M &= \left(\frac{\partial x}{\partial u}\right)^2+\left(\frac{\partial y}{\partial u}\right)^2+\left(\frac{\partial z}{\partial u}\right)^2,\\ N &= \frac{\partial x}{\partial u}\cdot\frac{\partial x}{\partial v}+\frac{\partial y}{\partial u}\cdot\frac{\partial y}{\partial v}+\frac{\partial z}{\partial u}\cdot\frac{\partial z}{\partial v},\\ P &= \left(\frac{\partial x}{\partial v}\right)^2+\left(\frac{\partial y}{\partial v}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2. \end{aligned}\right\} \tag{65}$$

With the help of the identities

$$D_\xi\frac{\partial\xi}{\partial u}+D_\eta\frac{\partial\eta}{\partial u}+D_\zeta\frac{\partial\zeta}{\partial u}=0$$

and

$$D_\xi\frac{\partial\xi}{\partial v}+D_\eta\frac{\partial\eta}{\partial v}+D_\zeta\frac{\partial\zeta}{\partial v}=0$$

one finds from the Eq. (41)

$$\frac{\partial x}{\partial u}=\frac{1}{D^2}(r\eta-q\zeta),\quad \frac{\partial y}{\partial u}=\frac{1}{D^2}(p\zeta-r\xi),$$
$$\frac{\partial z}{\partial u}=\frac{1}{D^2}(q\xi-p\eta),$$

where

$$D=\xi D_\xi+\eta D_\eta+\zeta D_\zeta,$$

$$p=D_\zeta\frac{\partial D_\eta}{\partial u}-D_\eta\frac{\partial D_\zeta}{\partial u},$$

$$q=D_\xi\frac{\partial D_\zeta}{\partial u}-D_\zeta\frac{\partial D_\xi}{\partial u},$$

$$r=D_\eta\frac{\partial D_\xi}{\partial u}-D_\xi\frac{\partial D_\eta}{\partial u}.$$

Similarly, one finds

$$\frac{\partial x}{\partial v}=\frac{1}{D^2}(r'\eta-q'\zeta),\quad \frac{\partial y}{\partial u}=\frac{1}{D^2}(p'\zeta-r'\xi),$$
$$\frac{\partial z}{\partial u}=\frac{1}{D^2}(q'\xi-p'\eta),$$

where

$$p'=D_\zeta\frac{\partial D_\eta}{\partial v}-D_\eta\frac{\partial D_\zeta}{\partial v},$$

$$q'=D_\xi\frac{\partial D_\zeta}{\partial v}-D_\zeta\frac{\partial D_\xi}{\partial v},$$

$$r'=D_\eta\frac{\partial D_\xi}{\partial v}-D_\xi\frac{\partial D_\eta}{\partial v}.$$

The substitution of the values of $\partial x/\partial u$, etc. into Eq. (65) and then into Eq. (64) can be left out here.

11. Closing considerations

Under the media considered here, the second [orthorhombic] medium offers but scant interest for seismology, since rocks with three planes of symmetry are presumably rare; much more important is the transversely isotropic medium, since most layered rocks (and many massive rocks) should behave like a transversely isotropic medium. The axis of symmetry is normal to the layers.

We have seen in Sections 3–5 that in such a medium, the wave surface consists of three sheets: the first is an ellipsoid of rotation with semiaxes $\sqrt{c_{44}}$ in the direction of the axis and $\sqrt{c_{66}}$ in the equatorial plane (for beryl $\sqrt{c_{66}}>\sqrt{c_{44}}$, but in rocks we could have $\sqrt{c_{66}}<\sqrt{c_{44}}$). The second sheet is also a surface of rotation with respect to the z-axis. It intersects the z-axis in the same distance as the ellipsoid of rotation, so that the two sheets touch at the z-axis. The intersection of the second sheet with the equatorial plane is a circle with radius $\sqrt{c_{44}}$. If among the elastic constants the relation

$$c_{11}c_{33} - c_{44}(c_{11} + c_{33}) = c_{13}^2 + 2c_{13}c_{44}$$

holds, the second sheet becomes a sphere of radius $\sqrt{c_{44}}$. Generally, it differs from a sphere; it can even have a sort of bulges, as seen in Section 4. There are regions (see Fig. 1, between the lines OS and OQ) where the second sheets meets every point three times. Since the same point is met by each of the other two sheets once, the points in the region SOQ are affected five times by the wave. The third sheet has no such bulges. It is also a surface of rotation about the z-axis. It intersects the z-axis at distance $OK=\sqrt{c_{33}}$ and the equatorial plane in a circle with radius $\sqrt{c_{11}}$ (for beryl $\sqrt{c_{11}}>\sqrt{c_{33}}$, but in rocks one may have $\sqrt{c_{11}}>\sqrt{c_{33}}$). The meridional curve of the third sheet is an oval that looks like an ellipsoid of rotation with the semiaxes $\sqrt{c_{33}}$ and $\sqrt{c_{11}}$; it coincides with this ellipsoid if and only if

$$c_{11}c_{33} - c_{44}(c_{11} + c_{33}) = c_{13}^2 + 2c_{13}c_{44}$$

The velocity of propagation of the third sheets is highest, it arrives thus before the other two. If $\sqrt{c_{66}}>\sqrt{c_{44}}$, the second sheet follows the third and the first arrives last; however, if $\sqrt{c_{66}}<\sqrt{c_{44}}$, the first sheet arrives before the second. In the direction of the z-axis (the axis of symmetry), the first and the second sheet propagate with the same velocity.

The deformation during the passage of a wave consists simultaneously of a dilatation and a torsion. Only if special relations hold between the elastic constants, dilatational oscillations propagate separately from torsional oscillations. Such relations hold, e.g., in isotropic media. In isotropic solids, only two waves propagate (the second sheet coincides with the first): viz., the dilatational and the torsional wave.

In my textbook "Physics of the Earth" (chapter 6, Leipzig 1911, Tauchnitz), as at other occasions, I have stressed that at large depth—in the central part of the earth—the material should be (close to) isotropic. But close to the surface, the material is certainly largely anisotropic, and there is no reason that for such media the special relations between the elastic constants hold that would allow separate propagation of dilatational and torsional waves. The deformation during the passage of a wave must thus have a mixed character, i.e., it must consist simultaneously of a dilatation and a torsion. Moreover, there should be in the crust not two, but three distinct waves (not counting the Lamb–Rayleigh surface wave). In certain direction, one and the same wave can meet one and the same point several times.

To fix our thoughts, we think of the situation where the oscillations pass from an isotropic medium into a transversely isotropic medium. We shall disregard all special complications; we only want to stress that refraction at the interface multiplies the wave. Each wave coming from the isotropic medium generates three waves in the transversely isotropic medium. If only the two primary waves—one dilatational and one torsional—reach the interface, there must be six waves in the transversely isotropic medium. Of course, the second and third wave (first and second sheet) generated by the dilatational wave follow each other closely; the same holds for the second and third wave generated by the torsional wave.

Since in seismology, there exists the deplorable habit to regard anisotropic materials as isotropic, one generally calls the first arriving wave train as dilatational, the second torsional. It is well known that the arrival of the second train is not easily determined: it is difficult to decide where on the seismogram the onset of the second train can be assumed. Next to other difficulties, one cause of this indeterminacy may be the fact that the train that is thought to represent a single wave consists really of two waves, viz. the second and the first sheet of the wave surface. Depending on the direction of arrival, the two trains can arrive with different separations. There may be a single onset or two onsets; in directions where the bulge of the wave surface passes through the point of observation, there may be even a sequence of three or four rapidly following onsets.

We speak here of wave trains and of onsets of new wave trains, since due to dispersion and other causes, one observes trains instead of individual waves. A closer inspection of this aspect lies outside the scope of this paper.

Since we started with the classical equations for oscillations of small amplitude (disregarding squares and products of deformation components), the results of this study does not apply to oscillations of large amplitudes.

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