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On the propagation of an elastic surface wave in a transversely isotropic medium

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1. Differential equations

Here, I will consider a transversely isotropic medium of the type that has been investigated from a different aspect in my mémoire "Parametric Representation of the Elastic Wave" (Bull. Acad. of Cracow, A. October 1911, pp. 503–536). It is known that such a medium has an axis of symmetry, that all directions normal to this axis are equivalent, and that it is described by five elastic constants; in certain cases, this number is reduced to four or even three (see, e.g., A.E.H. Love, Treatise on the Mathematical Theory of Elasticity, 2nd edition, Cambridge 1906. I adopt—as in "Parametric Representation"—Love's notation). In that paper, I dealt with an elastic wave propagating indefinitely in any direction of space; in the current memoir, I will consider the surface wave that has been discovered, as we know, by Lord Rayleigh. However, Lord Rayleigh (London, Math. Proceedings XVIII p4 et seq. I have introduced the Lord Rayleigh's theory in my "Physik der Erde, Leipzig 1911, p. 151 et seq.) and after him M. Lamb (On the propagation of tremors, Phil. Trans. R.S. London, Ser. A, vol 203, 1904, 1–42) have assumed an isotropic medium (with two elastic constants), while I assume an anisotropic medium (with five elastic constants).

In order to keep the derivations as simple as possible, we shall assume that the medium is bounded by a horizontal plane (thus, neglecting the curvature of the earth's surface), and that the axis of symmetry of the medium is perpendicular to the plane [A geologist would say that the medium is layered, and that the layers are horizontal]. Since we are interested only in the velocity of propagation and the ratio of the vertical to the horizontal displacement, we can assume that the wave propagates parallel to an arbitrary vertical plane (obviously permissible under our assumption). We shall also restrict the discussion to particular integrals.

If we choose the xz-plane as the vertical plane parallel to the direction of propagation, the motion becomes independent of y and the differential equations that have to be satisfied reduce to

$$\frac{\partial^2 u}{\partial t^2} = c_{11}\frac{\partial^2 u}{\partial x^2} + c_{44}\frac{\partial^2 u}{\partial z^2} + (c_{13}+c_{44})\frac{\partial^2 w}{\partial x \partial z}$$
$$\frac{\partial^2 w}{\partial t^2} = c_{44}\frac{\partial^2 w}{\partial x^2} + c_{33}\frac{\partial^2 w}{\partial z^2} + (c_{13}+c_{44})\frac{\partial^2 u}{\partial x \partial z} \qquad (1)$$

As usual, the letters u and w indicate the components of the displacement in the two princi-

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pal directions x and z, respectively. The z-axis is vertical, as a consequence, parallel to the symmetry axis of the medium. We assume that the positive direction is downward and that the free surface coincides with the plane $z=0$. The symbols c_{11}, c_{33}, ... are the ordinary elastic constants *divided by the density* ρ *of the medium* (this means that c_{11} simply stands for c_{11}/ρ. In this form, the c_{11}, ... are the squares of velocities). The constant c_{66}, does not show up at all.

2. Integrals suitable for the representation of the surface waves

The surface wave can be expressed by the following functional form:

$$u = u_0 e^s, \quad w = w_0 e^s,$$

where

$$s = -rz + i(fx + pt), \quad i = \sqrt{-1}; \tag{2}$$

The wave is thus characterized by a rapid decrease of amplitudes with depth.

On substituting, expression (2) into the differential Eq. (1), one finds

$$\begin{aligned} u_0(p^2 - c_{11}f^2 + c_{44}r^2) + w_0(c_{13} + c_{44})rf &= 0 \\ -u_0(c_{13} + c_{44})rf + w_0(p^2 - c_{44}f^2 + c_{33}r^2) &= 0 \end{aligned} \tag{3}$$

It is well known that these equations are equivalent to

$$(p^2 - c_{11}f^2 + c_{44}r^2)(p^2 - c_{44}f^2 + c_{33}r^2) + (c_{13} + c_{44})^2 r^2 f^2 = 0$$

$$u_0 = k(p^2 - c_{44}f^2 + c_{33}r^2)$$

$$w_0 = k(c_{13} + c_{44})rf, \tag{4}$$

where k is an undetermined coefficient.

Evidently, p/f is nothing but the velocity of propagation. Let us denote this velocity with V, i.e., we set

$$\frac{p}{f} = V \tag{5}$$

With the further abbreviations

$$\frac{r}{f} = \rho$$

$$\frac{c_{11} - V^2}{c_{44}} + \frac{c_{44} - V^2}{c_{33}} - \frac{(c_{13} + c_{44})^2}{c_{33}c_{44}} = 2m \tag{6}$$

$$\frac{c_{11} - V^2}{c_{44}} \frac{c_{44} - V^2}{c_{33}} = n^2$$

we write the first Eq. (4) in the form

$$\rho^4 - 2m\rho^2 + n^2 = 0. \tag{7}$$

The roots of this equation

$$\rho_1^2 = m + \sqrt{m^2 - n^2} \text{ and}$$
$$\rho_2^2 = m - \sqrt{m^2 - n^2} \tag{8}$$

depend only on the propagation velocity and on the constants c_{11}, c_{33}, ... But if there are two roots for ρ, there will also be two different values for r; thus, one must write in place of expression (2)

$$u = u_{01}e^{s_1} + u_{02}e^{s_2}, \quad w = i(w_{01}e^{s_1} + w_{02}e^{s_2}), \tag{9}$$

or (with suppression of the now superfluous factors f^2)

$$\begin{aligned} u_{01} &= k_1(V^2 - c_{44} + c_{33}\rho_1^2), \\ u_{02} &= k_2(V^2 - c_{44} + c_{33}\rho_2^2) \\ w_{01} &= k_1(c_{13} + c_{44})\rho_1, \; w_{02} = k_2(c_{13} + c_{44})\rho_2 \\ s_1 &= (-\rho_1 z + i(x + Vt))f, \; s_2 = (-\rho_2 z + i(x + Vt))f. \end{aligned} \tag{10}$$

Evidently, ρ_1, ρ_1, and t must be real and positive whereas V only has to be real.

3. Boundary conditions at the free surface

At the free surface, i.e., in the plane $z=0$, the two tangential stresses must vanish, and the normal stress must be equal to the atmospheric pressure. Since the latter is insignificant in comparison to the elastic stresses, we can neglect it and write

$$Z_x = 0,\ Z_{y,} = 0,\ Z_z = 0.$$

The second condition is identically satisfied; as for as the two other conditions are concerned, they reduce to

$$\frac{\partial u}{\partial z} + \frac{\partial w}{\partial z} = 0, \quad c_{13}\frac{\partial u}{\partial x} + c_{33}\frac{\partial w}{\partial z} = 0 \ (\mathit{for}\ z = 0)$$

If one introduces the values Eq. (9) and suppresses common factors (note that for $z = 0$ $s_1 = s_2$), the equations become

$$k_1\rho_1(V^2 + c_{13} + c_{33}\rho_1^2) + k_2\rho_2(V^2 + c_{13} + c_{33}\rho_2^2) = 0$$
$$k_1(c_{13}(V^2 - c_{44}) - c_{33}c_{44}\rho_1^2) + k_2(c_{13}(V^2 - c_{44}) - c_{33}c_{44}\rho_2^2) = 0 \tag{11}$$

In order to have Eq. (11) compatible, one must have

$$\rho_1(V^2 + c_{13} + c_{33}\rho_1^2)(c_{13}(V^2 - c_{44}) - c_{33}c_{44}\rho_2^2) = \rho_2(V^2 + c_{13} + c_{33}\rho_2^2) \times (c_{13}(V^2 - c_{44}) - c_{33}c_{44}\rho_1^2), \tag{12}$$

$$k_1 = K(c_{13}(V^2 - c_{44}) - c_{33}c_{44}\rho_2^2),$$

$$k_2 = -K(c_{13}(V^2 - c_{44}) - c_{33}c_{44}\rho_1^2). \tag{13}$$

The coefficient K (as before the coefficient f) is available if one wants to form the general solution and satisfy the initial conditions. With the help of the relations (8), one can eliminate ρ_1 and ρ_2 from Eq. (12), which then contains only V^2 and the elastic constants. One sees immediately that Eq. (12) has the same roots as the equation

$$m^2 - n^2 = 0.$$

Besides, by squaring, substituting m^2 and n^2 from Eq. (6) and rearranging, one finds that Eq. (7) can be written as

$$\sqrt{m^2 - n^2}(V^2 - c_{44})((V^2 - c_{44}) \times (c_{33}(V^2 - c_{11}) + c_{13}^2)^2 - c_{33}c_{44}(V^2 - c_{11})) = 0 \tag{12 bis}$$

4. Velocity of propagation of the surface wave

The Eq. (12 bis) gives the velocity of propagation of the surface wave. One sees that it can be decomposed into three equations, the first of which

$$m^2 - n^2 = 0. \tag{14}$$

can be written as

$$(c_{33} - c_{44})^2V^4 - 2((c_{33} - c_{44})(c_{11}c_{33} - c_{44}^2) - (c_{33} + c_{44})(c_{13} + c_{44})^2)V^2 + (c_{11}c_{33} - c_{44}^2)^2 - 2(c_{11}c_{33} + c_{44}^2)(c_{13} + c_{44})^2 + (c_{13} + c_{44})^4 = 0 \tag{14 bis}$$

This a quadratic equation in V^2 with the discriminant

$$4c_{33}c_{44}(c_{13} + c_{44})^2((c_{13} + c_{44})^2 - (c_{11} - c_{44})(c_{33} - c_{44})).$$

The discriminant will often be negative, all constants—with the exception of c_{13}, which may be negative—must be positive (see "Parametric Representation ..." p. 518), and c_{11}, c_{33} are generally considerably larger than c_{13} and c_{44}. In this case, the two complex roots of Eq. (14) have no physical significance, since V^2 must be real and positive. But there is something else. If $m^2 - n^2 = 0$, one has from Eq. (7) $\rho_1 = \rho_2$ and $s_1 = s_2$ at all depths. It is not necessary to impose the double condition (2) and if one tries to satisfy the surface conditions for $z = 0$, one finds that the coefficients u_0 and w_0 must vanish. The same result is, of course, obtained if one starts from Eq. (11); indeed for $\rho_1 = \rho_2$, these give $k_1 + k_2 = 0$, which combined with $s_1 = s_2$ (for all depths) leads to the conclusion that u and w are identical zero. For $m^2-n^2<0$, ρ_1^2 and ρ_2^2—and consequently ρ_1 and ρ_1—are complex. Nevertheless, if all the real parts are positive, the motion is not impossible. But this is not the place to discuss this aspect.

Let us turn to the linear equation

$$V^2 - c_{44} = 0, \tag{15}$$

which always give a real velocity that is equal to that of the horizontal velocity of the second sheet of the ordinary elastic wave [see ''Parametric representation...'' p. 533 et seq. In passing, I notice that Fig. 2 of p. 534 corresponds to the case $c_{66}<c_{44}$]. If one inserts this velocity into Eqs. (6) and (8), one finds at once:

$$n^2 = 0,$$
$$\rho_1^2 = 2m = \frac{c_{33}(c_{11}-c_{44})-(c_{13}+c_{44})^2}{c_{33}c_{44}},$$
$$\rho_2^2 = 0.$$

Under ordinary circumstances ρ_1^2 is positive; but from Eqs. (10) and (13) one finds that the four coefficients u_{01}, u_{02}, w_{01}, w_{02} become zero, thus the wave is evanescent, as in the previous case.

Let us turn finally to the equation of third order in V^2

$$(V^2-c_{44})(c_{33}(V^2-c_{11})+c_{13}^2)^2 - c_{33}c_{44}V^4(V^2-c_{11}) = 0 \tag{16}$$

As one sees easily, this equation has always a positive root between 0 and c_{44} and, for realistic elastic constants, two further positive roots, both larger than c_{11}. Nevertheless, these latter roots have no physical meaning, since, c_{11} being greater than c_{44}, a root greater than c_{11} makes n^2 positive and m^2 negative (see Eq. (6)). Consequently (see Eq. (8)), ρ_1^2 and ρ_2^2 are negative or complex (with negative real parts), and ρ_1 and ρ_2 are purely imaginary or complex. With respect to the root smaller than c_{44}, it is easy to convince oneself that it makes ρ_1^2 and ρ_2^2 real and positive, and the coefficients $u_{01}\ldots$ different from zero. Thus, similar to the situation in isotropic media [in isotropic media c_{44} is the square of the torsional wave velocity], we have found a single velocity of propagation; as for isotropic media, this velocity is less than $\sqrt{c_{44}}$.

5. Ratio of horizontal to vertical amplitude

At the surface $z=0$, the horizontal displacement becomes

$$u = (u_{01}+u_{02})\cos f(x+Vt)$$

and the vertical displacement becomes

$$w = -(w_{01}+w_{02})\sin f(x+Vt)$$

(note that, of course, only the real parts have been written out). A point at the surface thus describes an ellipse with the horizontal semi-axis $|(u_{01}+u_{02})|$ and the vertical semi-axis $|(w_{01}+w_{02})|$. The ratio R of vertical to horizontal semi-axes is (according to Eqs. (10) and (13))

$$R = \left|\frac{w_{01}+w_{02}}{u_{01}+u_{02}}\right| = \left|\frac{c_{13}(V^2-c_{44})+c_{33}c_{44}\rho_1\rho_2}{c_{33}(V^2-c_{44})(\rho_1+\rho_2)}\right| \tag{17}$$

Since (see Eq. (6))

$$\rho_1\rho_2 = n, \quad \rho_1+\rho_2 = \sqrt{2(m+n)}$$

one must determine the root (smaller than c_{44}) of Eq. (16) and the numbers m and n. As an example, we select the same constants

$$c_{11} = 5c_{44}, \quad c_{33} = 4c_{44} \text{ and } c_{13} = c_{44},$$

which were chosen for the numerical calculations for the "Parametric Representation ...". We find

$$V^2 = 0.93844\ldots c_{44}, \qquad V = 0.96872\ldots\sqrt{c_{44}},$$

$$n = \rho_1\rho_2 = 0.25001\ldots, \quad \sqrt{2(m+n)} = \rho_1 + \rho_2 = 1.89129\ldots,$$

and R= 2.015....

Thus, the ratio of the vertical to horizontal semi-axes is even greater than for an isotropic medium [see "Parametric representation..." p. 511] with Poisson's ratio $v = 0.25$, where $R = 1.468\ldots$.

One should not attach too much importance to these results. In the absence of knowledge of the constants suitable for rocks, I have arbitrarily adopted numerical values close to those of beryl. However, it appears that one cannot expect significantly lower values of R. Let us look at the problem from a different point of view and try to determine R from observational data. The observed velocity of surface waves is, in average, 3.4 km/s. According to Zoeppritz and Geiger, the value for $\sqrt{c_{44}}$ for the terrestrial crust is 4.01 km/s. We assume

$$V = 0.85\sqrt{c_{44}}, \quad V^2 = 0.7225 c_{44}$$

In order to determine the other constants, we take Eq. (16), which must be satisfied in any case. Since this equation is not sufficient, let us consider the limiting case, where the waves become evanescent, and let $m = n$. In this way, we obtain two equations. One cannot separate c_{11} and c_{33}. but one can determine c_{13} and

$$h = \frac{c_{33}(c_{11} - V^2)}{c_{44}}.$$

For c_{13}, one obtains an equation of fourth degree with a double root

$$c_{13} = -c_{44}$$

and two simple roots

$$c_{13} = 0.1410\ldots c_{44} \text{ and } c_{13} = -0.3766\ldots c_{44};$$

(remember that c_{13} is the only constant that may be negative). During the investigation of h, all values that correspond to the double root

$$c_{13} = -c_{44}$$

must be rejected and also two others far outside the range of values observed in nature. Finally, one finds

$$h = 2.7816\ldots c_{44} \text{ with } c_{13} = 0.1410\ldots c_{44}$$

and

$$h = 1.3230\ldots c_{44} \text{ with } c_{13} = -0.3766\ldots c_{44}.$$

In order to eliminate c_{33} from Eq. (17), a supplementary hypothesis was necessary. We assume for simplicity's sake

$$c_{33} = c_{11};$$

the relation that connects h with c_{11} and c_{33} becomes then

$$c_{11}^2 - c_{11}V^2 - hc_{44} = 0.$$

With the assumed numerical values, one gets

$$\left(\frac{c_{11}}{c_{44}}\right)^2 - 0.7225\left(\frac{c_{11}}{c_{44}}\right) - 2.7816\ldots = 0 \text{ for } \frac{c_{13}}{c_{44}} = 0.1410\ldots$$

and

$$\left(\frac{c_{11}}{c_{44}}\right)^2 - 0.7225\left(\frac{c_{11}}{c_{44}}\right) - 1.3230\ldots = 0 \text{ for } \frac{c_{13}}{c_{44}} = -0.3766\ldots$$

Both equations have a positive root, the first the root

$$c_{11} = c_{33} = 2.0677\ldots c_{44},$$

and the second the root

$$c_{11} = c_{33} = 1.5666\ldots c_{44}.$$

In the first case, one finds

$$R = 1.112\ldots,$$

and in the second,

$$R = 1.317\ldots.$$

Both values of R are in the neighborhood of the upper limit of the values observed by Prince B. Galitzin, which lie between 0.46 and 1.26 (Observations of the vertical component of the ground's motion (in German), Bull de l'Acad. d. Sc. de St. Petersbourg, 1911, 983–1006). What is the cause of this disagreement? We do not know. There may be various causes. Prince Galitzin thinks of extinction and of interference by Wiechert's "transverse surface waves". I doubt the efficacy of this second cause, since the "transverse surface waves" are evanescent (see the discussion in Section 4 concerning the root of Eq. (15)).

One might wonder what would happen if the axis of symmetry of the medium is horizontal instead of being vertical (a geologist would say that the layers are turned up). We shall not endeavor to discuss this problem, which should, by the way, not be difficult.

6. Case of an isotropic medium

For an isotropic medium , Eq. (17) should reproduce the simple Eq. (26) on page 154 of my "Physics of the Earth" (Ch. H. Tauchnitz, Ed., Leipzig 1911). In view of certain reduction applied in the deduction of Eq. (17), it is rather difficult to pass directly from Eq. 17 to the equation in "Physics of the Earth". A roundabout track easier to follow is to go back to Eqs. (10) and (13). One uses the abbreviations

$$A_1 = V^2 + c_{13} + c_{33}\rho_1^2$$

$$B_1 == c_{13}(V^2 - c_{44}) - c_{33}c_{44}\rho_1^2$$

$$L_1 == V^2 - c_{44} + c_{33}\rho_1^2$$

and writes

$$R = \frac{w_{01} + w_{02}}{u_{01} + u_{02}} = (c_{13} + c_{44})\frac{\rho_1 B_2 - \rho_2 B_1}{L_1 B_2 - L_2 B_1},$$

where B_2, L_2, A_2 are similar to B_1, L_1, A_1 and differ only in that they contain ρ_2 instead of ρ_1. Now Eq. (12) has the form

$$\rho_1 A_1 B_2 = \rho_2 A_2 B_1.$$

With the help of the last equation, one brings the expression for R into the form

$$R = (c_{13} + c_{44})\rho_1 \frac{B_2}{A_2}\frac{A_2 - A_1}{L_1 B_2 - L_2 B_1}.$$

After some simple reductions, one finds an expression that is equivalent to Eq. (17), namely

$$R = \rho_1 \frac{c_{13}(V^2 - c_{44}) - c_{33}c_{44}\rho_2^2}{(V^2 - c_{44})(V^2 + c_{13} + c_{33}\rho_2^2)} \qquad (17\text{ bis})$$

For an isotropic medium

$$c_{33} = c_{11}, \quad c_{13} = c_{11} - 2c_{44};$$

the roots of Eq. (7) are thus

$$\rho_1^2 = 1 - \frac{V^2}{c_{44}}, \quad \rho_2^2 = 1 - \frac{V^2}{c_{11}}.$$

With these identities, Eq. (17 bis) reduces to

$$R = \frac{1 - \frac{1}{2}\frac{V^2}{c_{44}}}{\sqrt{1 - \frac{V^2}{c_{44}}}}. \qquad (17\text{ ter})$$

With the notation of "Physics of the Earth", one has

$$\frac{p^2}{m^2} = V^2, \quad h^2 = \frac{1}{c_{11}}, \quad k^2 = \frac{1}{c_{44}}$$

and one sees that Eq. (17 ter) and Eq. (26) on page 154 of my "Physics of the Earth" are identical.

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