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On shear-wave triplication in transversely isotropic media

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Abstract

The exact solution to the problem of qSV triplication in homogenous transversely isotropic media has been long known, but the result is algebraically complex and is seldom applied in practice. We present an appropriate approximation (not assuming weak qSV-anisotropy) that simplifies the conditions for the onset of off-axis triplication as anisotropy is increased, identifying the anisotropy parameter σ as the controlling parameter. It follows that commonly reported surface-seismic P-wave move-out measurements imply that many formations in the earth's sedimentary crust support off-axis qSV triplications. For typical Vp/Vs velocity ratios and a horizontally stratified earth, however, off-axis qSV triplications appear to only occur for shear-wave incidence angles too far from the vertical to be sampled by surface-seismic converted-wave survey geometries.

Keywords: Anisotropy; Shear-wave triplication; Isotropic media

1. Introduction

The phenomenon of "qSV triplication" has been well known, especially for transversely isotropic media, since the time of Rudzki (1911). Rediscovering this early work, Helbig (1958) and Dellinger (1991) presented inequalities (of sixth order in the elastic moduli C_{11}, C_{33}, C_{13}, C_{55}) that give the conditions under which such triplications occur for homogenous transversely isotropic media. They found that triplications could occur for qSV waves propagating at three different angles to the axis of symmetry: parallel to the symmetry axis, perpendicular to the symmetry axis, or at an angle inbetween, and presented an inequality governing each case. The only algebraically difficult case is the last of the three, "off-axis" triplication.

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With the problem apparently already completely solved, there seems to be little to add, so the purpose of this present contribution needs to be stated clearly. The inequality governing off-axis triplication is algebraically complicated enough that it is not possible to understand it intuitively. Unfortunately, it is also the case of geological significance. In particular, (1) although it is straightforward to categorize whether a particular case is triplicating off axis or not, it is not easy to see how to generalize any one case to others; instead, one must recalculate each case individually. (2) It is easily understood that triplications is transversely isotropic media occur only when the "anisotropy is sufficiently large", but no simple specific understanding of what this means has been established. In particular, it is not clear just what measure(s) of anisotropy should be large (and how large), in order for the phenomenon of off-axis triplication to occur. (3) In order to apply the extract theory, one must measure at least three elastic moduli; this is only

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possible in a laboratory setting. In a (geophysical) field setting, ordinarily only certain combinations of elastic parameters can be measured; it is an incomplete set which precludes the calculation of the underlying elastic moduli, so that the theory may not be applied in such a context. (4) When we measure large P-wave anisotropy, it is not clear whether or not this implies that the same material should support qSV triplications. (5) Because of these characteristics of the exact solution, the whole subject remains an esoteric one, and nobody knows whether or not the phenomenon may have some utility, for example, in physical characterization of subsurface rock formations.

In this paper, we abandon the exact solution in favor of an approximate solution, which turns out to be so simple that it is easily understood intuitively, and leads easily to generalizations of any particular case. It turns out that the controlling anisotropy parameters are familiar from other contexts and can be deduced from other measurements not involving qSV triplication explicitly. Through use of this approximation, we hope that the newly accessible phenomenon may be found useful in applied studies.

2. Statement of the problem, and the exact solution

Let $V(\theta)$ give the phase velocity as a function of phase propagation angle θ for a wave mode in a homogenous anisotropic medium. The medium is assumed to be *transversely isotropic*, with a symmetry axis at $\theta=0$. Then, for a given plane wave propagating at an angle θ to the symmetry axis, the associated ray propagation direction is given by

$$\phi = \theta + \arctan\left(\frac{\mathrm{d}V(\theta)/\mathrm{d}\theta}{V(\theta)}\right), \tag{1}$$

and the ray propagates in that direction with a ray velocity v given by

$$v^2(\phi) = V^2(\theta) + (\mathrm{d}V(\theta)/\mathrm{d}\theta)^2. \tag{2}$$

These equations follow directly from the Pythagorean theorem: the ray velocity is the vector sum of motion perpendicular to the wavefront, $V(\theta)$, and motion parallel to the wavefront, $\mathrm{d}V(\theta)/\mathrm{d}\theta$ (Dellinger, 1991).

Triplications occur when one ray direction is associated with more than one phase direction of the same wave type. For transversely isotropic media, triplications occur when the ray angle ϕ moves forward, then backtracks, and then moves forward again as the phase angle θ uniformly increases, so that a range of ϕ is encountered three times instead of once. Mathematically, "backtracking" (and thus triplication) occurs when

$$\frac{\mathrm{d}\phi}{\mathrm{d}\theta} < 0. \tag{3}$$

Using Eq. (1) to eliminate ϕ, inequality (3) is equivalent to

$$\frac{d^2V(\theta)}{\mathrm{d}\theta^2} + V(\theta) < 0, \tag{4}$$

where $V(\theta)$ is the phase-velocity function (Dellinger, 1991). It should be noted that Eqs. (3) to (4) are derived from geometrical considerations only. They do not consider amplitudes and ignore the complex waveform effects associated with triplications at finite frequencies (Burridge, 1967).

The exact expression for the qSV phase (plane-wave) velocity in a homogenous transversely isotropic medium is

$$V_{\mathrm{qSV}}^2(\theta) = \frac{1}{2\rho}\Big[C_{55} + C_{33}\cos^2\theta + C_{11}\sin^2\theta - \sqrt{((C_{33}-C_{55})\cos^2\theta-(C_{11}-C_{55})\sin^2\theta)^2+4(C_{13}+C_{55})^2\sin^2\theta\cos^2\theta}\Big], \tag{5}$$

where the $C_{\alpha\beta}$ are elastic moduli (Thomsen, 1986). We will limit our discussion in this paper to normally polarized transversely isotropic media, i.e., those with $C_{11}>C_{55}$, $C_{33}>C_{55}$, and $C_{13}>-C_{55}$. Otherwise, the terminology "qSV mode" ceases to make physical sense, and some of the equations governing qSV triplication presented in this paper become invalid (Dellinger, 1991).

We wish to categorize sets of elastic constants as either representing "triplicating" or "non-triplicating" media. The borderline case occurs when both the first and second derivatives of ϕ with respect to θ are simultaneously zero for some θ. At such an "incipient triplication", the ray angle ϕ pauses in its forward motion as θ increases ($\mathrm{d}\phi/\mathrm{d}\theta=0$), but does not then backtrack. Instead, after pausing, it continues forward again ($d^2\phi/d\theta^2=0$).

Dellinger (1991) examined the behavior of qSV waves in homogenous transversely isotropic media as a function of C_{13}. He observed that incipient triplication can only occur when one of the two terms under the square root in Eq. (5) is zero. The second term is zero when either $\theta=0°$ or $\theta=90°$, so this term governs "on-axis" triplication. For $\theta=0°$, triplication occurs when

$$(C_{13}+C_{55})^2-C_{11}(C_{33}-C_{55})>0, \tag{6}$$

and for $\theta=90°$, triplication occurs when

$$(C_{13}+C_{55})^2-C_{33}(C_{11}-C_{55})>0 \tag{7}$$

(Musgrave, 1970). On-axis triplication becomes more pronounced as C_{13} is increased.

The first term under the square root in Eq. (5) is zero when

$$\sin^2\theta_i=\frac{C_{33}-C_{55}}{C_{33}+C_{11}-2C_{55}}, \tag{8}$$

which provides a formula for the angle θ_i at which incipient "off-axis" triplication occurs. If $C_{33}=C_{11}$, the incipient triplication occurs at exactly $\theta=45°$.

Evaluated at θ_i, inequality (4) becomes

$$(C_{13}+C_{55})^2-3C_{55}^2+C_{55}(C_{33}+C_{11})-3C_{11}C_{33}+2\sqrt{(C_{33}-C_{55})(C_{11}-C_{55})}\,\frac{C_{33}C_{11}-C_{55}^2}{C_{13}+C_{55}}>0 \tag{9}$$

(Payton, 1983). Off-axis triplication becomes more pronounced as C_{13} is decreased.

Off-axis triplications may be important because interbedded thin layers are often a major contributor to observed geological anisotropy at seismic wavelengths, and transverse isotropy due to thin isotropic layering may give rise to off-axis qSV triplications, but not on-axis ones (Berryman, 1979). Fig. 1 shows a qSV wavefront calculated for Greenhorn shale (Jones and Wang, 1981), a rock made up of innumerable thin layers. It exhibits off-axis triplication near 45°, extending towards the symmetry axes and ending in cusps.

Inequality (9) gives a combination of elastic parameters that, if positive, indicates that a particular transversely isotropic medium supports off-axis qSV triplications. The problem is that this particular com-

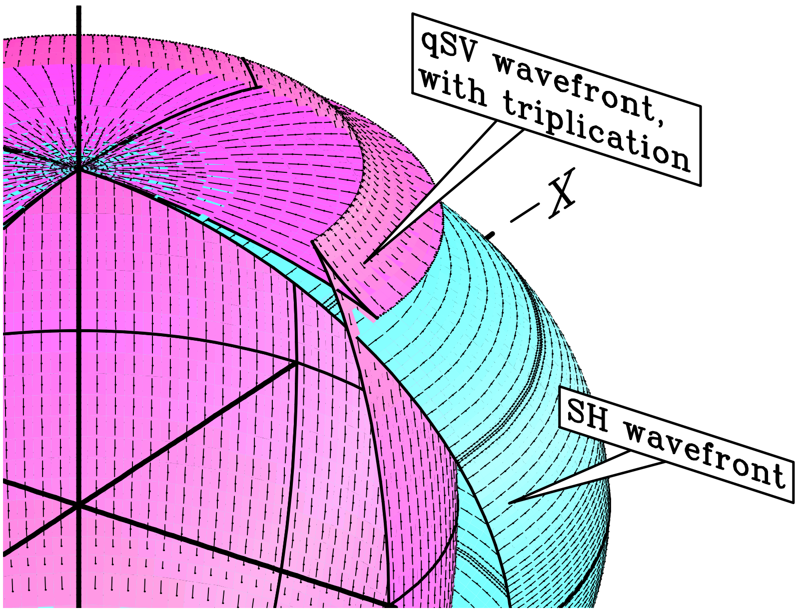

Fig. 1. A shear-wave group-velocity surface that has an off-axis qSV triplication, near 45°. The small line segments indicate the particle-motion polarization direction.

bination is both complicated and unfamiliar from other contexts, so that no intuitive understanding results.

3. An approximate solution

The combinations of elastic moduli defined by Thomsen (1986),

$$\epsilon \equiv \frac{C_{11} - C_{33}}{2C_{33}} \tag{10}$$

and

$$\delta \equiv \frac{(C_{13} + C_{55})^2 - (C_{33} - C_{55})^2}{2C_{33}(C_{33} - C_{55})}, \tag{11}$$

and by Tsvankin and Thomsen (1994),

$$\sigma \equiv \frac{C_{33}}{C_{55}}(\epsilon - \delta), \tag{12}$$

have been found useful in the context of weakly transversely isotropic media, because they reduce to zero in the limiting case of isotropy. For example, to first order in δ and ϵ, Eq. (5) reduces to

$$V^2_{\mathrm{qSV}}(\theta) = \frac{C_{55}}{\rho}(1 + 2\sigma \sin^2\theta \cos^2\theta), \tag{13}$$

a considerable simplification.

The "weak-anisotropy" parameters δ, ϵ, and σ have also proven useful for analyzing *strongly anisotropic* media. For example, Eq. (13) is also valid for small θ regardless of the magnitudes of δ, ϵ, and σ. The parameter σ therefore governs the paraxial behavior of qSV waves whether the anisotropy is weak or strong. The parameter δ similarly controls the paraxial behavior of qP-waves.

Represented in terms of σ and δ, Eq. (6), the condition for triplication at $\theta = 0°$ becomes

$$\sigma < -\frac{1}{2} \tag{14}$$

without approximation. Eq. (7), the condition for triplication at $\theta = 90°$ becomes (again without approximation)

$$\sigma < -\frac{1}{2} - \delta + \frac{1}{2\Gamma_0^2}, \tag{15}$$

where $\Gamma_0 \equiv \sqrt{C_{33}/C_{55}}$ is the vertical Vp/Vs velocity ratio.

Inequality (9), the inequality governing off-axis triplication, can also be exactly rewritten in terms of δ, σ, and Γ_0, albeit in a somewhat expanded form. Some approximation is necessary to produce a result simple enough to be useful. Inequalities (14) and (15) already demonstrated that σ is the primary measure of anisotropy governing the phenomenon of qSV triplication for the on-axis cases of $\theta = 0$ and $\theta = 90°$. Inequality (15) further suggests that an expansion in terms of δ and $1/\Gamma_0^2$ might be fruitful.

We therefore begin by assuming that the P-wave anisotropy parameter δ is small, and that the squared Vp/Vs velocity ratio $\Gamma_0^2 = C_{33}/C_{55}$ is large, and expand inequality (9) retaining only linear terms in δ and $1/\Gamma_0^2$. This yields

$$\sigma\left[1 + \frac{3\sigma}{4}\left(1 + \left\{\frac{2-\sigma}{3\Gamma_0^2}\right\} - 2\delta\right) - \delta\right] > 1 \tag{16}$$

(Thomsen, 2002). Note we have not made any particular assumption about the size of σ, which Eq. (12) shows contains the term $\delta\Gamma_0^2$ in its definition.

Eq. (16) is a cubic inequality in σ, which can be solved analytically, although only one of the three solutions is physically meaningful. If we further approximate the solution by setting $\delta = 0$ and $\Gamma_0 = \infty$, the resulting quadratic inequality has the physically meaningful solution

$$\sigma > \frac{2}{3}. \tag{17}$$

We can refine the approximation by replacing σ in the (small) term in curly brackets in Eq. (16) by 2/3,

$$\left\{\frac{2-\sigma}{3\Gamma_0^2}\right\} \approx \left\{\frac{2-2/3}{3\Gamma_0^2}\right\} = \left\{\frac{4}{9\Gamma_0^2}\right\}, \tag{18}$$

and then solving the remaining quadratic to yield

$$\sigma > \sigma_{\mathrm{critical}} \equiv \frac{2}{3}\left(1 + \delta - \frac{1}{9\Gamma_0^2}\right) \tag{19}$$

as the condition for triplications to occur. This is the primary result reached here; it is a simple approximation to the exact result, inequality (9).

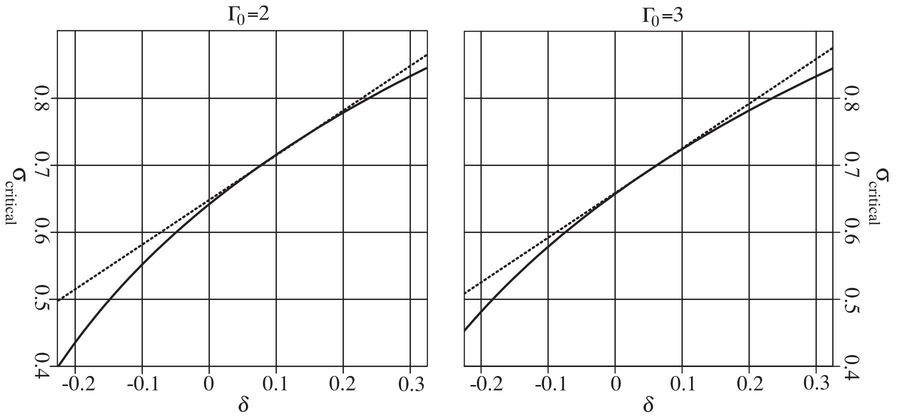

Fig. 2. The condition for onset of off-axis triplication for two different values of Γ_0. Media with elastic parameters that plot above the solid curve support off-axis triplication. The dotted line shows the approximation of inequality (9).

This result can be conveniently reproduced using any modern symbolic algebra manipulation program. Start from Eq. (9) and recast it in terms of the dimensionless parameters δ, σ, and Γ_0^{-2}. The number of parameters is then reduced by one because the overall scale of the elastic constants does not matter for this problem and divides out. The resulting cubic equation may be solved for σ analytically. Expanding the one physically meaningful solution in a power series in δ and $1/\Gamma_0^2$ and dropping terms of order higher than one reproduces Eq. (19).

Fig. 2 illustrates the accuracy of this approximation, comparing it to realizations of the exact inequality (9) for two different values of Γ_0. The accuracy of the approximation depends only very weakly on Γ_0. The approximation is also apparently accurate over the range $[-0.1<\delta<0.2]$, which includes most cases of geophysical interest (see for example Wang (2002)). Fig. 2 thus confirms that σ is the primary measure of anisotropy governing the phenomenon of qSV triplication for the off-axis as well as the on-axis case, and justifies our choice of expansion.

Our approximation does start to significantly diverge from the exact solution for $\delta<-0.15$, but this should not be a problem in practice, as we can see by rewriting inequality (9) as a relationship between δ and ϵ, as shown in Fig. 3. We expect to always have $\epsilon\geq 0$ for cases of geophysical interest (Thomsen, 2002). Examining Fig. 3, we can then see that rocks with a strongly negative δ should always lie well inside the region of off-axis triplication.

The exact curves in Figs. 2 and 3 were calculated by fixing C_{11}, C_{33}, and C_{55} and then varying C_{13} to make the curve. It is not clear from inequality (9) how other cases would appear with these moduli fixed at different values, or with a different set of three moduli fixed. The approximation of inequality (19) makes it immediately clear that σ is the primary parameter governing the phenomenon, followed by δ and Γ_0

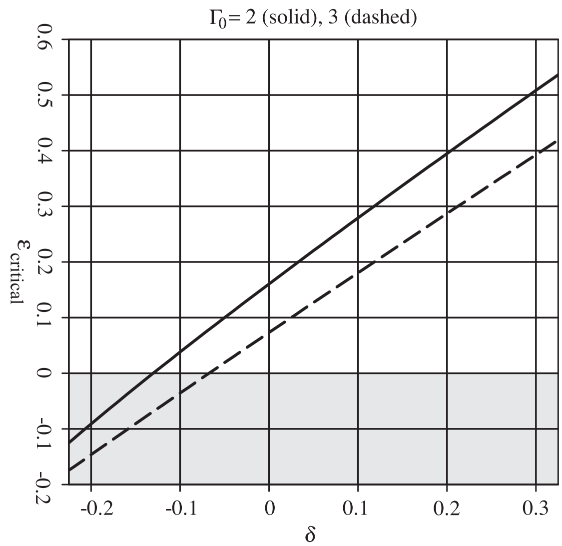

Fig. 3. The condition for onset of off-axis triplication for two different values of Γ_0, expressed as a function of $\epsilon=\delta+\sigma/\Gamma_0^2$. Media with elastic parameters that plot above the relevant curve support off-axis triplication.

in that order; this was not obvious from the weak-anisotropy relation (13).

4. Some implications

With the simple approximate condition (19) for the onset of off-axis triplication, it is possible to immediately see some implications. For example, it is clear from Eq. (12) that inequality (19) is also a condition on the difference $(\epsilon-\delta)$. Noting from Fig. 2 that the approximation always *underestimates* the extent of triplication, we can therefore state that whenever a transversely isotropic medium has

$$(\epsilon-\delta)\geq(\epsilon-\delta)_{\text{critical}}\equiv\sigma_{\text{critical}}/\Gamma_0^2, \tag{20}$$

then that material should support off-axis qSV triplications.

For example, for $\Gamma_0=3$ (a typical value for marine sediments), the value of $(\epsilon-\delta)_{\text{critical}}$ is near 0.07. This is not a large value; in fact, it lies in the range where perturbation theory can be used. This is important because this difference $(\epsilon-\delta)$ is familiar in other contexts, as it controls the departure of the qP-wavefront from the elliptical. In fact, Alkhalifah and Tsvankin (1995) showed that for transversely isotropic media with a vertical symmetry axis, the parameter

$$\eta=\frac{(\epsilon-\delta)}{1+2\delta} \tag{21}$$

controls all non-hyperbolic time-domain kinematic seismic effects in surface-reflection qP-wave data, even for the case of strong anisotropy.

Following this work, many studies have estimated η from seismic reflection data (see for example Alkhalifah and Rampton (2001)), and most of these have concluded that many formations of the sedimentary crust have values of η greater than 0.07. It follows from Eq. (20) that these formations would exhibit qSV triplication in data properly acquired to look for this effect. In fact, triplications have already been observed in Vertical Seismic Profile data acquired for oil-industry purposes (Slater et al., 1993). If the intuitive understanding enabled by this work led to the perception that triplications gave a sensitive measure of something useful, then it is likely that many more observations would be attempted and some would be successful.

5. Locating off-axis triplications

How close to vertical should we expect to observe off-axis triplication? In a principle, we can answer this question exactly and completely by simply inserting Eq. (5) into Eq. (1), and then looking to see over what range of ray angles ϕ that the triplication condition given by inequality (3) is satisfied for a given set of elastic constants. Unfortunately, the exact result is extremely complex; to gain insight, some approximation is necessary.

The problem considerably simplifies if we only attempt to find a formula for the ray angle at which *incipient triplication* occurs, ϕ_{i}. We begin by fixing $\sigma=\sigma_{\text{critical}}$, using the approximation of Eq. (19). We then perform the necessary derivatives with respect to θ needed to calculate ϕ. We can then fix $\theta=\theta_{\text{i}}$ using the exact value given by Eq. (8), obtaining an extremely accurate approximation for ϕ_{i}.

Because Eq. (8) for θ_{i} does not involve C_{13}, it can be expressed purely in terms of Γ_0 and ϵ:

$$\sin^2\theta_{\text{i}}=\frac{1}{2}\left(\frac{\Gamma_0^2-1}{(1+\epsilon)\Gamma_0^2-1}\right). \tag{22}$$

This suggests that the critical parameter controlling the position of the off-axis triplication is ϵ. We therefore use Eq. (12) to eliminate δ from Eq. (19) as well, obtaining

$$\sigma_{\text{critical}}=\frac{18+18\epsilon-2/\Gamma_0^2}{27+18/\Gamma_0^2}. \tag{23}$$

Our approximate ϕ_{i} can then be expressed as a function of just two parameters, ϵ and Γ_0.

Unfortunately, the resulting formula is still unwieldy. To simplify it further, we expand it in a Taylor series to first order in $1/\Gamma_0^2$ and second order in ϵ, and discard small cross terms. We then obtain the concise approximation

$$\phi_i\approx\frac{\pi}{4}-\epsilon\left(\frac{(1-\epsilon)}{4}+\frac{1}{3\Gamma_0^2}\right), \tag{24}$$

where ϕ_i is measured in radians from the (vertical) symmetry axis. Note that it is the P-wave anisotropy parameter ϵ which apparently controls this qSV phenomenon!

Fig. 4 demonstrates the accuracy of this approximation. The top two rows in Fig. 4 are calculated for $\sigma = \sigma_{critical}$ as given by Eq. (19), for two different Γ_0 and four different ϵ spanning the full range of values of likely geophysical interest. We can see from the top two rows that Eq. (19) does a good job of predicting the value of σ for which incipient triplication occurs, and Eq. (24) does a very good job of predicting the ray angle of that incipient triplication.

The second two rows repeat the same values of Γ_0 and ϵ, but have σ scaled up by a factor of 1.82 so that the media strongly triplicate. The arbitrary values of $\Gamma_0 = 2.05$ and $\sigma/\sigma_{critical} = 1.82$ were chosen so that one of the entries would correspond to real a rock, Greenhorn shale (Jones and Wang, 1981). Comparing the two sets of examples, we can see that as σ increases the triplication grows larger and also shifts position, but the incipient ray angle ϕ_i still lies well within the zone of triplication. More importantly, comparing the different examples in the second set, we can see that the triplications are all approximately the same size despite having very different ϵ and Γ_0. We conclude that the ratio $\sigma/\sigma_{critical}$ is the dominant factor predicting the extent of off-axis triplication.

Snell's law ensures that upcoming shear waves in typical converted-wave surveys over horizontally stratified reflectors will travel at a limited angle away from the vertical, usually much smaller than 30°. We expect that the triplications depicted in the lower two rows of Fig. 4 are "large", at least for geophysical purposes. Yet, even these large triplications do not extend to within 30° of the symmetry axis, to where they could be kinematically detected in a typical converted-wave survey. We therefore conclude that

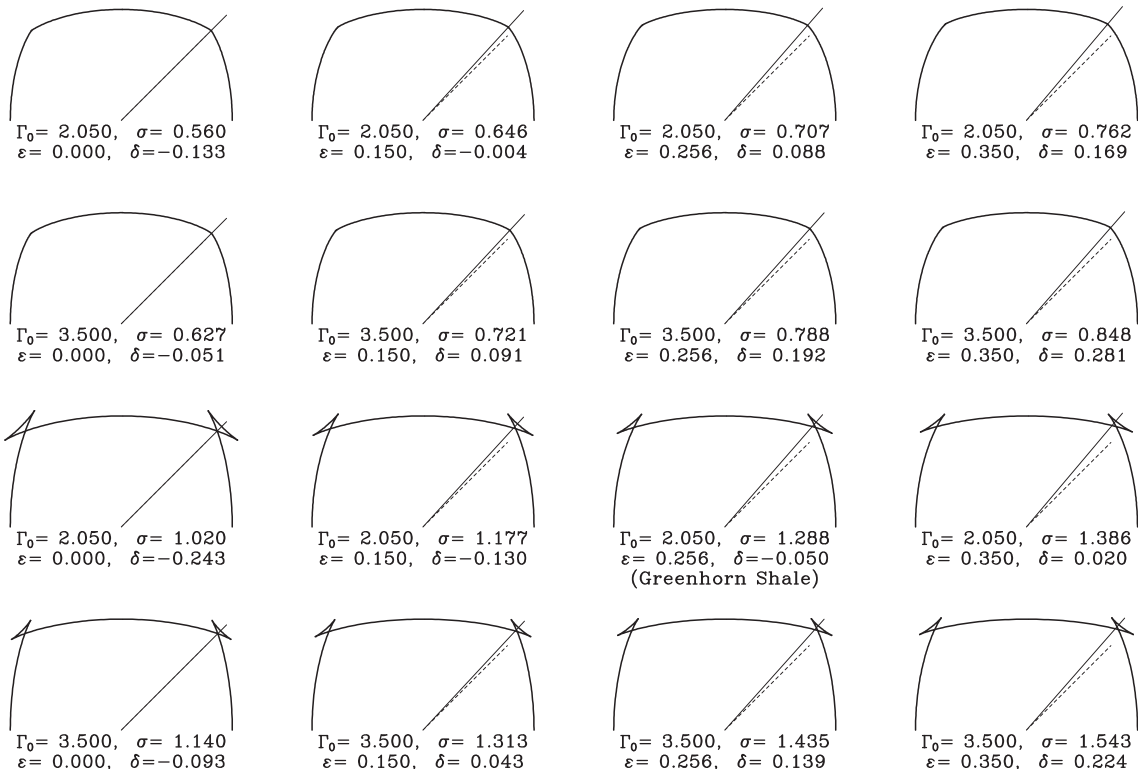

Fig. 4. Group-velocity surfaces $v(\phi)$ for qSV waves in transversely isotropic media with a vertical symmetry axis, for a range of ϵ (increasing, by a column, from left to right) and two different values of Γ_0 and $\sigma/\sigma_{critical}$ (varying by row). The solid tilted line gives the direction of the incipient triplication ϕ_i as approximated by Eq. (24). The shorter dashed line is at a fixed angle of 45° to the symmetry axis.

shear-wave triplications are unlikely to be observed in converted-wave surveys unless a survey is specifically designed to look for them.

6. Conclusions

We present an approximate solution to an old problem, for which the exact solution is long known, but is algebraically complex and impractical to evaluate. With appropriate approximations (not assuming weak qSV-anisotropy), we simplify the conditions for the onset of triplication, identifying the previously defined anisotropy parameter σ as the controlling measure of anisotropy. It follows that commonly measured P-wave kinematics often imply that many formations of the earth's sedimentary crust support qSV triplications. However, the triplications appear to only occur at angles of shear-wave incidence farther from the vertical than are typically probed by surface-seismic converted-wave surveys.

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