Mathematical Geology, Vol. 35, No. 7, October 2003 (© 2003)

Elliptical Descriptors: Some Simplified Morphometric Parameters for the Quantification of Complex Outlines[1]

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Among morphometric methods, a method called Elliptical Fourier Analysis, has been developed to decompose complex outlines into Fourier series. This elliptical Fourier formulation has been rarely applied to date, which is probably explainable because of the mathematical complexity and the difficulty of translating the Fourier coefficients into simple geometrical concepts. Utilizing elliptical analysis, a simplified geometrical approach to the Fourier decomposition is proposed in this study. We showed that the geometrical locus of the points associated with each harmonic used in the Fourier decomposition is an ellipse. The contribution of each harmonic was then characterized with four new geometrical parameters called elliptical descriptors. These are: the half-length of the major axis (L_{A_j}), the half-length of the minor axis (L_{B_j}), the orientation of the major axis, and the phase angle. These descriptors, in contrast to classical Fourier coefficients, possess geometrical significance, and allow for an estimate of each ellipse consisting of: (1) the size of the ellipse (proportional to the product $L_{\mathrm{A}_j} \cdot L_{\mathrm{B}_j}$), (2) the anisotropy of the ellipse (characterized by the ratio $L_{\mathrm{A}_j}/L_{\mathrm{B}_j}$), and (3) the orientation of the ellipse given by the orientation of the elliptical axes. These parameters completely define the geometry of the ellipse associated with each harmonic, and provide an evaluation of the importance of the harmonic contribution in the description of the form studied. Using these elliptical descriptors, an outline can be described, as well as reconstructed. A methodology is then proposed to characterize and to compare complex outlines using these elliptical descriptors. This new methodology allows the quantification of any form, regardless of their degree of complexity, and allows the translation of the morphological differences into simple geometrical concepts, a procedure difficult to carry out with conventional Fourier coefficients.

KEY WORDS: elliptical Fourier analysis, morphometry.

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INTRODUCTION

Among the morphometric methods (Lestrel, 1997a), Fourier methods have been extensively utilized to analyze outlines. Ehrlich and Weinberg (1970) were among the first to develop, using Fourier series, a method to quantify simple outlines. This method has been applied with considerable success in diverse disciplines such as geology (Anstey and Delmet, 1973; Christopher and Waters, 1974; Ehrlich and Weinberg, 1970; Flook, 1987; Kaesler and Waters, 1972), biology (Strojny and others, 1987), osteology, and physical anthropology (Jacobshagen, 1986; Johnson and others, 1992; Le Minor, Pister, and Kahn, 1989; O'Higgins and Johnson, 1993; O'Higgins and Williams, 1987; Schmittbuhl and others, 1998, 1999), to name just of few of the numerous studies now available.

A more sophisticated method called Elliptical Fourier Analysis, was developed by Kuhl and Giardina (1982) to decompose complex outlines into Fourier series. This elliptical Fourier analysis (Diaz and others, 1989; Ferrario and others, 1994; Ferrario and others, 1996; Le Minor and Schmittbuhl, 1999; Lestrel, 1997b; Lestrel, Bodt, and Swindler, 1993; Lestrel and Kerr, 1993; Lowe and others, 1994; Schmittbuhl and others, 1997) has been seldomly applied to date. This is probably explainable due to: (1) its mathematical complexity and (2) the difficulty of translating the Fourier coefficients into simple geometrical concepts.

Utilizing the theoretical developments of Kuhl and Giardina (1982), we defined a new set of parameters which provides for a more direct connection with the geometry of form. These new parameters called elliptical descriptors allow not only the quantification of the form, but also allow for a fuller understanding of the geometrical contribution of the Fourier harmonics.

THE CLASSICAL METHOD OF KUHL AND GIARDINA

Justification of the Method

Simple forms or holomorphic forms contain no reentrants (Beddow and Meloy, 1980), and therefore all the radii emanating from the centroid intersect once and only once the outline (see example in Fig. 1). Such holomorphic outline can be represented in polar coordinates, and expanded into a Fourier series since the polar function characterizing the outline is bijective (transformation which is a one-to-one correspondence).

Complex forms or nonholomorphic forms are characterized by the fact that the radii can intersect the outline at more than one place on the outline (see example in Fig. 2). In this case, the Fourier expansion of the polar function describing the outline becomes impossible, since this function is no longer bijective. In order to

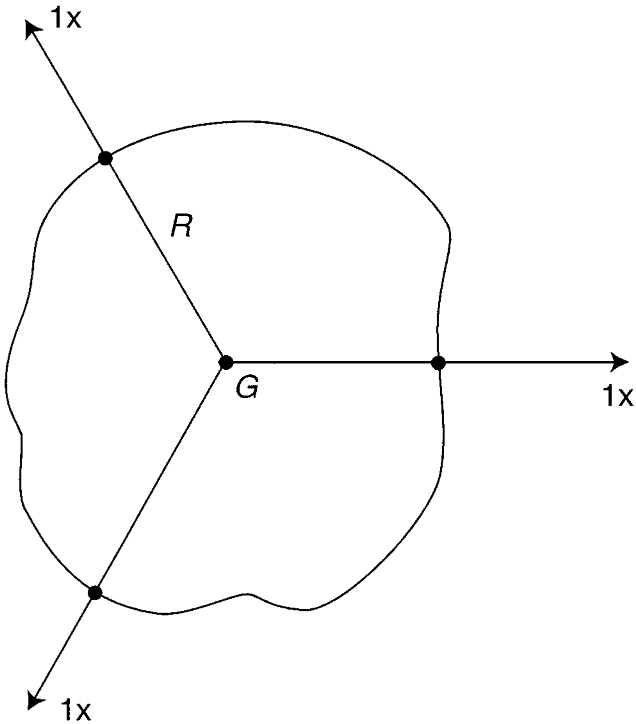

Figure 1. Simple or holomorphic outline. All the radii (R) emanating from the centroid (G) intersect once and only once the outline. Example of the foraminifer *Globoquadrina primitiva*.

fit a Fourier function to a complex outline now in Cartesian coordinates (instead of polar), Kuhl and Giardina (1982) introduced two new bijective functions, describing the outline from the curvilinear coordinates of the points of the outline. These new functions are parametric; that is, functions of a third variable (see next section).

From Cartesian to Curvilinear Coordinates

To fit a Fourier function to a nonholomorphic outline, Kuhl and Giardina (1982) introduced a new expression for the outline in which each x-coordinate and y-coordinate of a point on the outline is expressed as a function of its position on this outline. This means that each x- and y-coordinate is expressed as a function of the curvilinear coordinate (t) of the point being considered (see example in Fig. 3). The outline is thus characterized by the parametric functions $x(t)$ and $y(t)$. Since

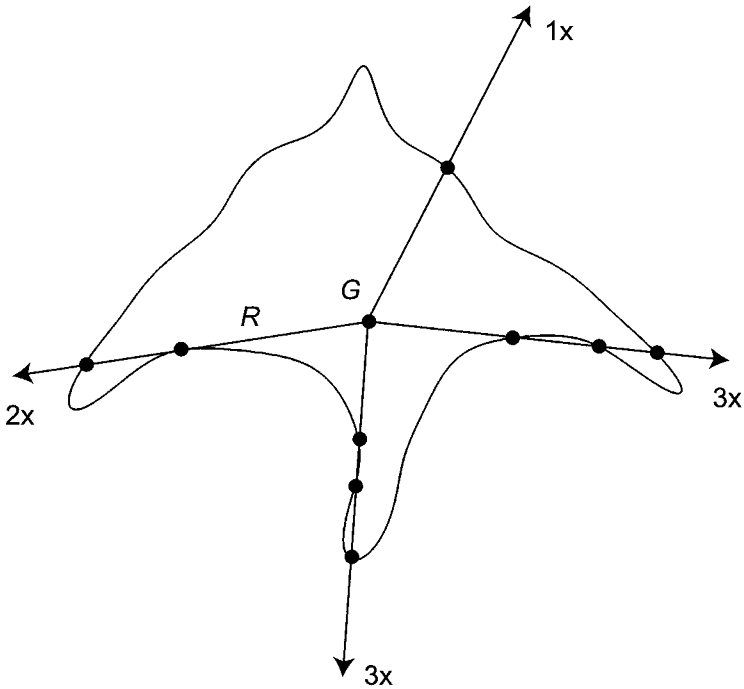

Figure 2. Complex or nonholomorphic outline. The radii (R) emanating from the centroid (G) can intersect the outline more than once. Example of the radiolaria *Pterocanium gravidum*.

these two functions are always bijective and piecewise periodic, they can then each be expanded into Fourier series.

Fourier Expansion of a Nonholomorphic Outline

The Fourier expansion of a nonholomorphic outline can be written as:

$$x_{\mathrm{f}}(t) = a_0 + \sum_{j=1}^{k} a_j \cos\left(\frac{2j\pi t}{T}\right) + \sum_{j=1}^{k} b_j \sin\left(\frac{2j\pi t}{T}\right)$$

and

$$y_{\mathrm{f}}(t) = c_0 + \sum_{j=1}^{k} c_j \cos\left(\frac{2j\pi t}{T}\right) + \sum_{j=1}^{k} d_j \sin\left(\frac{2j\pi t}{T}\right),$$

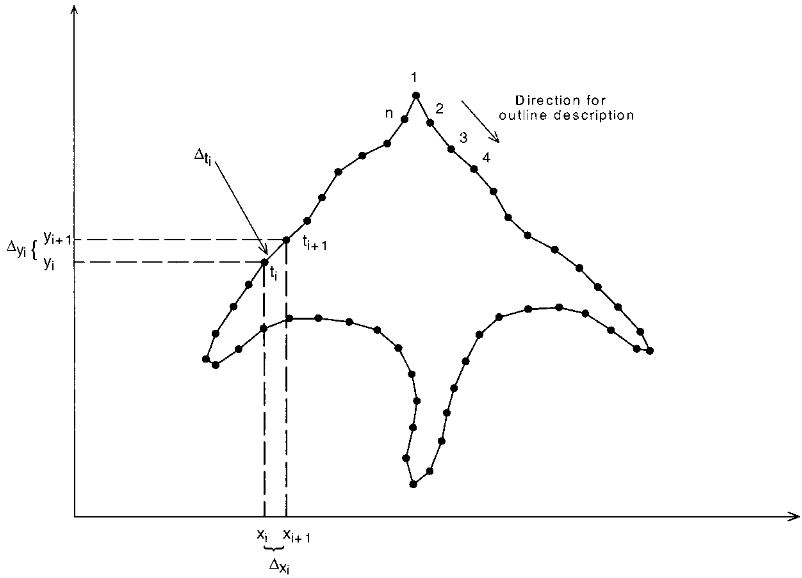

Figure 3. Description of a nonholomorphic outline by the functions $x(t)$ and $y(t)$. (Example of the radiolaria *Pterocanium gravidum*). The x- and y-coordinates of each of the n-sampled points were expressed in parametric fashion as a function of the curvilinear coordinate (t) of the considered point.

where the four Fourier coefficients a_j, b_j, c_j, d_j of the jth harmonic are obtained as:

$$a_j = \frac{1}{2j^2\pi^2} \sum_{i=1}^{n} \frac{\Delta x_i}{\Delta t_i} \cdot \left[\cos\left(\frac{2j\pi t_i}{T}\right) - \cos\left(\frac{2j\pi t_{i-1}}{T}\right)\right],$$

$$b_j = \frac{1}{2j^2\pi^2} \sum_{i=1}^{n} \frac{\Delta x_i}{\Delta t_i} \cdot \left[\sin\left(\frac{2j\pi t_i}{T}\right) - \sin\left(\frac{2j\pi t_{i-1}}{T}\right)\right],$$

$$c_j = \frac{1}{2j^2\pi^2} \sum_{i=1}^{n} \frac{\Delta y_i}{\Delta t_i} \cdot \left[\cos\left(\frac{2j\pi t_i}{T}\right) - \cos\left(\frac{2j\pi t_{i-1}}{T}\right)\right], \quad \text{and}$$

$$d_j = \frac{1}{2j^2\pi^2} \sum_{i=1}^{n} \frac{\Delta y_i}{\Delta t_i} \cdot \left[\sin\left(\frac{2j\pi t_i}{T}\right) - \sin\left(\frac{2j\pi t_{i-1}}{T}\right)\right],$$

and the constant terms a_0 and c_0, respectively for the $x(t)$ and $y(t)$-expansions, are calculated as:

$$a_0 = \frac{1}{T} \sum_{i=1}^{n} \frac{1}{2} \cdot \frac{\Delta x_i}{\Delta t_i} \cdot \left(t_i^2 - t_{i-1}^2\right) - \frac{\Delta x_i}{\Delta t_i} \cdot t_i,$$

and

$$c_0 = \frac{1}{T}\sum_{i=1}^{n}\frac{1}{2}\cdot\frac{\Delta y_i}{\Delta t_i}\cdot\left(t_i^2 - t_{i-1}^2\right) - \frac{\Delta y_i}{\Delta t_i}\cdot t_i,$$

where j = order of the harmonic; k = maximum number of harmonics used for the decomposition; T = perimeter of the outline; Δx_i = module of the step between points i and $i+1$ of the outline projected onto the x-axis; Δy_i = module of the step between points i and $i+1$ of the outline projected onto the y-axis; Δt_i = module of the step between points i and $i+1$ of the outline; t_i = curvilinear coordinate of the point i; n = number of points sampled on the outline (see example in Fig. 3).

METHODOLOGICAL EXTENSIONS TO THE KUHL AND GIARDINA ALGORITHM

Matrix Formulation of the Kuhl and Giardina Equations

The Kuhl and Giardina equations (1) were interpreted as a system of linear relations describing the transformation of a point $M(x_i(t), y_i(t))$ into a point $N(x_f(t), y_f(t))$. We chose to develop these linear relations as follows:

$$\begin{aligned} x_f(t) &= a_0 + a_1\cos\left(\frac{2\pi t}{T}\right) + a_2\cos\left(\frac{4\pi t}{T}\right) + \cdots + a_k\cos\left(\frac{2k\pi t}{T}\right) \\ &\quad + b_1\sin\left(\frac{2\pi t}{T}\right) + b_2\sin\left(\frac{4\pi t}{T}\right) + \cdots + b_k\sin\left(\frac{2k\pi t}{T}\right) \\ y_f(t) &= c_0 + c_1\cos\left(\frac{2\pi t}{T}\right) + c_2\cos\left(\frac{4\pi t}{T}\right) + \cdots + c_k\cos\left(\frac{2k\pi t}{T}\right) \\ &\quad + d_1\sin\left(\frac{2\pi t}{T}\right) + d_2\sin\left(\frac{4\pi t}{T}\right) + \cdots + d_k\sin\left(\frac{2k\pi t}{T}\right) \end{aligned} \tag{1}$$

and collecting the terms of same order:

$$\begin{aligned} x_f(t) &= a_0 + a_1\cos\left(\frac{2\pi t}{T}\right) + b_1\sin\left(\frac{2\pi t}{T}\right) + a_2\cos\left(\frac{4\pi t}{T}\right) \\ &\quad + b_2\sin\left(\frac{4\pi t}{T}\right)\cdots + a_k\cos\left(\frac{2k\pi t}{T}\right) + b_k\sin\left(\frac{2k\pi t}{T}\right) \end{aligned}$$

$$y_f(t) = c_0 + c_1 \cos\left(\frac{2\pi t}{T}\right) + d_1 \sin\left(\frac{2\pi t}{T}\right) + c_2 \cos\left(\frac{4\pi t}{T}\right) + d_2 \sin\left(\frac{4\pi t}{T}\right) \cdots + c_k \cos\left(\frac{2k\pi t}{T}\right) + d_k \sin\left(\frac{2k\pi t}{T}\right) \quad (2)$$

These two relations were then expressed in matrix form:

$$\begin{pmatrix} x_f(t) \\ y_f(t) \end{pmatrix} = \begin{pmatrix} a_0 \\ c_0 \end{pmatrix} + \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \cdot \begin{pmatrix} \cos(\frac{2\pi t}{T}) \\ \sin(\frac{2\pi t}{T}) \end{pmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \cdot \begin{pmatrix} \cos(\frac{4\pi t}{T}) \\ \sin(\frac{4\pi t}{T}) \end{pmatrix} + \cdots \begin{bmatrix} a_k & b_k \\ c_k & d_k \end{bmatrix} \cdot \begin{pmatrix} \cos(\frac{2k\pi t}{T}) \\ \sin(\frac{2k\pi t}{T}) \end{pmatrix} \quad (3)$$

corresponding to the matrix relation:

$$[N(t)] = [T_0] + \sum_{j=1}^{j=k} [T_j] \cdot [M_j(t)] \quad (4)$$

where:

$[M_j(t)]$ was the column matrix $\begin{pmatrix} \cos(\frac{2j\pi t}{T}) \\ \sin(\frac{2j\pi t}{T}) \end{pmatrix}$,

$[N(t)]$ was the column matrix $\begin{pmatrix} x_f(t) \\ y_f(t) \end{pmatrix}$,

$[T_j]$ was the matrix of Fourier coefficients of the jth harmonic $\begin{bmatrix} a_j & b_j \\ c_j & d_j \end{bmatrix}$, and

$[T_0]$ was the column matrix $\begin{pmatrix} a_0 \\ c_0 \end{pmatrix}$.

Geometrical Interpretation of the jth Fourier Harmonic

In the equation above (4), the right member was composed of a column matrix and a sum of matrix products. The jth term of the sum corresponded to the product of a 2×2 matrix (Fourier coefficients of jth order) by the column matrix (circular coordinates of the point $M_j(t)$). The result of this matrix product was another column matrix, which could be interpreted as the linear transformation $[T_j]$ of a point $M_j(t)$) of coordinates $(\cos(\frac{2j\pi t}{T}), \sin(\frac{2j\pi t}{T}))$ into the point $N_j(t)$ of coordinates $(x'_j(t), y'_j(t))$ and shown as:

$$[M_j(t)] \xrightarrow{[T_j]} [N_j(t)], \tag{5}$$

where:

$$[N_j(t)] = [T_j] \cdot [M_j(t)]. \tag{6}$$

Geometrical locus of the coordinate transform $[T_j]$.

A geometrical locus is a set of points verifying a given property (Walker, 1978). The locus of the $M_j(t)$ points $(\cos(\frac{2j\pi t}{T}), \sin(\frac{2j\pi t}{T}))$ was a unit circle of radius one, since the ratio $\frac{2j\pi t}{T}$ varied from 0 to $2j\pi$. Thus, the property of the geometrical locus of the $M_j(t)$ points located on the unit circle of radius one could be expressed in matrix notation as follows:

$$^t[M_j(t)] \cdot [M_j(t)] = 1 \qquad \text{unit circle of radius one} \tag{7}$$

and

$$<=> \qquad x_j^2(t) + y_j^2(t) = 1 \tag{8}$$

This provided the geometrical interpretation of the jth term of the Fourier series and could be interpreted as the geometrical locus corresponding to the coordinate transformation $[T_j]$ of the M_j points into N_j points. The matrix $[T_j]$ transformed the coordinates of the $M_j(t)$ points into $N_j(t)$ and can be shown as:

$$M_j(x_j(t), y_j(t)) \xrightarrow{[T_j]} N_j(x'_j(t), y'_j(t)), \tag{9}$$

or, in matrix notation:

$$[N_j(t)] = [T_j] \cdot [M_j(t)]. \tag{10}$$

The equation of the geometrical locus of the $N_j(t)$ points was obtained by replacing $[M_j(t)]$ by $[T_j^{-1}] \cdot [N_j(t)]$ in the equation of the circle (7) yielding:

$$^t[N_j(t)] \cdot \underbrace{^t\left[T_j^{-1}\right] \cdot \left[T_j^{-1}\right]}_{[D_j]} \cdot [N_j(t)] = 1, \tag{11}$$

$$\text{because } ^t[M_j(t)] = {}^t[N_j(t)] \cdot {}^t\left[T_j^{-1}\right]. \tag{12}$$

$[D_j]$ is a symmetrical matrix since the product of a matrix by its transposed form is always a symmetrical matrix (Frazer, Duncan, and Collar, 1960).

Thus, the equation of the geometrical locus of the $N_j(t)$ points was equal to:

$$^t[N_j(t)] \cdot [D_j] \cdot [N_j(t)] = 1. \tag{13}$$

This equation was the expression of a quadratic form represented by the matrix $[D_j]$, and corresponded geometrically to a conic. The nature of this conic depends on the sign of the determinant of $[D_j]$ (Bix, 1998). The determinant $|D_j|$ could be expressed as a function of $[T_j]$:

$$|D_j| = \left|^t\left[T_j^{-1}\right] \cdot \left[T_j^{-1}\right]\right| = \left|^t\left[T_j^{-1}\right]\right| \cdot \left|\left[T_j^{-1}\right]\right| = \left|T_j^{-1}\right| \cdot \left|T_j^{-1}\right| = \left|T_j^{-1}\right|^2, \tag{14}$$

$$\text{because } |^t[T^{-1}]| = |T^{-1}|. \tag{15}$$

The determinant of $[D_j]$ was always positive showing that the conic was an ellipse. Thus, the geometrical locus of the $N_j(t)$ points, image of $M_j(t)$ points by the transformation $[T_j]$, formed an ellipse centered on the origin (Q.E.D.).

Geometrical Description of the jth Ellipse

The equation of the geometrical locus of the jth ellipse was simplified by writing it in the base defined by the axes of the ellipse. This was obtained by rotating the ellipse in order to align its axes with the axes of the original coordinate system. Mathematically, this operation corresponded to the diagonalization of the matrix $[D_j]$. The matrix $[D_j]$ was always diagonalizable since $[D_j]$ was symmetrical (Frazer, Duncan, and Collar, 1960). The diagonal matrix $[D_{jD}]$ was obtained as follows:

$$[D_{jD}] = \left[\alpha_j^{-1}\right] \cdot [D_j] \cdot [\alpha_j], \tag{16}$$

where $[\alpha_j]$ was the rotation matrix.

This diagonalization also allowed the determination of the eigenvalues and eigenvectors of the matrix $[D_j]$. The eigenvalues $(\lambda_{A_j}, \lambda_{B_j})$ represented respectively, the half-length of the major axis, and the half-length of minor axis of the ellipse; while the eigenvectors defined the rotation angle α_j.

Ordination of the Points on the jth Ellipse

In this derivation, the matrix $[D_j]$ associated to the equation of the geometrical locus of the points $N_j(t)$, was composed of three independent parameters, whereas the matrix $[T_j]$, associated with the coordinate transformation of the points $M_j(t)$ into $N_j(t)$, was defined by four independent parameters corresponding to the four Fourier coefficients (a_j, b_j, c_j, d_j). The consideration of the geometrical locus, when compared to the coordinate transformation, involved the loss of one piece of information. This information is to the positioning of the starting point on the geometrical locus; this positioning is expressed as a phase angle determinating the position of the starting point on each ellipse.

This phase angle was determined by factorizing the matrix $[T_j]$, since only the matrix $[T_j]$ contained all the information. As every transformation defined by a 2×2 asymmetrical matrix can be expressed as the product of a general rotation $[\omega_j]$, with a nonrotational transformation (transformation for which the axes of the ellipse are invariant by the transformation) represented by a symmetrical matrix $[P_j]$) (Hobbs, Means, and Williams, 1976; Martin, 1982), the matrix $[T_j]$ was expressed as follows:

$$[T_j] = [P_j] \cdot [\omega_j], \tag{17}$$

where $[P_j]$ was obtained by the following relation (Schmittbuhl and others, 1997):

$$[P_j] = \left[\alpha_j^{-1}\right] \cdot \left[D_{jD}^{-\frac{1}{2}}\right] \cdot [\alpha_j], \qquad \text{(see (16))} \tag{18}$$

then

$$[T_j] = \left[\left[\alpha_j^{-1}\right] \cdot \left[D_{jD}^{-\frac{1}{2}}\right] \cdot [\alpha_j]\right] \cdot [\omega_j]. \tag{19}$$

Consequently, the general rotation $[\omega_j]$ representing the phase angle, was obtained as follows:

$$[\omega_j] = \left[[\alpha_j] \cdot \left[D_{jD}^{\frac{1}{2}}\right] \cdot \left[\alpha_j^{-1}\right]\right] \cdot [T_j]. \tag{20}$$

Elliptical Descriptors of the jth Harmonic

The jth harmonic was then characterized by a series of four new parameters called the elliptical descriptors:

- half-length of the semi-major axis of the jth ellipse (L_{A_j}) corresponding to ($\lambda_{\mathrm{A}_j}^{-1/2}$),
- half-length of the semi-minor axis of the jth ellipse (L_{B_j}) corresponding to ($\lambda_{\mathrm{B}_j}^{-1/2}$),
- angle of rotation (α_j) determining the orientation of the major axis of the jth ellipse with the axes of the original coordinate, and
- angle of phase (ω_j) corresponding to the positioning of the points on the jth ellipse.

The elliptical descriptors above ($L_{\mathrm{A}_j}, L_{\mathrm{B}_j}, \alpha_j, \omega_j$), in contrast to the Fourier coefficients (a_j, b_j, c_j, d_j), possess geometrical significance. They provide the following estimates for each ellipse:

- the size of the ellipse, proportional to the product of the half-lengths of the axes ($L_{\mathrm{A}_j} \cdot L_{\mathrm{B}_j}$),
- the anisotropy of the ellipse, characterized by the ratio of the half-lengths of the axes ($L_{\mathrm{A}_j}/L_{\mathrm{B}_j}$),
- the orientation of the ellipse characterized by the angle of rotation α_j.

These last three characteristics completely define the geometry of each Fourier ellipse, and provide an evaluation of the importance of the harmonic contribution in the description of the form studied (see example in Fig. 4). The description of the ellipse from the elliptical descriptors constitutes a really simplified approach compared to those proposed by some authors (Diaz and others, 1989; Kuhl and Giardina, 1982; Rohlf and Archie, 1984).

Reconstruction of an Outline Using the Elliptical Descriptors

Using the elliptical descriptors, it becomes possible to reconstruct an outline. This reconstruction is carried out point by point (Fig. 5). Each point $M_j(t)$ belonging to the unit circle of radius one is transformed into a point $N_j(t)$ by the following matrix relation:

$$[N(t)] = \sum_{j=1}^{k} [P_j] \cdot [\omega_j] \cdot [M_j(t)], \tag{21}$$

$$\text{with} \qquad [N(t)] = \begin{pmatrix} x_f(t) \\ y_f(t) \end{pmatrix}, \tag{22}$$

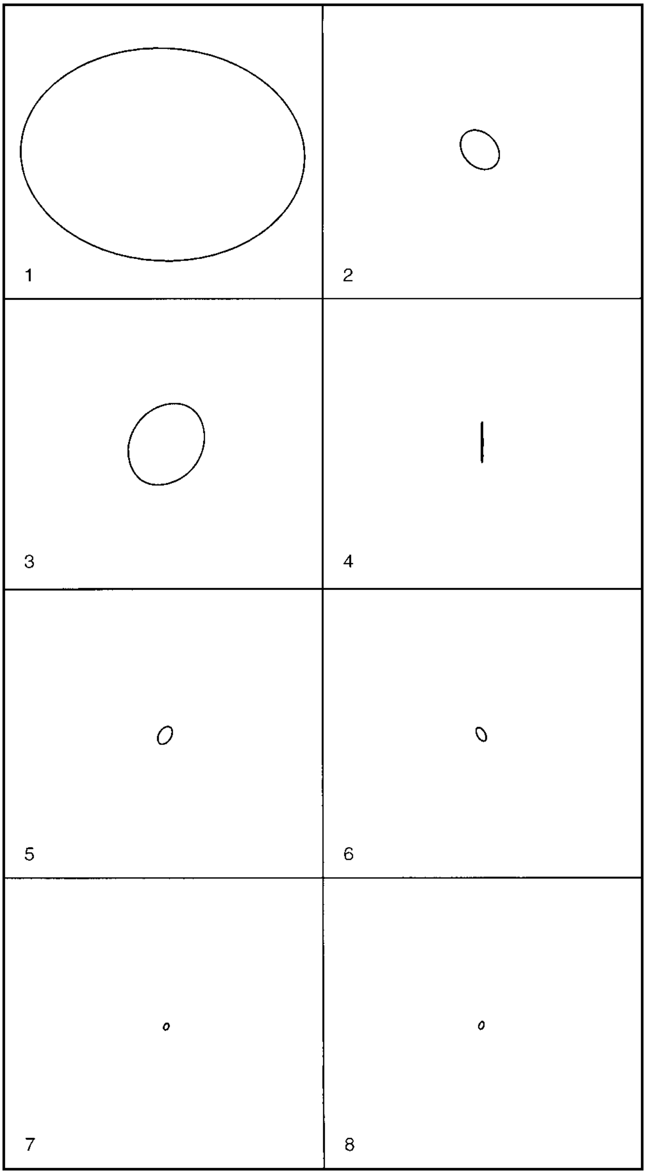

Figure 4. Fourier ellipses. Example of representations of the first eight harmonical ellipses determined from the elliptical Fourier analysis of the outline of the radiolaria *Pterocanium gravidum*. The evaluation of the importance of the contribution of each harmonic in the description of the Radiolaria is provided by the simultaneous examination of elliptic size and anisotropy.

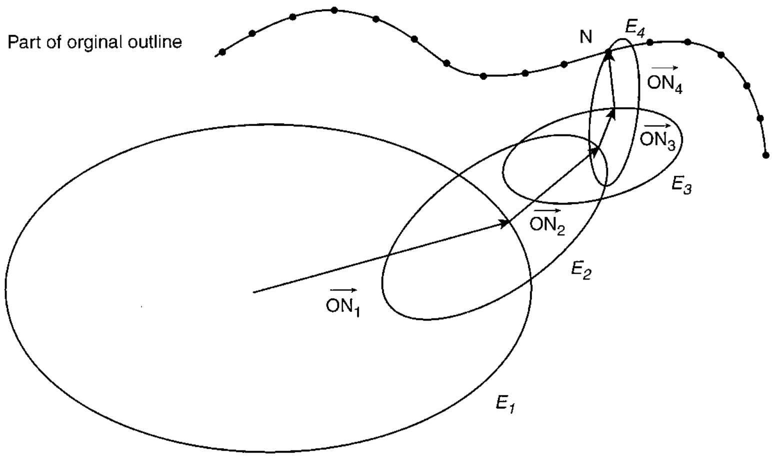

Figure 5. Steps of reconstruction of an outline using elliptical descriptors. As the vectors $\overrightarrow{ON}_{j(j\in\{1,\ldots,k\})}$ are summed, the point N corresponding to the extremity of the sum vector $(\sum \overrightarrow{ON}_j)$ converges towards its homologue on the original outline. (E_1: 1st ellipse; E_2: 2nd ellipse; E_3: 3rd ellipse; E_4: 4th ellipse).

$$[M(t)] = \begin{pmatrix} x_i(t) \\ y_i(t) \end{pmatrix} = \begin{pmatrix} \cos(\frac{2j\pi t}{T}) \\ \sin(\frac{2j\pi t}{T}) \end{pmatrix}, \tag{23}$$

$$\text{and} \qquad [P_j] = \left[\alpha_j^{-1}\right] \cdot \left[D_{jD}^{-\frac{1}{2}}\right] \cdot [\alpha_j] \tag{see 16}$$

The first step of reconstruction is to calculate, for each harmonic, the angular step (Δ_{ang}) between the $M_j(t)$ prints on the unit circle of radius one. This angular step (Δ_{ang}) is a function of the harmonic order (j) and of the number of points (n) used for the reconstruction:

$$\Delta_{\text{ang}} = \frac{2\pi j}{n} \tag{24}$$

The angular step increases proportionally with respect to the order of the harmonic (j).

The second step of the reconstruction for each harmonic consists of the calculation of the transformation of the $M_j(t)$ points, into $N_j(t)$ points:

$$\overrightarrow{OM}_j \xrightarrow{[P_j]\cdot[\omega_j]} \overrightarrow{ON}_j \tag{25}$$

with $\overrightarrow{OM_j}$: vector directed from O (center of the unit circle of radius one) to $M_j(t)$ (point belonging to the circle)

$\overrightarrow{ON_j}$: vector directed from O to $N_j(t)$.

This transformation is obtained by a rotation $[\omega_j]$ of the vectors $\overrightarrow{OM_j}$, followed by a nonrotational transformation $[P_j]$ of these vectors. This is equivalent to an initial rotation of the points $M_j(t)$ (rotation of the starting point), followed by a transformation corresponding, from the consideration of the geometrical locus, to the transformation of circle into ellipse.

The last step corresponds to the summation of the $\overrightarrow{ON_j}$ vectors for each of the harmonics considered (Fig. 5). As the vectors of each harmonic are summed, the point $N(t)$, corresponding to the extremity of the sum vector, converge towards its original homologue on the outline. The harmonics of high order allow for the reconstruction of morphological details (Fig. 6). The accuracy of the reconstruction is therefore increased with increasing number of harmonics (see example in Fig. 7). However, the maximum number of harmonics (k) used for reconstruction, which is a function of the number of sampled points (n), is equal to $n/2$ due to Nyquist frequency considerations (Press and others, 1992).

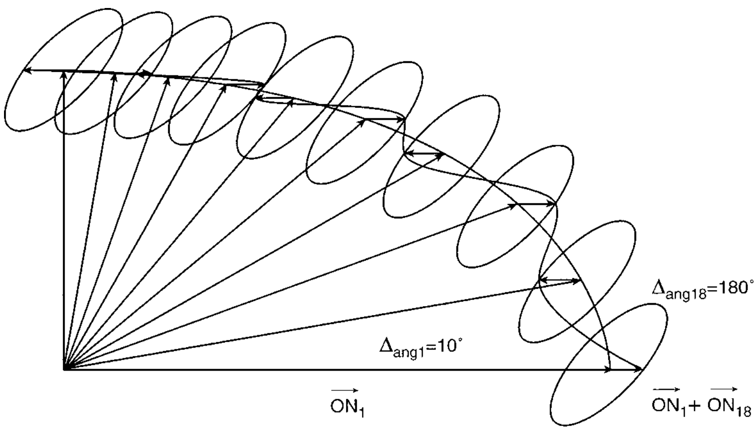

Figure 6. Reconstruction of morphological details. The harmonics of high order allow for the reconstruction of the morphological details; the 18th harmonic is presented as example (Δ_{ang1}, Δ_{ang18}: angular step of the 1st and 18th ellipse; bold line: part of the reconstructed outline ; simple line: part of the 1st ellipse).

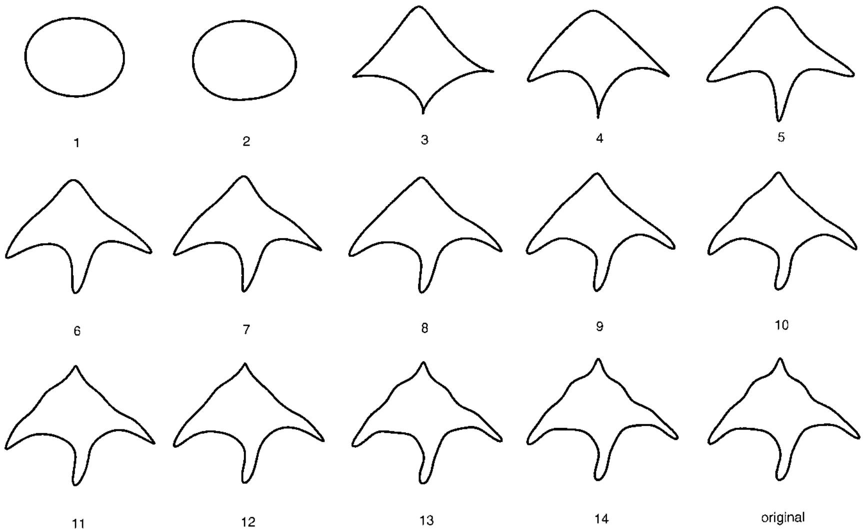

Figure 7. Stepwise reconstructions. Example of the outline of radiolaria *Pterocanium gravidum* using the first 14 harmonics, and representation of the original outline. The accuracy of the reconstruction is therefore increased with increasing number of harmonics.

METHODOLOGY FOR COMPARISON OF COMPLEX OUTLINES USING ELLIPTICAL DESCRIPTORS

The purpose of this part is to present a methodology for the comparison of outlines using elliptical descriptors.

Preliminary Operations of Normalization

Before the morphological comparison of a series of forms using elliptical descriptors can be carried out, a number of preliminary normalization operations are required.

Size Normalization

A size normalization of the enclosed outline area is required in order to compare the importance of the elliptical contributions. The importance of the elliptical contributions, indeed, is related to the size of the ellipses, ellipse size being proportional to the area enclosed by the outlines. The normalization is performed by scaling the area to unit value (1.0); this procedure being already used by other authors (Lestrel 1989a,b; Lestrel, Bodt, and Swindler, 1993; Tanaka and others, 2000).

Angular Normalization

A common orientation of the outlines is needed in order to compare the orientation of the ellipses of one specimen with those of another. The orientation of the ellipses, related to the orientation of the elliptical axes, is a function of the outline orientation. The angular normalization chosen in the present approach consists in the alignment of each outline according to its conventional position of reference. Thus, this angular normalization allows a comparison of forms with respect to their reference positions, such comparisons being difficult to carry out with the procedure of orientation developed by Kuhl and Giardina (1982), and used by other authors (Ferrario and others, 1994, 1996; Lestrel, Bodt, and Swindler, 1993).

Phase Normalization

A same position of each starting point on the outline is required in order to compare the phase angles of one specimen with those of another. This normalization is obtained by shifting the starting point on each outline so that it is located on the x- or y-axis (Kuhl and Giardina, 1982).

Calculation of the Elliptical Descriptors and Stepwise Reconstructions of Outlines

The elliptical descriptors (L_{A_j}, L_{B_j}, α_j, ω_j) are calculated for each harmonic of the Fourier decomposition (Table 1). From these parameters, the size and the anisotropy of each ellipse are estimated (Table 2, see example in Fig. 4).

Stepwise reconstructions of outlines are performed using an increasing number of harmonics (see example in Fig. 7). At each step of the reconstruction, the convergence between the reconstructed outline and the original outline can be estimated visually and quantified by a fit index. The fit index chosen was obtained from the sum of the squared distances between the reconstructed points and the corresponded points on the original outline.

Morphological Comparisons of Outlines Using Elliptical Descriptors and Stepwise Reconstructions

Using the elliptical Fourier analysis, precise morphological comparisons of series of outlines become possible and reproducible. The corresponding statistical

Table 1. Elliptical Descriptors of the Outline of Radiolaria *Pterocanium gravidum*

Harmonics (j)	Major axis half-length (L_{A_j})	Minor axis half-length (L_{B_j})	Orientation of the major axis (α_j)	Phase angle (ω_j)
1	718.27	587.89	5.37	97.26
2	124.72	89.13	60.35	151.55
3	229.43	180.87	113.76	160.00
4	110.01	0.44	91.16	184.75
5	54.18	30.38	114.55	17.77
6	40.28	18.35	71.78	18.57
7	20.07	10.07	121.08	178.93
8	21.03	12.45	115.69	207.86
9	16.96	4.68	97.89	17.04
10	15.12	7.25	79.75	10.33
11	14.07	8.00	104.86	172.57
12	4.08	3.27	176.93	277.42
13	13.99	9.17	93.28	10.12
14	6.78	1.72	75.19	188.68

Note. Results for the first 14 harmonics.

analyses provide for intraspecific and/or interspecific comparisons of elliptical descriptors associated to each harmonic. Multivariate statistical analyses (Reyment, Blackitch, and Campbell, 1984; Sharma, 1996) of the elliptical descriptors can be used to perform such comparisons. Stepwise reconstructions (Fig. 7) allow the

Table 2. Elliptical Descriptors of the Outline of Radiolaria *Pterocanium gravidum*

Harmonics (j)	Size	Anisotropy
1	422266.21	1.22
2	11116.59	1.40
3	41497.72	1.27
4	48.65	248.79
5	1645.75	1.78
6	738.91	2.20
7	202.02	1.99
8	261.92	1.69
9	79.39	3.62
10	109.65	2.08
11	112.49	1.76
12	13.34	1.25
13	128.32	1.52
14	11.68	3.93

Note. Size and anisotropy of the ellipses associated to the first 14 harmonics.

establishment of direct relationships between elliptical descriptors variability and real morphological variability. Previous works, based only on Fourier coefficients, could not provide these interpretations and comparisons.

CONCLUSION

The elliptical descriptors developed in this study, not only allow for the quantification of form, whatever their degree of complexity, but also allow for the translation of morphological differences into simple geometrical concepts; a procedure difficult to carry out with the conventional Fourier coefficients.

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