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Transfer equations applying to layered media sounding problems

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SUMMARY

The sounding problem of 1-D layered media has been solved with wave energy transfer equations. Forward and backward wave propagation at any trajectory point are calculated. Analytical solutions for simple trajectories are derived. In the general case the solutions can be determined numerically. For certain active seismic sounding problem having good data coverage, this algorithm can be used to invert for characteristics of the layered media.

Key words: inverse problem, layered media, transfer equations, wave propagation.

1 INTRODUCTION

Wave equations, as well as the other methods of signal processing, permitting us to simplify signal decoding (Huang & Fehler 2000; Sato & Kennett 2000; de Smet *et al.* 2000; Kennett 2000; Toomey & Bean 2000; Mendel & Goutsias 1986) are commonly used for layered media sounding problems. At present wave coupling and scattering are still the main problem of deep seismic sounding, and propagation through heterogeneous media is still are of the most important problems of theoretical investigations (Pavlenkova 1999). Wave scattering theory is a key element. Two approaches have been adopted in multiple scattering theory. One of them is exact or analytical theory, and the other is transfer theory.

The exact approach is based on the wave equations. Scattering characteristics are determined, and then corresponding differential and integral equations for such statistical parameters as dispersion and correlation functions are derived. This approach is mathematically exact in the sense that, in principle, multi-scattering, diffraction and interference effects can be taken into account. However, it is almost impossible to construct the theory taking all effects into consideration completely. So all theories giving acceptable solutions are approximate and valid only for a certain parameter range.

On the other hand, transfer theory is not based on the wave equations, but is derived directly with energy transfer in a scattering medium (Shang & Gao 1988; Zeng *et al.* 1991; Sato 1993; Margerin *et al.* 2000; Chandrasekhar 1950; Ishimaru 1978). Such theory is constructed heuristically, and cannot be called mathematically vigorous. Diffraction and interference are important in scattering due to inhomogeneity, but transfer theory does not describe these effects accurately. It is assumed for transfer theory, that there is no correlation between fields.

In spite of the differences between the two approaches, there exist fundamental relationships, because the approaches describe the same phenomena. In particular, it is shown that the intensity for transfer theory and the mutual coherence function for exact theory are connected by the Fourier transformation (Ishimaru 1978). This means that, though transfer theory is based on intensities, it includes field correlation information. Polarization effects can be taken into consideration, too.

Besides that, we think that diffraction and interference effects are background components in propagating through layered media. The absence of such effects in transfer theory gives a possibility to get the solution in analytical form, which is used a reference for layered media studies. Diffraction and interference and other noise sources are considered in subsequent statistical analysis.

Energy transfer integral equations have been proposed by Shang & Gao (1988) and Zeng *et al.* (1991). Those equations are based on the knowledge of the medium Greens function, the choice of which involves a certain arbitrariness. Analytical solutions were deduced for the cases of 1-D media (Sato 1993) and 2-D media (Shang & Gao 1988). Energy transfer classical differential equations are used instead in our paper. We have succeeded in deriving complete solutions as well as analytical solutions for specific cases of the 1-D problem under rather general assumptions.

2 ENERGY TRANSFER EQUATIONS

Based on standard methods of transfer equation development (Chandrasekhar 1950; Ishimaru 1978), the following two functions describe the 1-D case: $p(t, z)$-the density of energy, propagating in the direction of increasing z, $q(t, z)$-the density of energy, propagating in opposite direction. We can write the following expressions if wave energy $\gamma(z)\,dz$ is absorbed in the path element $(z, z+dz)$, $\sigma(z)\,dz$ is reflected, and $1-\gamma(z)dz-\sigma(z)\,dz = 1-\varepsilon(z)\,dz$ remains and continues to propagate in the original direction:

$$p(t+dt,z) = p(t, z-V(z)\,dt)(1-\varepsilon(z)V(z)\,dt) + q(t, z+V(z)\,dt)\sigma(z)V(z)\,dt + f(t,z)\,dt, \tag{1}$$

$$q(t+dt,z) = q(t, z+V(z)\,dt)(1-\varepsilon(z)V(z)\,dt) + p(t, z-V(z)\,dt)\sigma(z)V(z)\,dt + g(t,z)\,dt. \tag{2}$$

In a differential form, we have

$$\frac{1}{V(z)}\frac{\partial p(t,z)}{\partial t} + \frac{\partial p(t,z)}{\partial z} + \varepsilon(z)p(t,z) = \sigma(z)q(t,z) + \frac{1}{V(z)}f(t,z), \tag{3}$$

$$\frac{1}{V(z)}\frac{\partial q(t,z)}{\partial t} - \frac{\partial q(t,z)}{\partial z} + \varepsilon(z)q(t,z) = \sigma(z)p(t,z) + \frac{1}{V(z)}g(t,z). \tag{4}$$

Here $V(z)$ is wave propagation velocity at the point z, $f(t,z)\,dt$ is the density of energy, generated by external transmitters (radiators) in the direction of increasing z for the time dt, and $g(t,z)\,dt$ is the analogous quantity in the opposite direction. If we consider undirected transmitters, then $g(t,z) = f(t,z)$.

2.1 Transfer equation solution in the case of layered (piecewise inhomogeneous) media

Let us consider that a transmitter produces at the point $z = a$ the energy pulse $f(t)$ in the direction of increasing z and the pulse $g(t)$ in the opposite direction; that is, $f(t,z) = \delta(z-a)f(t)$, $g(t,z) = \delta(z-a)g(t)$, where $\delta(z-a)$ is the delta function. After the Fourier transformation of (3) and (4), we can write

$$\tilde{\varepsilon}(i\omega,z)P(i\omega,z) + \frac{\partial P(i\omega,z)}{\partial z} = \sigma(z)Q(i\omega,z) + \frac{F(i\omega)}{V(z)}\delta(z-a), \tag{5}$$

$$\tilde{\varepsilon}(i\omega,z)Q(i\omega,z) - \frac{\partial Q(i\omega,z)}{\partial z} = \sigma(z)P(i\omega,z) + \frac{G(i\omega)}{V(z)}\delta(z-a), \tag{6}$$

where

$$\tilde{\varepsilon}(i\omega,z) = \frac{i\omega}{V(z)} + \varepsilon(z), \quad P(i\omega,z) = \int p(t,z)\mathrm{e}^{-i\omega t}\,dt, \quad Q(i\omega,z) = \int q(t,z)\mathrm{e}^{-i\omega t}\,dt,$$
$$F(i\omega) = \int f(t)\mathrm{e}^{-i\omega t}\,dt, \quad G(i\omega) = \int g(t)\mathrm{e}^{-i\omega t}\,dt. \tag{7}$$

We can make the following important conclusions, analysing the system of eqs (5) and (6). First, for homogeneous ($\varepsilon, \sigma, V =$ constant) media without transmitters, functions P and Q must obey the same general equation

$$\frac{\partial^2 P}{\partial z^2} - \lambda^2 P = 0, \tag{8}$$

where

$$\lambda^2 = \tilde{\varepsilon}^2 - \sigma^2, \quad \tilde{\varepsilon}(i\omega) = \frac{i\omega}{V} + \varepsilon \tag{9}$$

Secondly, $P(i\omega,z)$ and $Q(i\omega,z)$ are continuous of z, except at $z = a$, where each has a first order discontinuity. The solutions of the equation have the form

$$A\mathrm{e}^{\lambda z} + B\mathrm{e}^{-\lambda z}, \quad (\mathrm{Re}\lambda > 0), \tag{10}$$

where A and B are arbitrary constants.

2.2 Piecewise inhomogeneous media

Let us consider **a** piecewise homogeneous medium, having the following boundaries lines: $a_1 < a_2 < \cdots < a_N < a_{N+1} = \infty$. The medium is homogeneous in each interval (a_n, a_{n+1}), and is characterized by the following parameters: $\varepsilon_{n+1}, \sigma_{n+1}, \gamma_{n+1}$. Medium parameters are $\varepsilon_1, \sigma_1, \gamma_1$ on the interval $(-\infty, a_1)$. We can take the wave transmitter to be positioned at a boundary a_k; that is $a = a_k$. If the transmitter is located within some interval of homogeneity, interval parts to the left and right of the transmitter can be considered as different intervals, but their characteristics are the same. For example, the boundary a_1 is the Earth's surface, the layers $a_2, \ldots, a_N$ are located inside the Earth; or a_1 is the water surface, a_2 is the water bottom, and the layers $a_3, \ldots, a_N$ are located inside the Earth.

According to (10) we can write for the interval (a_n, a_{n+1})

$$P(i\omega,z) = A_{n+1}\mathrm{e}^{\lambda_{n+1}z} + B_{n+1}\mathrm{e}^{-\lambda_{n+1}z}, \tag{11}$$

$$Q(i\omega,z) = C_{n+1}\mathrm{e}^{\lambda_{n+1}z} + D_{n+1}\mathrm{e}^{-\lambda_{n+1}z}. \tag{12}$$

Substituting (11) and (12), then we have

$$(\tilde{\varepsilon}_{n+1} + \lambda_{n+1})A_{n+1}\mathrm{e}^{\lambda_{n+1}z} + (\tilde{\varepsilon}_{n+1} - \lambda_{n+1})B_{n+1}\mathrm{e}^{-\lambda_{n+1}z} = \sigma_{n+1}C_{n+1}\mathrm{e}^{\lambda_{n+1}z} - \sigma_{n+1}D_{n+1}\mathrm{e}^{-\lambda_{n+1}z}; \tag{13}$$

that is,

$$A_{n+1} = \frac{\sigma_{n+1}}{\tilde{\varepsilon}_{n+1} + \lambda_{n+1}} C_{n+1} = \frac{\tilde{\varepsilon}_{n+1} - \lambda_{n+1}}{\sigma_{n+1}} C_{n+1}, \tag{14}$$

$$B_{n+1} = \frac{\sigma_{n+1}}{\tilde{\varepsilon}_{n+1} - \lambda_{n+1}} D_{n+1} = \frac{\tilde{\varepsilon}_{n+1} + \lambda_{n+1}}{\sigma_{n+1}} D_{n+1}. \tag{15}$$

It should be noticed that the functions $P(i\omega, z)$ and $Q(i\omega, z)$ are continuous at the points $a_1 < a_2 < \cdots < a_{k-1}$ and $a_{k+1} < a_{k+2} < \cdots < a_N$, and they have a discontinuity at the point a_k:

$$P(i\omega, a_k + 0) - P(i\omega, a_k - 0) = \frac{1}{V_{k+1}} F(i\omega), \tag{16}$$

$$Q(i\omega, a_k + 0) - Q(i\omega, a_k - 0) = -\frac{1}{V_k} G(i\omega). \tag{17}$$

It follows that

$$A_{n+1} e^{\lambda_{n+1} a_n} + B_{n+1} e^{-\lambda_{n+1} a_n} = A_n e^{\lambda_n a_n} + B_n e^{-\lambda_n a_n} + \delta_{nk} \frac{F(i\omega)}{V_{k+1}} \tag{18}$$

$$C_{n+1} e^{\lambda_{n+1} a_n} + D_{n+1} e^{-\lambda_{n+1} a_n} = C_n e^{\lambda_n a_n} + D_n e^{-\lambda_n a_n} - \delta_{nk} \frac{G(i\omega)}{V_k}, \tag{19}$$

where

$$\delta_{nk} = \begin{cases} 1, & n = k \\ 0, & n \neq k. \end{cases}$$

Using the relations (14) and (15), we can rewrite the system of equations in backward propagation wave amplitudes C_n and D_n or in forward propagation wave amplitudes A_n and B_n. Taking into account only backward propagation wave amplitudes C_n and D_n, we obtain

$$C_{n+1} e^{\lambda_{n+1} a_n} + D_{n+1} e^{-\lambda_{n+1} a_n} = C_n e^{\lambda_n a_n} + D_n e^{-\lambda_n a_n} - \delta_{nk} \frac{1}{V_k} G(i\omega), \tag{20}$$

$$C_{n+1} \frac{\sigma_{n+1}}{\tilde{\varepsilon}_{n+1} + \lambda_{n+1}} e^{\lambda_{n+1} a_n} + D_{n+1} \frac{\sigma_{n+1}}{\tilde{\varepsilon}_{n+1} - \lambda_{n+1}} e^{-\lambda_{n+1} a_n} = C_n \frac{\sigma_n}{\tilde{\varepsilon}_n + \lambda_n} e^{\lambda_n a_n} + D_n \frac{\sigma_n}{\tilde{\varepsilon}_n - \lambda_n} e^{-\lambda_n a_n} + \delta_{nk} \frac{1}{V_{k+1}} F(i\omega). \tag{21}$$

Using matrix designations, the relations can be written as

$$\begin{pmatrix} e^{\lambda_{n+1} a_n} & e^{-\lambda_{n+1} a_n} \\ \dfrac{\sigma_{n+1}}{\tilde{\varepsilon}_{n+1} + \lambda_{n+1}} e^{\lambda_{n+1} a_n} & \dfrac{\sigma_{n+1}}{\tilde{\varepsilon}_{n+1} - \lambda_{n+1}} e^{-\lambda_{n+1} a_n} \end{pmatrix} \begin{pmatrix} C_{n+1} \\ D_{n+1} \end{pmatrix} = \begin{pmatrix} e^{\lambda_n a_n} & e^{-\lambda_n a_n} \\ \dfrac{\sigma_n}{\tilde{\varepsilon}_n + \lambda_n} e^{\lambda_n a_n} & \dfrac{\sigma_n}{\tilde{\varepsilon}_n - \lambda_n} e^{-\lambda_n a_n} \end{pmatrix} \begin{pmatrix} C_n \\ D_n \end{pmatrix} + \delta_{nk} \begin{pmatrix} -\dfrac{G(i\omega)}{V_k} \\ \dfrac{F(i\omega)}{V_{k+1}} \end{pmatrix}. \tag{22}$$

The solution of the recurrent eq. (22) has the form

$$\begin{aligned} \begin{pmatrix} e^{\lambda_{n+1} a_n} C_{n+1} \\ e^{-\lambda_{n+1} a_n} D_{n+1} \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ \dfrac{\sigma_{n+1}}{\tilde{\varepsilon}_{n+1} + \lambda_{n+1}} & \dfrac{\sigma_{n+1}}{\tilde{\varepsilon}_{n+1} - \lambda_{n+1}} \end{pmatrix}^{-1} g_n g_{n-1} \ldots g_2 \begin{pmatrix} 1 & 1 \\ \dfrac{\sigma_1}{\tilde{\varepsilon}_1 + \lambda_1} & \dfrac{\sigma_1}{\tilde{\varepsilon}_1 - \lambda_1} \end{pmatrix} \begin{pmatrix} e^{\lambda_1 a_1} C_1 \\ e^{-\lambda_1 a_1} D_1 \end{pmatrix} \\ &+ \eta_{nk} \begin{pmatrix} 1 & 1 \\ \dfrac{\sigma_{n+1}}{\tilde{\varepsilon}_{n+1} + \lambda_{n+1}} & \dfrac{\sigma_{n+1}}{\tilde{\varepsilon}_{n+1} - \lambda_{n+1}} \end{pmatrix}^{-1} g_n g_{n-1} \ldots g_{k+1} \begin{pmatrix} -\dfrac{G(i\omega)}{V_k} \\ \dfrac{F(i\omega)}{V_{k+1}} \end{pmatrix}, \end{aligned} \tag{23}$$

where

$$g_n = \begin{pmatrix} 1 & 1 \\ \dfrac{\sigma_n}{\tilde{\varepsilon}_n + \lambda_n} & \dfrac{\sigma_n}{\tilde{\varepsilon}_n - \lambda_n} \end{pmatrix} \begin{pmatrix} e^{\lambda_n (a_n - a_{n-1})} & 0 \\ 0 & e^{-\lambda_n (a_n - a_{n-1})} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \dfrac{\sigma_n}{\tilde{\varepsilon}_n + \lambda_n} & \dfrac{\sigma_n}{\tilde{\varepsilon}_n - \lambda_n} \end{pmatrix}^{-1}, \tag{24}$$

$$\eta_{nk} = \sum_{m=1}^{n} \delta_{mk} = \begin{cases} 0, & n < k, \\ 1, & n \geq k. \end{cases} \tag{25}$$

It should be noticed, that

$$\begin{aligned} g_n g_{n-1} \ldots g_2 &= E, \quad \text{for the case of } n = 1 \\ g_n g_{n-1} \ldots g_{k+1} &= E, \quad \text{for the case of } n = k, \end{aligned} \tag{26}$$

where E is the unit matrix.

Formuala (23) describes backward propagation were amplitudes C_{n+1} and D_{n+1} at the boundary a_n in terms backward propagation wave amplitudes C_1 and D_1 at the boundary a_1.

However, the amplitudes C_1 and D_1 are still unknown. From physical considerations we can make a conclusion that $D_1 = 0$, because the wave cannot tend to the infinity if $z \to -\infty$. Analogously, $C_{N+1} = 0$, because the wave cannot tend to the infinity if $z \to +\infty$. So, we have to assume $D_1 = 0$ in (23) and obtain C_1 from the condition that $C_{N+1} = 0$. A closely related method was used by Felderhof (1986) for solving of the 1-D problem of wave propagation along a randomly distributed array of scatterers.

Let us consider the product of $g_n g_{n-1} \ldots g_2$ for $n \geq 2$. Matrix g_n can be written in the following form

$$g_n = \frac{1}{2}\left\{ e^{\lambda_n \Delta a_n} \begin{pmatrix} \frac{\tilde{\varepsilon}_n}{\lambda_n} + 1 & -\frac{\sigma_n}{\lambda_n} \\ \frac{\sigma_n}{\lambda_n} & -\frac{\tilde{\varepsilon}_n}{\lambda_n} + 1 \end{pmatrix} + e^{-\lambda_n \Delta a_n} \begin{pmatrix} -\frac{\tilde{\varepsilon}_n}{\lambda_n} + 1 & \frac{\sigma_n}{\lambda_n} \\ -\frac{\sigma_n}{\lambda_n} & \frac{\tilde{\varepsilon}_n}{\lambda_n} + 1 \end{pmatrix} \right\}, \tag{27}$$

where $\Delta a_n = a_n - a_{n-1}$. Let us designate

$$x_n = \frac{1}{2}\left(\sqrt{\frac{\tilde{\varepsilon}_n + \sigma_n}{\lambda_n}} + \sqrt{\frac{\tilde{\varepsilon}_n - \sigma_n}{\lambda_n}} \right) = \frac{u_n + v_n}{2},$$
$$y_n = \frac{1}{2}\left(\sqrt{\frac{\tilde{\varepsilon}_n + \sigma_n}{\lambda_n}} - \sqrt{\frac{\tilde{\varepsilon}_n - \sigma_n}{\lambda_n}} \right) = \frac{u_n - v_n}{2}, \tag{28}$$

where

$$u_n = \sqrt{\frac{\tilde{\varepsilon}_n + \sigma_n}{\lambda_n}}, \quad v_n = \sqrt{\frac{\tilde{\varepsilon}_n - \sigma_n}{\lambda_n}}. \tag{29}$$

It should be noticed that

$$u_n v_n = 1, \quad \frac{u_n - v_n}{u_n + v_n} = \frac{\tilde{\varepsilon}_n - \lambda_n}{\sigma_n} = \frac{\sigma_n}{\tilde{\varepsilon}_n + \lambda_n}, \tag{30}$$

$$\frac{1}{2}\begin{pmatrix} \frac{\tilde{\varepsilon}_n}{\lambda_n} + 1 & -\frac{\sigma_n}{\lambda_n} \\ \frac{\sigma_n}{\lambda_n} & -\frac{\tilde{\varepsilon}_n}{\lambda_n} + 1 \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} (x_n, -y_n),$$
$$\frac{1}{2}\begin{pmatrix} -\frac{\tilde{\varepsilon}_n}{\lambda_n} + 1 & \frac{\sigma_n}{\lambda_n} \\ -\frac{\sigma_n}{\lambda_n} & \frac{\tilde{\varepsilon}_n}{\lambda_n} + 1 \end{pmatrix} = \begin{pmatrix} y_n \\ x_n \end{pmatrix} (-y_n, x_n). \tag{31}$$

It follows that

$$g_n = \left\{ e^{\lambda_n \Delta a_n} \begin{pmatrix} x_n \\ y_n \end{pmatrix} (x_n, -y_n) + e^{-\lambda_n \Delta a_n} \begin{pmatrix} y_n \\ x_n \end{pmatrix} (-y_n, x_n) \right\}$$
$$= \sum_{\theta=\pm 1} e^{\theta \lambda_n \Delta a_n} \begin{pmatrix} \frac{1+\theta}{2} x_n + \frac{1-\theta}{2} y_n \\ \frac{1-\theta}{2} x_n + \frac{1+\theta}{2} y_n \end{pmatrix} \left(\frac{1+\theta}{2} x_n - \frac{1-\theta}{2} y_n, \frac{1-\theta}{2} x_n - \frac{1+\theta}{2} y_n \right)$$
$$= \sum_{\theta=\pm 1} e^{\theta \lambda_n \Delta a_n} \begin{pmatrix} \frac{u_n + \theta v_n}{2} \\ \frac{u_n - \theta v_n}{2} \end{pmatrix} \left(\frac{v_n + \theta u_n}{2}, \frac{v_n - \theta u_n}{2} \right), \tag{32}$$

and then

$$g_n g_{n-1} \cdots g_2 = \sum_{\theta_2, \ldots, \theta_n = \pm 1} e^{\sum_{m=2}^{n} \theta_m \lambda_m \Delta a_m} \begin{pmatrix} \frac{u_n + \theta_n v_n}{2} \\ \frac{u_n - \theta_n v_n}{2} \end{pmatrix} \left(\frac{u_n + \theta_n v_n}{2}, \frac{u_n - \theta_n v_n}{2} \right)$$
$$\times \begin{pmatrix} \frac{u_{n-1} + \theta_{n-1} v_{n-1}}{2} \\ \frac{u_{n-1} - \theta_{n-1} v_n}{2} \end{pmatrix} \left(\frac{u_{n-1} + \theta_{n-1} v_{n-1}}{2}, \frac{u_{n-1} - \theta_{n-1} v_n}{2} \right) \cdots \begin{pmatrix} \frac{u_2 + \theta_2 v_2}{2} \\ \frac{u_2 - \theta_2 v_2}{2} \end{pmatrix} \left(\frac{v_2 + \theta_2 u_2}{2}, \frac{v_2 - \theta_2 u_2}{2} \right)$$
$$= \sum_{\theta_2, \ldots, \theta_n = \pm 1} e^{\sum_{m=2}^{n} \theta_m \lambda_m \Delta a_m} \prod_{m=3}^{n} \frac{1}{2} (u_{m-1} v_m + \theta_m \theta_{m-1} v_{m-1} u_m) \begin{pmatrix} \frac{u_n + \theta_n v_n}{2} \\ \frac{u_n - \theta_n v_n}{2} \end{pmatrix} \left(\frac{v_2 + \theta_2 u_2}{2}, \frac{v_2 - \theta_2 u_2}{2} \right), \tag{33}$$

where $\prod_{m=3}^{2}\ldots$ is changed to 1 in the case of $n = 2$. It should be reminded that the entire sum $\sum_{\theta_2,\ldots,\theta_n=\pm 1}\ldots$ is replaced with unit matrix in the case of $n = 1$.

Let us substitute expression (33) in (23), assuming that $D_1 = 0$. We obtain

$$\begin{aligned}
&\begin{pmatrix} 1 & 1 \\ \dfrac{\sigma_{n+1}}{\tilde{\varepsilon}_{n+1}+\lambda_{n+1}} & \dfrac{\sigma_{n+1}}{\tilde{\varepsilon}_{n+1}-\lambda_{n+1}} \end{pmatrix}^{-1} g_n g_{n-1}\cdots g_2 \begin{pmatrix} 1 & 1 \\ \dfrac{\sigma_1}{\tilde{\varepsilon}_1+\lambda_1} & \dfrac{\sigma_1}{\tilde{\varepsilon}_1-\lambda_1} \end{pmatrix} \begin{pmatrix} e^{\lambda_1 a_1} C_1 \\ e^{-\lambda_1 a_1} D_1 \end{pmatrix} \\
&= \frac{\sigma_{n+1}}{2\lambda_{n+1}} \begin{pmatrix} \dfrac{\sigma_{n+1}}{\tilde{\varepsilon}_{n+1}-\lambda_{n+1}} & -1 \\ -\dfrac{\sigma_{n+1}}{\tilde{\varepsilon}_{n+1}+\lambda_{n+1}} & 1 \end{pmatrix} \sum_{\theta_2,\ldots,\theta_n=\pm 1} e^{\sum_{m=2}^{n}\theta_m\lambda_m\Delta a_m} \prod_{m=3}^{n}\frac{1}{2}(u_{m-1}v_m+\theta_m\theta_{m-1}v_{m-1}u_m) \\
&\times \begin{pmatrix} \dfrac{u_n+\theta_n v_n}{2} \\ \dfrac{u_n-\theta_n v_n}{2} \end{pmatrix} \left(\frac{v_2+\theta_2 u_2}{2}, \frac{v_2-\theta_2 u_2}{2}\right) \begin{pmatrix} 1 & 1 \\ \dfrac{\sigma_1}{\tilde{\varepsilon}_1+\lambda_1} & \dfrac{\sigma_1}{\tilde{\varepsilon}_1-\lambda_1} \end{pmatrix} \begin{pmatrix} e^{\lambda_1 a_1} C_1 \\ 0 \end{pmatrix} \\
&= \sum_{\substack{\theta_2,\ldots,\theta_n=\pm 1 \\ \theta_1 = 1}} e^{\sum_{m=2}^{n}\theta_m\lambda_m\Delta a_m} \frac{\sigma_{n+1}}{2^{n-1}\lambda_{n+1}} \begin{pmatrix} \dfrac{u_n v_{n+1}+\theta_n u_{n+1} v_n}{u_{n+1}-v_{n+1}} \\ \dfrac{u_n v_{n+1}-\theta_n u_{n+1} v_n}{u_{n+1}+v_{n+1}} \end{pmatrix} \prod_{m=2}^{n}(u_{m-1}v_m+\theta_m\theta_{m-1}v_{m-1}u_m)\frac{1}{u_1+v_1} e^{\lambda_1 a_1} C_1
\end{aligned} \tag{34}$$

If $n \geq k+1$, we can deduce the above in a similar manner

$$\begin{aligned}
&\begin{pmatrix} 1 & 1 \\ \dfrac{\sigma_{n+1}}{\tilde{\varepsilon}_{n+1}+\lambda_{n+1}} & \dfrac{\sigma_{n+1}}{\tilde{\varepsilon}_{n+1}-\lambda_{n+1}} \end{pmatrix}^{-1} g_n g_{n-1}\cdots g_{k+1} \begin{pmatrix} -\dfrac{G(i\omega)}{V_k} \\ \dfrac{F(i\omega)}{V_{k+1}} \end{pmatrix} = \sum_{\theta_{k+1},\ldots,\theta_n=\pm 1} e^{\sum_{m=k+1}^{n}\theta_m\lambda_m\Delta a_m} \frac{\sigma_{n+1}}{2^{n-k+1}\lambda_{n+1}} \begin{pmatrix} \dfrac{u_n v_{n+1}+\theta_n u_{n+1} v_n}{u_{n+1}-v_{n+1}} \\ \dfrac{u_n v_{n+1}-\theta_n u_{n+1} v_n}{u_{n+1}+v_{n+1}} \end{pmatrix} \\
&\times \prod_{m=k+2}^{n}(u_{m-1}v_m+\theta_m\theta_{m-1}v_{m-1}u_m)\left(-(v_{k+1}+\theta_{k+1}u_{k+1})\frac{G(i\omega)}{V_k}+(v_{k+1}-\theta_{k+1}u_{k+1})\frac{F(i\omega)}{V_{k+1}}\right).
\end{aligned} \tag{35}$$

If $n = k$ we can write

$$\begin{pmatrix} 1 & 1 \\ \dfrac{\sigma_{k+1}}{\tilde{\varepsilon}_{k+1}+\lambda_{k+1}} & \dfrac{\sigma_{k+1}}{\tilde{\varepsilon}_{k+1}-\lambda_{k+1}} \end{pmatrix}^{-1} \begin{pmatrix} -\dfrac{G(i\omega)}{V_k} \\ \dfrac{F(i\omega)}{V_{k+1}} \end{pmatrix} = \frac{\sigma_{k+1}}{2\lambda_{k+1}} \begin{pmatrix} -\dfrac{G(i\omega)}{V_k}\dfrac{u_{k+1}+v_{k+1}}{u_{k+1}-v_{k+1}} - \dfrac{F(i\omega)}{V_{k+1}} \\ \dfrac{G(i\omega)}{V_k}\dfrac{u_{k+1}-v_{k+1}}{u_{k+1}+v_{k+1}} + \dfrac{F(i\omega)}{V_{k+1}} \end{pmatrix}. \tag{36}$$

Finally, we obtain the following results. If $n < k$:

$$\begin{pmatrix} e^{\lambda_{n+1}a_n} C_{n+1} \\ e^{-\lambda_{n+1}a_n} D_{n+1} \end{pmatrix} = \sum_{\substack{\theta_2,\ldots,\theta_n=\pm 1 \\ \theta_1 = 1}} e^{\sum_{m=2}^{n}\theta_m\lambda_m\Delta a_m} \frac{\sigma_{n+1}}{2^{n-1}\lambda_{n+1}} \begin{pmatrix} \dfrac{u_n v_{n+1}+\theta_n u_{n+1} v_n}{u_{n+1}-v_{n+1}} \\ \dfrac{u_n v_{n+1}-\theta_n u_{n+1} v_n}{u_{n+1}+v_{n+1}} \end{pmatrix} \prod_{m=2}^{n}(u_{m-1}v_m+\theta_m\theta_{m-1}v_{m-1}u_m)\frac{1}{u_1+v_1} e^{\lambda_1 a_1} C_1, \tag{37}$$

if $n = k$:

$$\begin{aligned}
\begin{pmatrix} e^{\lambda_{n+1}a_n} C_{n+1} \\ e^{-\lambda_{n+1}a_n} D_{n+1} \end{pmatrix} &= \sum_{\substack{\theta_2,\ldots,\theta_n=\pm 1 \\ \theta_1 = 1}} e^{\sum_{m=2}^{n}\theta_m\lambda_m\Delta a_m} \frac{\sigma_{n+1}}{2^{n-1}\lambda_{n+1}} \begin{pmatrix} \dfrac{u_n v_{n+1}+\theta_n u_{n+1} v_n}{u_{n+1}-v_{n+1}} \\ \dfrac{u_n v_{n+1}-\theta_n u_{n+1} v_n}{u_{n+1}+v_{n+1}} \end{pmatrix} \prod_{m=2}^{n}(u_{m-1}v_m+\theta_m\theta_{m-1}v_{m-1}u_m) \\
&\quad \times \frac{1}{u_1+v_1} e^{\lambda_1 a_1} C_1 + \frac{\sigma_{k+1}}{2\lambda_{k+1}} \begin{pmatrix} -\dfrac{G(i\omega)}{V_k}\dfrac{u_{k+1}+v_{k+1}}{u_{k+1}-v_{k+1}} - \dfrac{F(i\omega)}{V_{k+1}} \\ \dfrac{G(i\omega)}{V_k}\dfrac{u_{k+1}-v_{k+1}}{u_{k+1}+v_{k+1}} + \dfrac{F(i\omega)}{V_{k+1}} \end{pmatrix}
\end{aligned} \tag{38}$$

if $n > k$:

$$\begin{pmatrix} e^{\lambda_{n+1}a_n} C_{n+1} \\ e^{-\lambda_{n+1}a_n} D_{n+1} \end{pmatrix} = \sum_{\substack{\theta_2,\ldots,\theta_n=\pm1 \\ \theta_1=1}} e^{\sum_{m=2}^{n}\theta_m\lambda_m\Delta a_m} \frac{\sigma_{n+1}}{2^{n-1}\lambda_{n+1}} \begin{pmatrix} \dfrac{u_n v_{n+1}+\theta_n u_{n+1} v_n}{u_{n+1}-v_{n+1}} \\ \dfrac{u_n v_{n+1}-\theta_n u_{n+1} v_n}{u_{n+1}+v_{n+1}} \end{pmatrix} \prod_{m=2}^{n}(u_{m-1}v_m+\theta_m\theta_{m-1}v_{m-1}u_m)\frac{1}{u_1+v_1}e^{\lambda_1 a_1}C_1$$

$$+ \sum_{\theta_{k+1},\ldots,\theta_n=\pm1} e^{\sum_{m=k+1}^{n}\theta_m\lambda_m\Delta a_m} \frac{\sigma_{n+1}}{2^{n-k+1}\lambda_{n+1}} \begin{pmatrix} \dfrac{u_n v_{n+1}+\theta_n u_{n+1} v_n}{u_{n+1}-v_{n+1}} \\ \dfrac{u_n v_{n+1}-\theta_n u_{n+1} v_n}{u_{n+1}+v_{n+1}} \end{pmatrix}$$

$$\times \prod_{m=k+2}^{n}(u_{m-1}v_m+\theta_m\theta_{m-1}v_{m-1}u_m)\left[-(v_{k+1}+\theta_{k+1}u_{k+1})\frac{G(i\omega)}{V_k}+(v_{k+1}-\theta_{k+1}u_{k+1})\frac{F(i\omega)}{V_{k+1}}\right]. \quad (39)$$

Assuming in obtained formulae that $n = N$, we have $\begin{pmatrix} e^{\lambda_{N+1}a_N} C_{N+1} \\ e^{-\lambda_{N+1}a_N} D_{N+1} \end{pmatrix}$. Then, using the assumption $C_{N+1} = 0$, we can calculate C_1.

2.3 Backward propagation energy density at the point a_1 in the case of $n = N$, $k = N$

$$\begin{pmatrix} e^{\lambda_{N+1}a_N} C_{N+1} \\ e^{-\lambda_{N+1}a_N} D_{N+1} \end{pmatrix} = \sum_{\substack{\theta_2,\ldots,\theta_N=\pm1 \\ \theta_1=1}} e^{\sum_{m=2}^{N}\theta_m\lambda_m\Delta a_m} \frac{\sigma_{N+1}}{2^{N-1}\lambda_{N+1}} \begin{pmatrix} \dfrac{u_N v_{N+1}+\theta_N u_{N+1} v_N}{u_{N+1}-v_{N+1}} \\ \dfrac{u_N v_{N+1}-\theta_N u_{N+1} v_N}{u_{N+1}+v_{N+1}} \end{pmatrix}$$

$$\times \prod_{m=2}^{N}(u_{m-1}v_m+\theta_m\theta_{m-1}v_{m-1}u_m)\frac{1}{u_1+v_1}e^{\lambda_1 a_1}C_1 + \frac{\sigma_{N+1}}{2\lambda_{N+1}} \begin{pmatrix} -\dfrac{G(i\omega)}{V_N}\dfrac{u_{N+1}+v_{N+1}}{u_{N+1}-v_{N+1}} - \dfrac{F(i\omega)}{V_{N+1}} \\ \dfrac{G(i\omega)}{V_N}\dfrac{u_{N+1}-v_{N+1}}{u_{N+1}+v_{N+1}} + \dfrac{F(i\omega)}{V_{N+1}} \end{pmatrix}. \quad (40)$$

Using $C_{N+1} = 0$, we get

$$e^{\lambda_1 a_1}C_1 = Q(i\omega, a_1)|_{k=N} = 2^{N-2}(u_1+v_1)\frac{\frac{G(i\omega)}{V_N}(u_{N+1}+v_{N+1}) + \frac{F(i\omega)}{V_{N+1}}(u_{N+1}-v_{N+1})}{\sum_{\substack{\theta_2,\ldots,\theta_N=\pm1, \\ \theta_1=1,\theta_{N+1}=1}} e^{\sum_{m=2}^{N}\theta_m\lambda_m\Delta a_m}\prod_{m=2}^{N+1}(u_{m-1}v_m+\theta_m\theta_{m-1}v_{m-1}u_m)}. \quad (41)$$

2.4 Backward propagation energy density at the point a_1 in the case of $n = N$, $k < N$

$$\begin{pmatrix} e^{\lambda_{N+1}a_N} C_{N+1} \\ e^{-\lambda_{N+1}a_N} D_{N+1} \end{pmatrix} = \sum_{\substack{\theta_2,\ldots,\theta_N=\pm1 \\ \theta_1=1}} e^{\sum_{m=2}^{N}\theta_m\lambda_m\Delta a_m} \frac{\sigma_{N+1}}{2^{N-1}\lambda_{N+1}} \begin{pmatrix} \dfrac{u_N v_{N+1}+\theta_N u_{N+1} v_N}{u_{N+1}-v_{N+1}} \\ \dfrac{u_N v_{N+1}-\theta_N u_{N+1} v_N}{u_{N+1}+v_{N+1}} \end{pmatrix}$$

$$\times \prod_{m=2}^{N}(u_{m-1}v_m+\theta_m\theta_{m-1}v_{m-1}u_m)\frac{1}{u_1+v_1}e^{\lambda_1 a_1}C_1 + \sum_{\theta_{k+1},\ldots,\theta_N=\pm1} e^{\sum_{m=k+1}^{N}\theta_m\lambda_m\Delta a_m} \frac{\sigma_{N+1}}{2\lambda_{N+1}} \begin{pmatrix} \dfrac{u_N v_{N+1}+\theta_N u_{N+1} v_N}{u_{N+1}-v_{N+1}} \\ \dfrac{u_N v_{N+1}-\theta_N u_{N+1} v_N}{u_{N+1}+v_{N+1}} \end{pmatrix}$$

$$\times \prod_{m=k+2}^{N}(u_{m-1}v_m+\theta_m\theta_{m-1}v_{m-1}u_m)\left(-(v_{k+1}+\theta_{k+1}u_{k+1})\frac{G(i\omega)}{V_k}+(v_{k+1}-\theta_{k+1}u_{k+1})\frac{F(i\omega)}{V_{k+1}}\right). \quad (42)$$

Using $C_{N+1} = 0$

$$e^{\lambda_1 a_1}C_1 = Q(i\omega, a_1)|_{k<N}. = 2^{N-2}(u_1+v_1)\frac{\begin{array}{c}\sum_{\substack{\theta_{k+1},\ldots,\theta_N=\pm1 \\ \theta_{N+1}=1}} e^{\sum_{m=k+1}^{N}\theta_m\lambda_m\Delta a_m}\prod_{m=k+2}^{N+1}(u_{m-1}v_m+\theta_m\theta_{m-1}v_{m-1}u_m)\times \\ \times\left[(v_{k+1}+\theta_{k+1}u_{k+1})\frac{G(i\omega)}{V_k}-(v_{k+1}-\theta_{k+1}u_{k+1})\frac{F(i\omega)}{V_{k+1}}\right]\end{array}}{\sum_{\substack{\theta_2,\ldots,\theta_N=\pm1 \\ \theta_1=1,\theta_{N+1}=1}} e^{\sum_{m=2}^{N}\theta_m\lambda_m\Delta a_m}\prod_{m=2}^{N+1}(u_{m-1}v_m+\theta_m\theta_{m-1}v_{m-1}u_m)} \quad (43)$$

Expressions (41) and (43) describe the backward propagated energy density, detected at the point a_1. Formula (41) describes the case of a transmitter located at the last boundary a_N; that is, $k = N$. Formula (43) describes is for a transmitter located at any other boundary $a_k < a_N$; that is, $k < N$.

2.5 Forward propagation energy density at the point a_1 in the cases of $n = N, k = N$ and $n = N, k < N$

The expression for forward propagated energy can be calculated using (14), (15). One can see from (14) and (15), referring to (29), that

$$A_1 = \frac{u_1 - v_1}{u_1 + v_1} C_1, \quad \text{and} \quad B_1 = 0. \tag{44}$$

Hence

$$P(i\omega, a_1)|_{k=N} = 2^{N-2}(u_1 - v_1) \frac{\frac{G(i\omega)}{V_N}(u_{N+1} + v_{N+1}) + \frac{F(i\omega)}{V_{N+1}}(u_{N+1} - v_{N+1})}{\sum_{\substack{\theta_2,\dots,\theta_N = \pm 1, \\ \theta_1 = 1, \theta_{N+1} = 1}} e^{\sum_{m=2}^{N} \theta_m \lambda_m \Delta a_m} \prod_{m=2}^{N+1} (u_{m-1} v_m + \theta_m \theta_{m-1} v_{m-1} u_m)} \tag{45}$$

$$P(i\omega, a_1)|_{k<N} = 2^{N-2}(u_1 - v_1) \frac{\begin{array}{c}\sum_{\substack{\theta_{k+1},\dots,\theta_N = \pm 1 \\ \theta_{N+1} = 1}} e^{\sum_{m=k+1}^{N} \theta_m \lambda_m \Delta a_m} \prod_{m=k+2}^{N+1} (u_{m-1} v_m + \theta_m \theta_{m-1} v_{m-1} u_m) \times \\ \times \left[(v_{k+1} + \theta_{k+1} u_{k+1}) \frac{G(i\omega)}{V_k} - (v_{k+1} - \theta_{k+1} u_{k+1}) \frac{F(i\omega)}{V_{k+1}} \right]\end{array}}{\sum_{\substack{\theta_2,\dots,\theta_N = \pm 1 \\ \theta_1 = 1, \theta_{N+1} = 1}} e^{\sum_{m=2}^{N} \theta_m \lambda_m \Delta a_m} \prod_{m=2}^{N+1} (u_{m-1} v_m + \theta_m \theta_{m-1} v_{m-1} u_m)} \tag{46}$$

Since the detector unable is to distinguish forward propagation energy from backward propagation energy, we have the total detected energy $P_\Sigma(i\omega, a_1) = P(i\omega, a_1) + Q(i\omega, a_1)$. In this situation it is reasonable to assume that our transmitter is undirected too; that is, $G(i\omega, a_1) = F(i\omega, a_1)$. We obtain

$$P_\Sigma(i\omega, a_1)|_{k=N} = 2^{N-1} u_1 u_{N+1} \frac{F(i\omega)\left(\frac{1}{V_N} + \frac{1}{V_{N+1}}\right)}{\sum_{\substack{\theta_2,\dots,\theta_N = \pm 1, \\ \theta_1 = 1, \theta_{N+1} = 1}} e^{\sum_{m=2}^{N} \theta_m \lambda_m \Delta a_m} \prod_{m=2}^{N+1} (u_{m-1} v_m + \theta_m \theta_{m-1} v_{m-1} u_m)}. \tag{47}$$

3 SOME SOLUTIONS AND GEOPHYSICAL INTERPRETATION

Reconstruction of observed energy density timetable is conducted by the inverse Fourier transformation. It can be calculated analytically only for special simple cases.

3.1 The case of homogeneous medium, where a detector is located at the boundary a_1 and a transmitter is located at the boundary $a_2 = a_1 + \Delta a$ ($N = k = 2$)

We obtain from expressions (41) and (45):

$$Q(i\omega, a_1) = (u_1 + v_1) \frac{\frac{G(i\omega)}{V_2}(u_3 + v_3) + \frac{F(i\omega)}{V_3}(u_3 - v_3)}{\sum_{\theta_2 = \pm 1} e^{\theta_2 \lambda_2 \Delta a_2}(u_1 v_2 + \theta_2 v_1 u_2)(u_2 v_3 + \theta_2 v_2 u_3)} \tag{48}$$

$$P(i\omega, a_1) = (u_1 - v_1) \frac{\frac{G(i\omega)}{V_2}(u_3 + v_3) + \frac{F(i\omega)}{V_3}(u_3 - v_3)}{\sum_{\theta_2 = \pm 1} e^{\theta_2 \lambda_2 \Delta a_2}(u_1 v_2 + \theta_2 v_1 u_2)(u_2 v_3 + \theta_2 v_2 u_3)} \tag{49}$$

Assuming that three layers are identical ($u_1 = u_2 = u_3 = u$, $v_1 = v_2 = v_3 = v$, $V_2 = V_3 = V$), we write

$$Q(i\omega, a_1) = \frac{1}{4V} e^{-\lambda \Delta a} [G(i\omega)(u^2 + v^2 + 2) + F(i\omega)(u^2 - v^2)] = \frac{1}{2V} e^{-\lambda \Delta a} \left[G(i\omega)\left(\frac{\tilde{\varepsilon}}{\lambda} + 1\right) + F(i\omega)\frac{\sigma}{\lambda} \right], \tag{50}$$

$$P(i\omega, a_1) = \frac{1}{4V} e^{-\lambda \Delta a} [G(i\omega)(u^2 - v^2) + F(i\omega)(u^2 + v^2 - 2)] = \frac{1}{2V} e^{-\lambda \Delta a} \left[G(i\omega)\frac{\sigma}{\lambda} + F(i\omega)\left(\frac{\tilde{\varepsilon}}{\lambda} - 1\right) \right], \tag{51}$$

and then

$$P_\Sigma(i\omega, a_1) = \frac{1}{2V} e^{-\lambda \Delta a} \left[G(i\omega)\left(\frac{\tilde{\varepsilon} + \sigma}{\lambda} + 1\right) + F(i\omega)\left(\frac{\tilde{\varepsilon} + \sigma}{\lambda} - 1\right) \right]. \tag{52}$$

3.2 The case of medium sounding by directed δ-pulse, where a detector is located near a transmitter at the point a_1. ($G(i\omega) = 0$, $F(i\omega)$ = constant)

We obtain from expression (50):

$$Q(i\omega, a_1) = \frac{\sigma F}{2V} \frac{e^{-\Delta\alpha\lambda}}{\sqrt{\tilde{\varepsilon}^2 - \sigma^2}} \tag{53}$$

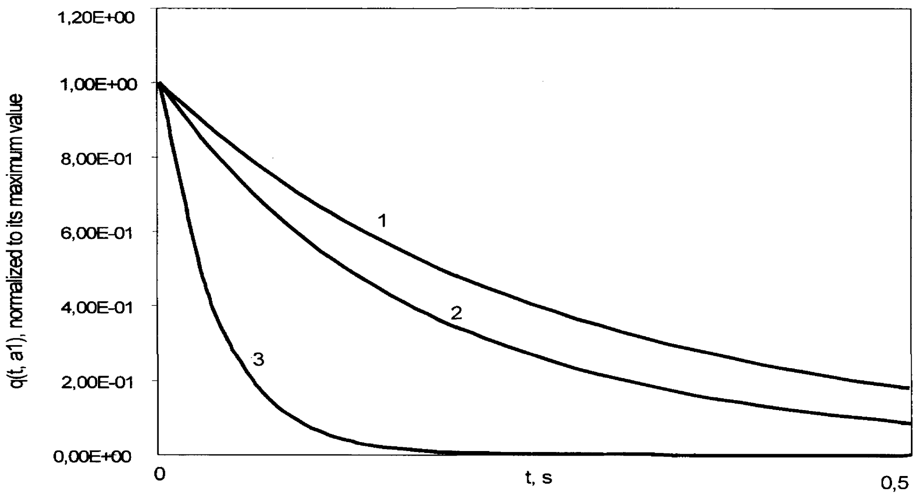

Figure 1. $q(t, a_1)$ described by (54) for the case $\Delta a = 0$, normalized to its maximum value. (1) $\sigma = 0{,}001\ \mathrm{m}^{-1}$, $\gamma = 0{,}001\ \mathrm{m}^{-1}$, $V = 2000\ \mathrm{m\ s}^{-1}$; (2) $\sigma = 0{,}001\ \mathrm{m}^{-1}$, $\gamma = 0{,}001\ \mathrm{m}^{-1}$, $V = 3000\ \mathrm{m\ s}^{-1}$; (3) $\sigma = 0{,}008\ \mathrm{m}^{-1}$, $\gamma = 0{,}001\ \mathrm{m}^{-1}$, $V = 3000\ \mathrm{m\ s}^{-1}$.

After the Fourier-transformation of (53), we obtain the following expression (Korn & Korn 1968, table 8.4.-1):

$$q(t, a_1) = \frac{1}{2\pi}\frac{\sigma F}{2V}\int_{-\infty}^{\infty} \mathrm{e}^{-\Delta a\sqrt{\left(\varepsilon + \frac{i\omega}{V}\right)^2 - \sigma^2}} \frac{\mathrm{e}^{i\omega t}}{\sqrt{\left(\varepsilon + \frac{i\omega}{V}\right)^2 - \sigma^2}}\, d\omega = \frac{F\sigma}{2}\eta\,(Vt - \Delta a)\,\mathrm{e}^{-\varepsilon Vt} I_0(\sigma\sqrt{(Vt)^2 - (\Delta a^2)}), \tag{54}$$

where $I_0(\sigma\sqrt{(Vt)^2 - (\Delta a^2)})$ is the 0th order 1st type modified Bessel function.

The nature of this solution (54) is illustrated in Fig. 1 for $\Delta a = 0$.

3.3 Detector is located at the point $a_1 = 0$, where a transmitter is located at the point $a_2 = a > 0$; the wave is radiated in the direction of the detector ($F(i\omega) = 0$, $G(i\omega) = $ constant)

In this case

$$Q(i\omega, a) = \mathrm{e}^{-\lambda a}\frac{G}{2V}\left(\frac{\tilde{\varepsilon}}{\lambda} + 1\right), \tag{55}$$

by using inverse Fourier transformation (Korn & Korn 1968, table 8.4.-1), we have

$$q(t, a) = e^{-\varepsilon a}\frac{G}{V}\delta\left(t - \frac{a}{V}\right) + \frac{G}{2}\sigma\mathrm{e}^{-\varepsilon Vt}\sqrt{\frac{Vt + a}{Vt - a}} I_1(\sigma\sqrt{(Vt)^2 - a^2}) \tag{56}$$

This solution is shown in Fig. 2.

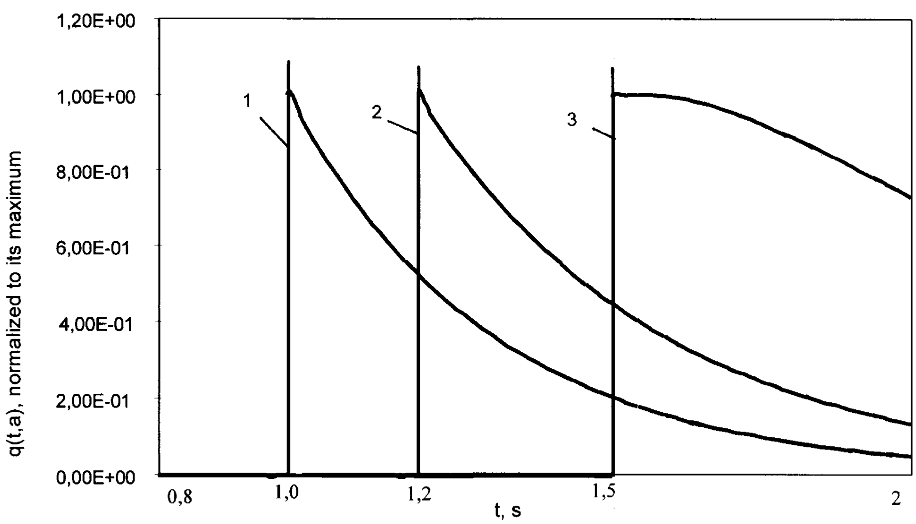

Figure 2. $q(t, a)$ described by (56), normalized to its maximum over the entire interval (with the exception of the point $t = \frac{\Delta a}{V}$). $a = 3000$ m, $\gamma = 0{,}001\ \mathrm{m}^{-1}$. (1) $\sigma = 0{,}001\ \mathrm{m}^{-1}$, $V = 3000\ \mathrm{m\ s}^{-1}$; (2) $\sigma = 0{,}001\ \mathrm{m}^{-1}$, $V = 2500\ \mathrm{m\ s}^{-1}$; (3) $\sigma = 0{,}002\ \mathrm{m}^{-1}$, $V = 2000\ \mathrm{m\ s}^{-1}$.

3.4 Undirected transmitter is located at the point a_1, where a detector is located at the point $a_2 = a_1 + \Delta a$

Let us consider $F(i\omega) = G(i\omega)=$ constant $=F$. From (50)–(52) it follows (Korn & Korn 1968, table 8.4.-1) that

$$q(t, a_1) = \frac{F}{V}e^{-\varepsilon V t}\delta\left(t - \frac{\Delta a}{V}\right) + \frac{F\sigma}{2}\eta(Vt - \Delta a)\mathrm{e}^{-\varepsilon V t}\left\{I_0[\sigma\sqrt{(Vt)^2 - (\Delta a)^2}] + \frac{Vt + \Delta a}{\sqrt{(Vt)^2 - (\Delta a)^2}}I_1[\sigma\sqrt{(Vt)^2 - (\Delta a)^2}]\right\}, \tag{57}$$

$$p(t, a_1) = \frac{F\sigma}{2}\eta(Vt - \Delta a)\mathrm{e}^{-\varepsilon V t}\left\{I_0[\sigma\sqrt{(Vt)^2 - (\Delta a)^2}] + \frac{Vt - \Delta a}{\sqrt{(Vt)^2 - (\Delta a)^2}}I_1[\sigma\sqrt{(Vt)^2 - (\Delta a)^2}]\right\} \tag{58}$$

$$p_\Sigma(t, a_1) = \frac{1}{2\pi}\frac{F}{V}\int_{-\infty}^{\infty}\mathrm{e}^{-\sqrt{\left(\varepsilon + \frac{i\omega}{V}\right)^2 - \sigma^2}\Delta a}\frac{\varepsilon + \frac{i\omega}{V} + \sigma}{\sqrt{\left(\varepsilon + \frac{i\omega}{V}\right)^2 - \sigma^2}}\mathrm{e}^{i\omega t}\,d\omega = \frac{F}{V}\mathrm{e}^{-\varepsilon V t}\delta\left(t - \frac{\Delta a}{V}\right)$$
$$+ F\sigma\eta(Vt - \Delta a)\mathrm{e}^{-\varepsilon V t}\left\{I_0[\sigma\sqrt{(Vt)^2 - (\Delta a)^2}] + \frac{Vt}{\sqrt{(Vt)^2 - (\Delta a)^2}}I_1[\sigma\sqrt{(Vt)^2 - (\Delta a)^2}]\right\}, \tag{59}$$

where

$$\eta(z) = \begin{cases} 0, & z < 0, \\ 1, & z \geq 0. \end{cases}$$

The solution described by (57) and (58) is shown in Fig. 3.

If we consider $\gamma = 0, \sigma V = \varepsilon V = \frac{\varsigma}{2}$ and $F = \frac{W}{2}$, we can easily see that formula (59) coincides with formula (11) on p. 142 of Sato (1993), where in designations by Sato ς is the reflection coefficient, and W is the total radiated energy. The formulae are not identical, because there is a freedom in choosing the Greens function for the model based on the integral transfer equation. If in Sato (1993) the Greens function is chosen to be $\frac{1}{V}\delta(t - \frac{\Delta a}{V})\mathrm{e}^{-\frac{\varsigma}{2}t - \varepsilon V t}$, $(\sigma V = \frac{\varsigma}{2})$, our (59) and Sato's (11) coincide completely each at her.

3.5 Undirected transmitter at the boundary of two media ($F(i\omega) = G(i\omega)=$ constant)

Assuming that $u_2 = u_1$ and $v_2 = v_1$ for (48) and replacing u_3 with u_2, v_3 with v_2, and V_3 with V_2 (the transmitter is located at the boundary of two media), we derive for the case $F(i\omega) = G(i\omega) =$ constant $= F$:

$$Q(i\omega, a_1) = \mathrm{e}^{\lambda_1 a_1}C_1 = \frac{F}{2}\mathrm{e}^{-\lambda_1\Delta a}\frac{\frac{\sigma_2}{\varepsilon_2 - \lambda_2}\frac{1}{V_1} + \frac{1}{V_2}}{\left(\frac{1}{1 - \frac{\varepsilon_2 - \sigma_2}{\lambda_2}}\right) - \left(\frac{1}{1 + \frac{\varepsilon_1 - \sigma_1}{\lambda_1}}\right)}. \tag{60}$$

The solution given by eq. (60) is shown in Fig. 4.

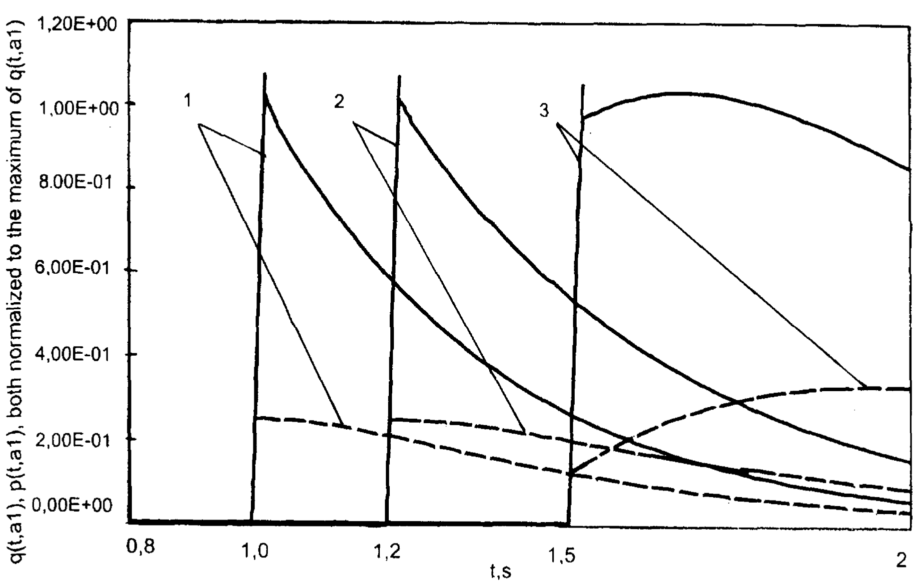

Figure 3. $q(t, a_1)$ described by (57) ——, and $p(t, a_1)$ described by (58) – – –, both normalized to the maximum value of $q(t, a_1)$ over the entire interval (with the exception of the point $t = \frac{\Delta a}{V}$).$\Delta a = 3000$ m, $\gamma = 0{,}001$ m^{-1}. (1) $\sigma = 0{,}001$ m^{-1}, $V = 3000$ m s^{-1}; (2) $\sigma = 0{,}001$ m^{-1}, $V = 2500$ m s^{-1}; (3) $\sigma = 0{,}002$ m^{-1}, $V = 2000$ m s^{-1}.

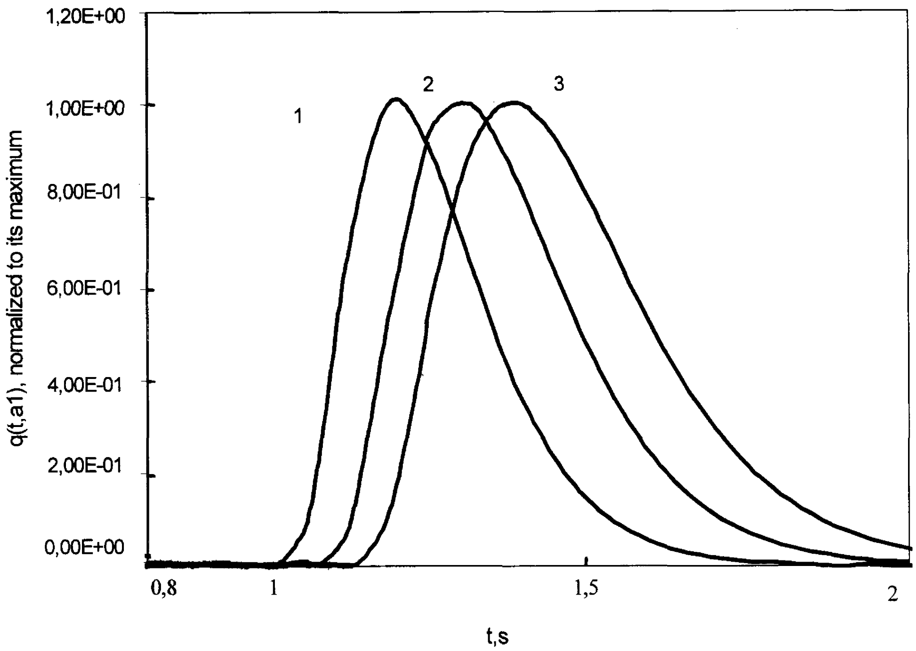

Figure 4. $q(t, a_1)$ described by (60), normalized by its maximum value; $\gamma_1 = \gamma_2 = 0{,}001\ \mathrm{m}^{-1}$, $\Delta a = 3000$ m. (1) $\sigma_1 = 0{,}005\ \mathrm{m}^{-1}$, $\sigma_2 = 0{,}001\ \mathrm{m}^{-1}$, $V_1 = V_2 = 2500\ \mathrm{m\,s}^{-1}$; (2) $\sigma_1 = 0{,}005\ \mathrm{m}^{-1}$, $\sigma_2 = 0{,}008\ \mathrm{m}^{-1}$, $V_1 = 2000\ \mathrm{m\,s}^{-1}$, $V_2 = 3000\ \mathrm{m\,s}^{-1}$; (3) $\sigma_1 = 0{,}005\ \mathrm{m}^{-1}$, $\sigma_2 = 0{,}008\ \mathrm{m}^{-1}$, $V_1 = 1000\ \mathrm{m\,s}^{-1}$, $V_2 = 3000\ \mathrm{m\,s}^{-1}$.

3.6 Undirected transmitter ($G(i\omega) = 0$, $F(i\omega) = $ constant), sounding layered medium

We obtain from (43), assuming $G(i\omega) = 0$ (the case of a directed (cumulative) transmitter):

$$Q(i\omega, a_1) = 2^{N-2}\left(1 + v_1^2\right) \times \frac{F(i\omega)}{V_{k+1}} \times \sum_{\substack{\theta_{k+1},\ldots,\theta_N = \pm 1 \\ \theta_{N+1} = 1}} \mathrm{e}^{\sum_{m=k+1}^{N} \theta_m \lambda_m \Delta a_m} \times \left(-1 + \theta_{k+1} u_{k+1}^2\right)$$

$$\times \prod_{m=k+2}^{N+1} \left(1 + \theta_m \theta_{m-1} v_{m-1}^2 u_m^2\right) \Bigg/ \sum_{\substack{\theta_2,\ldots,\theta_N = \pm 1 \\ \theta_1, \theta_{N+1} = 1}} \mathrm{e}^{\sum_{m=2}^{N} \theta_m \lambda_m \Delta a_m} \prod_{m=2}^{N+1} \left(1 + \theta_m \theta_{m-1} v_{m-1}^2 u_m^2\right). \tag{61}$$

Then, assuming that $v_1 \to 1$ and substituting u and v we derive

$$Q(i\omega, a_1) = 2^{N-1} \times \frac{F(i\omega)}{V_{k+1}}$$

$$\times \left[\frac{\tilde{\varepsilon}_{k+1} + \sigma_{k+1}}{\lambda_{k+1}} \cdot \frac{\sum_{\substack{\theta_{k+1},\ldots,\theta_N = \pm 1 \\ \theta_{N+1} = 1}} \mathrm{e}^{\sum_{m=k+1}^{N} \theta_m \lambda_m \Delta a_m} \theta_{k+1} \prod_{m=k+2}^{N+1} \left(1 + \theta_m \theta_{m-1} \left(\frac{\tilde{\varepsilon}_{m-1} - \sigma_{m-1}}{\lambda_{m-1}}\right)\left(\frac{\tilde{\varepsilon}_m + \sigma_m}{\lambda_m}\right)\right)}{\sum_{\substack{\theta_2,\ldots,\theta_N = \pm 1 \\ \theta_1, \theta_{N+1} = 1}} \mathrm{e}^{\sum_{m=2}^{N} \theta_m \lambda_m \Delta a_m} \prod_{m=2}^{N+1} \left(1 + \theta_m \theta_{m-1} \left(\frac{\tilde{\varepsilon}_{m-1} - \sigma_{m-1}}{\lambda_{m-1}}\right)\left(\frac{\tilde{\varepsilon}_m + \sigma_m}{\lambda_m}\right)\right)} \right.$$

$$\left. - \frac{\sum_{\substack{\theta_{k+1},\ldots,\theta_N = \pm 1 \\ \theta_{N+1} = 1}} \mathrm{e}^{\sum_{m=k+1}^{N} \theta_m \lambda_m \Delta a_m} \prod_{m=k+2}^{N+1} \left(1 + \theta_m \theta_{m-1} \left(\frac{\tilde{\varepsilon}_{m-1} - \sigma_{m-1}}{\lambda_{m-1}}\right)\left(\frac{\tilde{\varepsilon}_m + \sigma_m}{\lambda_m}\right)\right)}{\sum_{\substack{\theta_2,\ldots,\theta_N = \pm 1 \\ \theta_1, \theta_{N+1} = 1}} \mathrm{e}^{\sum_{m=2}^{N} \theta_m \lambda_m \Delta a_m} \prod_{m=2}^{N+1} \left(1 + \theta_m \theta_{m-1} \left(\frac{\tilde{\varepsilon}_{m-1} - \sigma_{m-1}}{\lambda_{m-1}}\right)\left(\frac{\tilde{\varepsilon}_m + \sigma_m}{\lambda_m}\right)\right)} \right] \tag{62}$$

This solution (62) with $v_1 = 1$ is represented in Fig. 5.

4 RESULTS AND DISCUSSION

We derived the rigorous solution of transfer equation for energy propagating through 1-D layered medium. Several cases of detector location are discussed. We assumed that characteristics of reflected energy density are determined by the following:

(1) propagation, depending on macroacoustic parameters such as 'average' velocity and 'average' absorption;
(2) reflection properties, depending on local parameters such as local changing of modulus of elasticity or density.

We applied such characteristics of absorption, velocity and reflection to the model of layered medium.

The solution is derived by means of original linear transformation.

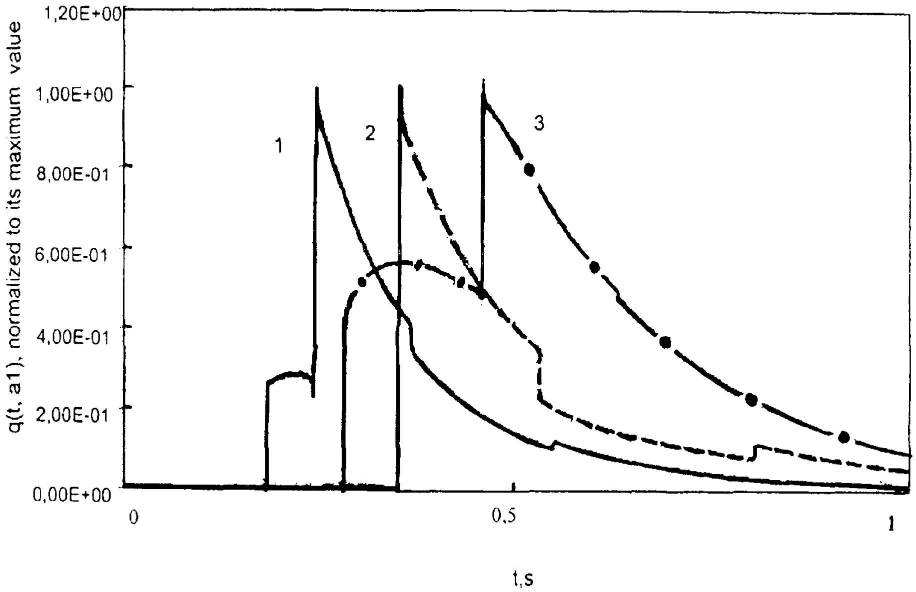

Figure 5. $q(t, a_1)$ described by (62), normalized to its maximum value; $N = 10$, $a_4 = 1200$ m, $a_5 = 1500$ m, $a_6 = 1800$ m, $a_7 = 2100$ m, $a_8 = 2400$ m, $a_9 = 2700$ m, $a_{10} = 3000$ m; $\gamma_n = 0{,}001$ m^{-1} for $n = 1, \ldots, 10$; (1) —— $a_2 = 600$ m, $a_3 = 900$ m, $k = 3$, $V = 2500$ m s^{-1}, $\sigma_2 = \sigma_4 = \sigma_6 = \sigma_8 = \sigma_{10} = 0{,}001$ m^{-1}, $\sigma_3 = \sigma_5 = \sigma_7 = \sigma_9 = 0{,}005$ m^{-1}; (2) - - - $a_2 = 200$ m, $a_3 = 700$ m, $k = 4$, $V = 2000$ m s^{-1}, $\sigma_2 = \sigma_4 = \sigma_6 = \sigma_8 = \sigma_{10} = 0{,}001$ m^{-1}, $\sigma_3 = \sigma_5 = \sigma_7 = \sigma_9 = 0{,}003$ m^{-1}; $n = 1, \ldots, 10$; (3) - • - $a_2 = 600$ m, $a_3 = 900$ m, $k = 5$, $V = 2500$ m s^{-1}, $\sigma_2 = \sigma_4 = \sigma_6 = \sigma_8 = \sigma_{10} = 0{,}001$ m^{-1}, $\sigma_3 = \sigma_5 = \sigma_7 = \sigma_9 = 0{,}005$ m^{-1}.

It is clear, that real media are 3-D, and signal scattering is multidirectional, but it is rather difficult to obtain analytical solutions for 3-D problem. We have discussed the ideal case in our paper, because the results, obtained by modelling and computing are not general enough. We think it is necessary to notice that 1-D model has been studied repeatedly in geophysics. Most significant result was derived by Sato (1993). In that paper the solution is obtained for almost homogeneous medium; absorbing and reflecting characteristics of the medium can not be separated by means of the involved approach, based on the integral transfer equation. Our solution derived for rather general case of layered medium. We applied differential equations of energy transfer; absorption and reflection coefficients were considered separately.

1-D model approximately describes the real medium scattering, even if we consider cumulative (directional transmitter). It is reasonable to assume that all the energy is propagated in the vertical direction, and velocity $V(z)$ is the velocity of P waves $V_p(z)$. S waves calculations, apparently, can be fulfilled by obtained formulas with the other parameters $V(z) - V_S(z)$. In connection with the fact that considered case is ideal, our results can not be applied directly to real signal processing, but the results can give valuable considerations for improving and simplifying of the algorithms of real geophysical signals and can be used as comparison standards (Denisov & Finicov 1997).

Analytical solution for 1-D media, evidently, gives a possibility to cause results obtaining for three-dimensional media, and to estimate correlation between P and S waves numerically.

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