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Logical assessment of observational knowledge in volcanology

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Abstract

A paradox of volcanology is that the most fundamental part of its knowledge – the observation of volcanic processes and objects – is less strict and less well organized compared with experimental and theoretical approaches determined by these observations. The object of this study is the knowledge resulting from volcanological observations, and the objective is to give this knowledge strict form. For this, we elaborate the methodology of assessment of observational knowledge by means of propositional logic. Two kinds of assessments are developed: (1) assessment of individual statements and rationales for truth and satisfiability; and (2) assessment of a set of statements for self-consistency and deducibility of a statement from the set. Propositional logic allows an analyst to find controversies and contradictions, build self-consistent domains of observational knowledge, and obtain strict inference within these domains by means of logical calculi. We believe that the results of reported work can be used in field volcanological studies for optimization of data interpretation, in hazard assessment for evaluation of recommendations of experts and in risk mitigation for improving communication between scientists and non-professionals.

Keywords: logic; volcanology; hazard assessment; consistency; conceptual uncertainty; inference; observations

1. Introduction

Observations of eruptive processes and their products are the primary, most valuable and reliable source of volcanological information. Modeling and theoretical considerations are given a basis by observations and make sense only in relation to them. Moreover, in many cases the volcanologic reconstruction, forecast and hazard assessment are based solely or almost solely on observations, current and preceding, particular and general. Simultaneously, the reasoning on observations remains pretty loose and 'woolly'. As was stated earlier, the tool for improving the situation is logic (Pshenichny and Moukhachov, 2001; Pshenichny, 2002). This paper aims to explore the opportunities of the simplest logical sys-

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tem – propositional logic – in assessment of reasoning on volcanological observations.

Theoretically, two kinds of assessments can be made based on propositional logic. These are the assessment of an individual statement for truth and assessment for deducibility of a statement from other statements (and/or self-consistency of a set of statements). We consider the application of these assessments to pieces of volcanological knowledge and discuss the theoretical and practical benefits of our approach.

The works on application of logic or related approaches in Earth sciences are very scarce (Sirotinskaya (1986); Cagnoli (1998); Klir (2002)) and consider the logic solely as a means of data interpretation but not as a tool to study the *reasoning* in a field of knowledge (Pshenichny and Moukhachov, 2001). Meanwhile, those scientists focusing on the structure of knowledge in particular domains of geology (e.g. in stratigraphy – Dienes, 1978 – or tectonics – Potts and Reddy, 1999) did not use logic in their work that restricted their results solely to these domains. However, there were attempts at constructing (or revealing?) universal patterns of thinking by means other than logic (see e.g. Gould, 1981; Fedorov, 1989), which doubtless are of great interest but, like any other knowledge except the logic itself, require an assessment by means of logic.

2. Propositional logic

Propositional logic in its modern form was elaborated in the second half of the 19th to the first half of the 20th century by Frege (1896), Whitehead and Russell (1910), Hilbert and Bernays (1956), Kleene (1952) and other researchers. We will give a brief account of it based dominantly on Kleene (1952).

This logic describes the relations between *statements*, or *propositions*. These are expressed by narrative sentences of natural language (English, Russian, Greek, etc.). A statement is the *sense* of narrative sentence. It is either true or false. These two characteristics are *logical values*, or *truth values*, of statements.

Verbal expressions like 'not true that', 'and', 'or', 'if ... then', 'if and only if ... then', 'either ... or' and some others have the sense of *logical connectives*. Propositional logic studies the exact sense of these expressions and general laws of their usage.

Those statements that do not include logical connectives are termed *simple*; those that do are *compound statements*, or *compounds*. Compounds also are true or false. Their logical values are determined, first, by the logical values of elementary statements, and second, by the logical connectives between them.

The language of propositional logic is an artificial language capable of disclosing the logical structure of compound statements.

The *alphabet* of this language includes three kinds of signs.

1. Propositional variables: p, q, r, s, t, p_1, q_1, r_1, s_1, t_1, p_2, q_2, ..., expressing elementary statements.
2. Logical connectives: $\neg$ negation, & conjunction, $\vee$ disjunction, $\supset$ implication, $\equiv$ equivalence and, possibly, some others.
3. Technical signs: (, left bracket;), right bracket.

The basic concept in propositional logic is the *propositional formula*. It is defined as follows.

1. Propositional variable is propositional formula (e.g. p is a formula).
2. If A is a propositional formula, then $\neg$A is a formula too.
3. If A and B are propositional formulae, then (A&B), (A$\vee$B), (A$\supset$B), (A$\equiv$B) are formulae too.

A and B are called *metaletters*. This means that $\neg$A, (A&B), (A$\vee$B), (A$\supset$B), (A$\equiv$B) are not necessarily $\neg p$, $(p\&q)$, $(p\vee q)$, $(p\supset q)$, $(p\equiv q)$, correspondingly, but just principal schemes of formulae, in which another formulae of any length can occupy the places of A and B.

The connective expressed in this scheme is called the *main connective* of the propositional formula. Any part of a given propositional formula, which is a propositional formula itself, is called a *subformula* of this formula.

By accepted convention, each connective, except negation, requires a pair of left and right brackets, though the entire formula may be not taken into brackets.

Table 1
General truth table for basic logical connectives

A	B	¬A	A&B	A∨B	A⊃B	A≡B
TRUE	TRUE	FALSE	TRUE	TRUE	TRUE	FALSE
FALSE	TRUE	TRUE	FALSE	TRUE	TRUE	TRUE
TRUE	FALSE		FALSE	TRUE	FALSE	TRUE
FALSE	FALSE		FALSE	FALSE	TRUE	FALSE

3. Methodology of logical assessments of observational knowledge

3.1. *Assessment of rationales for truth by truth tables*

The exact sense of the logical connectives is defined by the following *truth table* (see Table 1).

The truth value of a formula is that of its main connective. There are formulae which can have only one truth value (i.e. are always true or always false) with any values of variables; e.g. the formula $(p \vee \neg p)$ is always true and $(p \& \neg p)$ is always false (see Table 2).

If a formula is always true, it is called a *logical law*, or *tautology*. If it is always false, it is a *controversy* (not to be confused with the controversy as the relation between any two statements, one of which is the negation of the other; the conjunction of these statements gives the controversy in the sense meant here; to avoid ambiguity, we will call this relation a *contradiction*). Some basic tautologies are listed in Appendix. All the rest of the formulae may take both truth values and are called *neutral*, or *satisfiable*.

For instance, the following rationale is tautology. 'If the lava is slowly ejecting, it piles up in a dome and does not flow, or, while ejecting, it forms a short thick viscous flow.' Let us denote every simple statement with a propositional variable:

Table 2
Truth table for most simple tautology $(p \vee \neg p)$ and controversy $(p \& \neg p)$

p	$\neg p$	$p \vee \neg p$	$p \& \neg p$
TRUE	FALSE	TRUE	FALSE
FALSE	TRUE	TRUE	FALSE

p 'Lava is slowly ejecting';
q 'Lava piles up in a dome';
r 'There is lava flow'.

Then we will have this rationale in the following form:

$$((p \supset q) \supset \neg r)) \vee (p \supset r) \quad (1)$$

Its truth table (Table 3a) shows that it is always true.

At the same time, the statement 'Either slowly ejecting lava piles up in a dome or forms a short thick viscous flow' is satisfiable:

$$(p \supset q) \vee (p \supset r) \quad (2)$$

and the statement 'Slowly ejecting lavas both pile up in domes and behave in different mode' is a controversy:

$$(p \equiv q) \& (p \equiv \neg q) \quad (3)$$

(see Tables 3b and c, respectively).

We should always keep in mind that statements in the logical sense may have various expressions in a natural language, such as English. A statement may be expressed not by a sentence but by a shorter phrase, and a compound or even several compound statements can be 'packed up' in one sentence of natural language. Simultaneously, one, even simple, logical statement can appear 'scattered' in more than one sentence of natural language. Logical connectives may be diversely expressed in verbal form. Therefore, we should be very careful when extracting the sense, always aiming to figure out what is actually *meant* in the sentence, written, or said, or thought.

If formulae have similar truth values with the same truth values of variables (e.g. see Table 4), they are called *equivalent* and can be substituted by one another, e.g. formulae $(p \supset q)$ and $(\neg p \vee q)$:

Table 3a
Truth table for tautology $((p \supset q) \supset \neg r)) \vee (p \supset r)$, T – TRUE, F – FALSE

p	q	r	$p \supset q$	$\neg r$	$(p \supset q) \supset \neg r$	$p \supset r$	$((p \supset q) \supset \neg r)) \vee (p \supset r)$
T	T	T	T	F	F	T	T
F	T	T	T	F	F	T	T
T	F	T	F	F	T	T	T
F	F	T	T	F	F	T	T
T	T	F	T	T	T	F	T
F	T	F	T	T	T	T	T
T	F	F	F	T	T	F	T
F	F	F	T	T	T	T	T

$$(p \supset q) \equiv (\neg p \vee q)$$

This means, for instance, that the statement 'If there is a fallout of fine ash, there will be lung diseases in people living in the fallout area' is equivalent to the statement 'Either there is no fallout of fine ash, or people have lung diseases'. Again, this equivalence is not clearly seen in natural language but is obvious in the language of propositional logic. Equivalent substitutions are used to express one logical connective by others and/or construct strict inference (see below).

Naturally, all tautologies are equivalent to each other, and similarly all controversies are equivalent to each other. Basic equivalences of propositional logic are listed in the Appendix.

3.2. *Assessment of rationales for truth by reduction to normal forms*

If the formula is long enough, the truth table is not useful. However, truth tables are not the only way to assess formulae for truth. The structure of a formula and the set of truth values it takes are interrelated. Therefore, it is possible to say just by the appearance of a formula, whether it is tautology, controversy or satisfiable.

Using equivalent substitutions (see Appendix), any formula can be brought to its *conjunctive normal form* (CNF). This is a form in which all implication and equivalence signs are substituted by combination of conjunction, disjunction and negation and the formula becomes a conjunction of elementary disjunctions. 'Elementary' means that disjunctions disjoin only individual variables and the negation refers only to variables too. Similarly, the formula can be brought to a *disjunctive normal form* (DNF), which is the disjunction of elementary conjunctions; the rule for negation is the same.

It is easy to show that:

(1) every formula in propositional logic has at least one (maybe more) CNF and DNF;

(2) the formula is a tautology if every elementary disjunction in its CNF includes at least one elementary disjunction of a variable and its negation; and

(3) the formula is a controversy if its DNF includes at least one elementary conjunction of a variable and its negation

Table 3b
Truth table for satisfiable formula $(p \supset q) \vee (p \supset r)$

p	q	r	$p \supset q$	$p \supset r$	$(p \supset q \vee (p \supset r)$
TRUE	TRUE	TRUE	TRUE	TRUE	TRUE
FALSE	TRUE	TRUE	TRUE	TRUE	TRUE
TRUE	FALSE	TRUE	FALSE	TRUE	TRUE
FALSE	FALSE	TRUE	TRUE	TRUE	TRUE
TRUE	TRUE	FALSE	TRUE	FALSE	TRUE
FALSE	TRUE	FALSE	TRUE	TRUE	TRUE
TRUE	FALSE	FALSE	FALSE	FALSE	FALSE
FALSE	FALSE	FALSE	TRUE	TRUE	TRUE

Table 3c
Truth table for controversy $(p \equiv q)\&(p \equiv \neg q)$

p	q	$\neg q$	$p \equiv q$	$p \equiv \neg q$	$(p \equiv q)\&(p \equiv \neg q)$
TRUE	TRUE	FALSE	TRUE	FALSE	FALSE
FALSE	TRUE	FALSE	FALSE	TRUE	FALSE
TRUE	FALSE	TRUE	FALSE	TRUE	FALSE
FALSE	FALSE	TRUE	TRUE	FALSE	FALSE

(for a strict proof, see e.g. Hilbert and Bernays, 1956).

This can be illustrated by the corresponding forms of formulae 1–3.

The CNF for formula 1, which is a tautology, is $(\underline{p} \vee \underline{\neg r} \vee \underline{\neg p} \vee \underline{r})\&(\neg q \vee \underline{\neg r} \vee \neg p \vee \underline{r})$. The CNF for a satisfiable formula (formula 2) is $\neg p \vee q \vee r$. DNFs are longer. We will not give the DNF for formula 2, for it is not illustrative, and a somewhat simplified DNF for a controversy (formula 3) is:

$$(\neg p \& \underline{\neg q} \& \underline{q}) \vee (\underline{\neg p} \& \neg q \& \underline{p}) \vee (\underline{\neg p} \& \underline{p} \& q) \vee$$

$$(\underline{\neg p} \& \underline{p}) \vee (\underline{\neg p} \& \underline{p} \& \underline{\neg q} \& \underline{q}) \vee (\underline{\neg p} \& \underline{p} \& \neg q) \vee$$

$$(\underline{q} \& \underline{\neg q} \& \neg p) \vee (\underline{q} \& \underline{\neg q} \& \underline{\neg p} \& \underline{p})$$

Underlined symbols are variables, with their negations occurring within elementary disjunctions or conjunctions.

3.3. *Assessment of rationales for deducibility and consistency*

If we want to know whether a standpoint or decision is substantiated well enough and is firmly supported by data and general concepts, we need to check it for *deducibility* from what we take for premises (i.e. these very data and concepts). This task is the same as *strict proof and inference* of statements that is elaborated well in propositional logic. If the given statement is deducible, or can be inferred, from some set of statements, the question arises whether it is possible to infer its negation from the same set. If not, then this set of statements is called *self-consistent* and valid for reasoning, if yes, it is *inconsistent* and requires correction (adding, removal or re-formulation of the statements it consists of).

Table 4
Truth table for the formulae $(p \supset q)$ and $(\neg p \vee q)$

p	q	$\neg p$	$p \supset q$	$\neg p \vee q$
TRUE	TRUE	FALSE	TRUE	TRUE
FALSE	TRUE	TRUE	TRUE	TRUE
TRUE	FALSE	FALSE	FALSE	FALSE
FALSE	FALSE	TRUE	TRUE	TRUE

Assessment for deducibility and self-consistency can be made by *logical calculi*. Logical calculus is a transformation or a succession of transformations of formulae in accordance with some *rules of inference*, which reveals that formula B *follows* from formulae A_1, A_2, ..., A_n. (As before, A, B and other capital letters are metaletters meaning any kind of propositional formula.) This means that B has the value TRUE if and only if each of A_1, A_2, ..., A_n has this value. Logical consequence is denoted $A_1, A_2, ..., A_n \rightarrow B$. Formulae A_1, A_2, ..., A_n are called premises, B consequence.

Reduction to normal forms is the simplest calculus, in which there is one premise and the only kind of rules of inference is equivalent substitution. In trivial cases, so-called principal and shortened conjunctive and disjunctive normal forms help find all consequences (conjunctive forms) and hypotheses (disjunctive forms) of given formulae. However, if the case is more than one assumption (that is, actually, the case in most cases), more powerful calculi (sequential, natural-sequential and others) with specific rules of inference apply.

We will demonstrate strict inference from volcanological observations by the example of natural-sequential calculus elaborated by Gentzen (1934). It is described below.

1.1. *Sequence* is expression $A_1, A_2, ..., A_m \rightarrow B$, where A_1, A_2, ..., A_m, B are propositional formulae. Formulae A_1, A_2, ..., A_m are front members of the sequence, and B is the back member. There may be no front members at all, but the back one must always be present: $\rightarrow B$.

1.2. *Inference in natural-sequential calculus* consists of a number of sequences. Each of these either is '*main sequence*' (see below) or is derived from a previous one by a structural transformation (see below) or a rule of inference (see below). The last sequence of an inference has no front members and its back member is a finite formula.

1.3. There are two types of *main sequences*, called '*logical*' and '*mathematical*' ones. A *logical main sequence* is a sequence of general form $C \rightarrow C$, where C is a propositional formula (this sequence arises if the inference is based on assumption expressed by C). A *mathematical main sequence* is a sequence of general form $\rightarrow D$, where C is an axiom of mathematics.

1.4. Allowed structural transformations (for propositional logic; a horizontal line means that the below sequence follows from the above sequences).

1.4.1. Transposition of two front members:

$$\frac{C,\ D,\ \Gamma \rightarrow \Delta}{D,\ C,\ \Gamma \rightarrow \Delta}$$

1.4.2. Withdrawal of a front member, which is the same as another front member:

$$\frac{C,\ C,\ \Gamma \rightarrow \Delta}{C,\ \Gamma \rightarrow \Delta}$$

1.4.3. Addition of any propositional formula to front members:

$$\frac{\Gamma \rightarrow \Delta}{C,\ \Gamma \rightarrow \Delta}$$

There is a specific rule for predicate logic, which we do not cite here.

1.5. Rules of inference (for propositional logic).

Let A, B and C denote any propositional formulae and Γ, Δ and Θ any (possibly empty) lists of formulae divided by commas. The formulae of these lists are front members of some sequences.

The following rules of inference of natural-sequential calculus are applicable to propositional logic.

Introduction of conjunction (henceforth Rule IC):

$$\frac{\begin{array}{l}\Gamma \rightarrow A\\ \Delta \rightarrow B\end{array}}{\Gamma,\ \Delta \rightarrow (A\&B)}$$

Elimination of conjunction (henceforth Rule EC):

$$\frac{\Gamma \rightarrow A\&B}{\Gamma \rightarrow A} \qquad \frac{\Gamma \rightarrow A\&B}{\Gamma \rightarrow B}$$

Introduction of disjunction (henceforth Rule ID):

$$\frac{\Gamma \rightarrow A}{\Gamma \rightarrow A \vee B} \qquad \frac{\Gamma \rightarrow B}{\Gamma \rightarrow A \vee B}$$

Elimination of disjunction (henceforth Rule ED):

$$\frac{\begin{array}{l}\Gamma \rightarrow A \vee B\\ A, \Delta \rightarrow C\\ B,\ \Theta \rightarrow C\end{array}}{\Gamma,\ \Delta,\ \Theta \rightarrow C}$$

Introduction of implication (henceforth Rule II):

$$\frac{A,\ \Gamma \rightarrow B}{\Gamma \rightarrow A \supset B}$$

Elimination of implication (henceforth Rule EI):

$$\frac{\begin{array}{l}\Gamma \rightarrow A\\ \Delta \rightarrow A \supset B\end{array}}{\Gamma,\ \Delta \rightarrow B}$$

Introduction of negation (henceforth Rule IN):

$$\frac{\begin{array}{l}A,\ \Gamma \rightarrow B\\ A, \Delta \rightarrow \neg B\end{array}}{\Gamma,\ \Delta \rightarrow \neg A}$$

Elimination of double negation (henceforth Rule EN):

$$\frac{\Gamma \rightarrow \neg\neg A}{\Gamma \rightarrow A}$$

In the above expressions, the symbols put to the left of the arrow can be regarded as the 'memory' of the inference, and those to the right are its 'working part'.

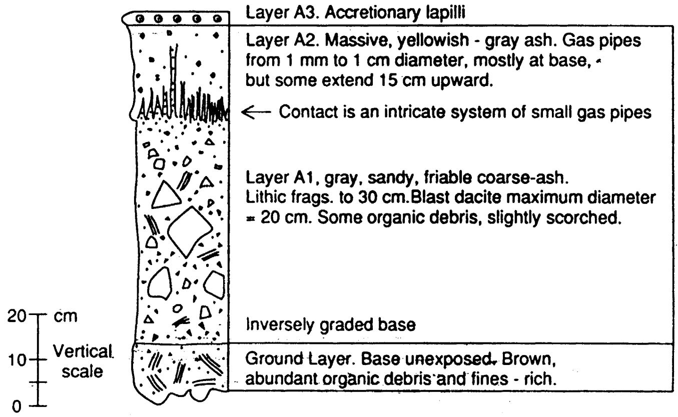

Fig. 1. (Figure 6 in Fisher (1990).) Locality N33.3, 9.5 km from the source. Upper contact of layer A1 shows innumerable tiny gas pipes extending upward into layer A2.

4. Application

We apply this calculus to a fragment of paper by Fisher (1990) on turbulent pyroclastic flow deposits of the Mt. St. Helens May 1980 eruption. The fragment consists of a paragraph of text (see below) and a figure (see Fig. 1), which gives information also involved in reasoning.

'The contact relationship between layer A2 and A1 suggests that layer A1 was gas-rich during transport, but because it lacked abundant fines it was permeable, gases were usually expelled quickly during transport. Except in a few places, gas loss occurred before layer A2 was fully in place, allowing development of a sharp contact by movement of A2 after A1 had come to rest. At a few localities, however, layer A2 was emplaced fast enough to seal off the gas within layer A1 before it completely escaped, and the gas pressure was great enough to cause continued upward movement of gas into layer A2, resulting in preservation of the gas pipes' (Fisher, 1990, p. 1042).

Let us denote relevant statements of this fragment.

p 'Gas escapes from the layer A1' (or, in terms of Fisher, 'gases escaped quickly during transport');

q 'Layer A2 is less permeable than the layer A1' (this is not explicitly said but shown in the figure (see Fig. 1) as finer grain size of layer A2 than A1 and can be expressed as the equivalence of the following statements: 'Layer A2 has finer grain size' ≡ 'Layer A2 is less permeable', but for simplicity of proof we omit this equivalence;

r 'Gas pipes form' (we may also add, '... form distinctly enough to be preserved by the moment of observation');

s 'Material of layer A1 was gas-rich' (again, here we actually have the equivalence, ' Material of layer A1 was gas-rich' ≡ 'The gas pressure in layer 1 was high');

t 'Material of layer A2 was in place' (by the moment of formation of layer A1 in a given place);

u 'There is a sharp top of layer A1' (speaking about formation of the sharp contact, Fisher says that, in his view, it is sharp exactly because first it was *not* a contact between two layers but just the top of one of them).

Having introduced such statements, we can formulate the *assumptions* that led Richard Fisher to the conclusions he made.

$$s \supset p \rightarrow s \supset p \qquad \text{(assumption 1)}$$

'If the layer A1 was gas-rich, gas had to be escaping from it'. Fisher obviously means this, though he does not put this explicitly (perhaps because he considers this notion obvious):

$$(p\&q) \supset r \rightarrow (p\&q) \supset r. \qquad \text{(assumption 2)}$$

'If gas was escaping from layer A1 and layer A2 is less permeable than A1, gas pipes must form'. The third assumption is:

$$(p\&\neg t)\supset u \rightarrow (p\&\neg t)\supset u \quad \text{(assumption 3)}$$

'If gas was escaping from layer A1 and layer A2 was not yet in place, A1 will form a sharp top.'

In addition, we can make some obvious assumptions in the form accepted in natural-sequential calculus:

$$p \rightarrow p \quad \text{(assumption 4)}$$

$$q \rightarrow q \quad \text{(assumption 5)}$$

$$s \rightarrow s \quad \text{(assumption 6)}$$

$$\neg t \rightarrow \neg t \quad \text{(assumption 7)}$$

Then the conclusions of Fisher will look like:

$$s\supset r \quad \text{(conclusion 1)}$$

('if layer A1 was gas-rich, then there will form gas pipes'), and

$$s\supset u \quad \text{(conclusion 2)}$$

('if layer A1 was gas-rich, then this layer will have a sharp top').

The first one can be proved based on assumptions 1, 2, 5 and 6, using structural transformations where necessary. The inference is below.

1. $s\supset p \rightarrow s\supset p$ (assumption 1)
2. $(p\&q)\supset r \rightarrow (p\&q)\supset r$ (assumption 2)
3. $q \rightarrow q$ (assumption 5)
4. $s \rightarrow s$ (assumption 6)
5. $s\supset p,\ s \rightarrow p$ (Rule EI for lines 1, 4)
6. $s\supset p,\ s,\ q \rightarrow p\&q$ (Rule IC for lines 3, 5)
7. $s\supset p,\ s,\ q,\ (p\&q)\supset r \rightarrow r$ (Rule EI for lines 6, 2)
8. $s\supset p,\ q,\ (p\&q)\supset r \rightarrow s\supset r$ (Rule II for line 7)
9. $s\supset p,\ q \rightarrow ((p\&q)\supset r)\supset(s\supset r)$ (Rule II for line 8)
10. $s\supset p \rightarrow (q\supset(((p\&q)\supset r)\supset(s\supset r)))$ (Rule II for line 9)
11. $\rightarrow ((s\supset p)\supset(q\supset(((p\&q)\supset r)\supset(s\supset r))))$ (Rule II for line 10)

The back member of the final sequence is a tautology. Conclusion 1 is proved.

The second conclusion is also deducible, from assumptions 1, 3, 6 and 7 (with appropriate structural transformations).

1. $s\supset p \rightarrow s\supset p$ (assumption 1)
2. $(p\&\neg t)\supset u \rightarrow (p\&\neg t)\supset u$ (assumption 3)
3. $s \rightarrow s$ (assumption 6)
4. $\neg t \rightarrow \neg t$ (assumption 7)
5. $s\supset p,\ s \rightarrow p$ (Rule EI for lines 1, 3)
6. $s\supset p,\ s,\ \neg t \rightarrow p\&\neg t$ (Rule IC for lines 4, 5)
7. $s\supset p,\ s,\ \neg t,\ (p\&\neg t)\supset u \rightarrow u$ (Rule EI for lines 3, 6)
8. $s\supset p,\ \neg t,\ (p\&\neg t)\supset u \rightarrow s\supset u$ (Rule II for line 7)
9. $s\supset p,\ \neg t \rightarrow ((p\&\neg t)\supset u)\supset(s\supset u)$ (Rule II for line 8)
10. $s\supset p \rightarrow (\neg t\supset((p\&\neg t)\supset u)\supset(s\supset u))$ (Rule II for line 9)
11. $\rightarrow (s\supset p)\supset(\neg t\supset((p\&\neg t)\supset u)\supset(s\supset u))$ (Rule II for line 10)

The back member of the final sequence is a tautology. Conclusion 2 is proved. This means that the conclusions of Fisher on the relationship between the two tuff layers are strictly proved.

However, this does not mean that they have become an absolute truth. Let us widen the set of assumptions by adding few new statements (propositional variables) reflecting some obvious facts of volcanology.

The variables are:

f – pyroclastic material flows,

g – pyroclastic material degasses quickly.

(Following Fisher, herewith we consider pyroclastic flows only, not tephra, hence the negation of f, $\neg f$, would mean that pyroclastic material becomes a layer.) Then we can make the following assumptions:

$$f \rightarrow f \quad \text{(assumption 8)}$$

$$g \rightarrow g \quad \text{(assumption 9)}$$

$$g\supset\neg s \rightarrow g\supset\neg s \quad \text{(assumption 10)}$$

$$(f\&g)\supset\neg s \rightarrow (f\&g)\supset\neg s \quad \text{(assumption 11)}$$

Assumption 10 means that a natural measure of 'quickness' is that every next time the material is notably poorer in gas than before. Assumption 11 means that if pyroclastic material flows and degasses quickly, the material of the resulting layer (A1 in our case) is not gas-rich (otherwise the degassing cannot be considered quick indeed – see above).

With these assumptions, the following can be proved (keeping in mind the structural transformations):

1. $f \to f$ (assumption 8)
2. $g \to g$ (assumption 9)
3. $f, g \to f\&g$ (Rule IC for lines 1, 2)
4. $(f\&g) \supset \neg s \to (f\&g) \supset \neg s$ (assumption 11)
5. $f, g, (f\&g) \supset \neg s \to \neg s$ (Rule EI for lines 3, 4)
6. $g \supset \neg s \to g \supset \neg s$ (assumption 10)
7. $f, (f\&g) \supset \neg s \to \neg g$ (Rule IN for lines 5, 6)
8. $f, g, (f\&g) \supset \neg s \to \neg g \& \neg s$ (Rule IC for lines 5, 7)
9. $f, g, (f\&g) \supset \neg s \to f \& \neg g \& \neg s$ (Rule IC for lines 1, 8)
10. $f, g \to ((f\&g) \supset \neg s) \supset (f \& \neg g \& \neg s)$ (Rule II for line 9)
11. $f \to g \supset (((f\&g) \supset \neg s) \supset (f \& \neg g \& \neg s))$ (Rule II for line 10)
12. $\to f \supset (g \supset (((f\&g) \supset \neg s) \supset (f \& \neg g \& \neg s)))$ (Rule II for line 11)

Based on the main tautologies of propositional logic (see Appendix), it can be shown that the inferred formula f and $\neg g$ and $\neg s$ is equivalent to $\neg((f \text{ and } \neg g) \supset s)$. Literally, it means that 'it is not true that if pyroclastic material flows and not degasses quickly, the material of the resulting layer will be gas-rich' or, in simpler words, if pyroclastic flow degasses quickly, the resulting deposit *will not* be gas-rich. This contradicts the idea taken for assumption 1, thus exposing a hidden contradiction between the two premises formulated verbally as: (1) 'gases escaped quickly during transport', and (2) 'layer A1 was gas-rich upon deposition'. To avoid it, a statement was required that gases had not escaped during transport, which could invoke a special consideration why this had not happened, and so forth, until new self-consistency is achieved.

Other calculi than natural-sequential can also be applied in volcanology. There are theorems proved in propositional logic that no contradictive results can be deduced by two different calculi, hence, their choice is solely a matter of convenience. Their detailed consideration is outside the scope of this work. Special software is being elaborated for automatic inference of formulae for deducibility and processing of big sets of formulae by various calculi. First results of this work were reported by Moukhachov and Netchitailov (2001).

5. Discussion: practical benefits and future research

Assessment of individual statements for truth, if it were to become practice of field volcanological studies, would ensure a check on thinking and instant correction. As was shown here by a trivial example of formulae 1–3 (see Section 3.1), similar information may be given in many different ways, and recording it in terms of propositional logic helps us understand what we actually mean and whether what we mean is correct (i.e. expressed by satisfiable formulae or tautologies). It is unlikely that we will be able to formulate many of our ideas as tautologies. What is more probable is that a combination of different standpoints, or even the thoughts of similar scientists at different stages of research, can lead to controversy, which may pass unnoticed if not revealed by truth tables or normal forms.

However, even more important is the assessment for deducibility and self-consistency by the procedure of strict inference. It allows us to explicitly formulate premises of reasoning, follow the pathway of thought and discern thinking from intuition. This is especially important in hazard assessment, where recommendations of different experts (often made by intuition) require objective and unbiased trial. Even formalized expert judgement procedures so far have been based on consensus or poll, possibly with statistical weighing of opinions (Aspinall and Cooke, 1998; Aspinall and Woo, 1994; Ross, 1989). However, an opinion, both individual and collective, may be influenced by purely psychological factors, and weighing demands a supreme 'experienced technical facilitator ... to supervise elicitations' of experts (Aspinall and Cooke, 1998, p. 2115), who would unavoidably rely on his/her own intuition, which cannot be verified. Raising no objection to scientific intuition, we think that it should be accompanied by strict assessment, especially for insurance and judicial items. 'Scientific uncertainty' pinned down by these authors or, to put it more explicitly, *conceptual uncertainty* (Pshenichny, 2002) underlies even quantitative methods of hazard and risk assessment (see below).

However, the revealing of 'illogical', contradictive concepts may enlighten not the fallacies of thinking but peculiarities of the studied object. In such cases, scientists should seek to eliminate the contradiction by re-examination of their views in a wider context; e.g. in the fragment of text analyzed above two incompatible pieces of evidence (sharp contact and gas pipes at the contact of similar layers) were interpreted jointly and did not lead to contradiction, although, as was shown in the same example, widening the context may readily lead to contradiction.

If no contradictions are encountered, the logical calculi allow us to build long chains of inference incorporating any kind of relevant knowledge, e.g. from seismic signals to social disorder. The more diverse knowledge is involved, the more the ultimate consequences of observed events can be quickly predicted. Logic may appear especially useful in predicting the hazards with long return periods, unique disasters, or catastrophes which have never happened in human history. When the data are scarce or absent, we can operate with our concepts. Even our fantasy, if given strict form, may become a tool for hazard assessment. For this, the following rationale can be suggested.

Any application of a logical calculus to the natural world is its *interpretation* by the latter, i.e. every propositional formula is an event or a group of related events, and each step of inference is a succession of events in time. But logic describes our reasoning about these events, not the events themselves and, contrary to mathematics, does not mean that the thought events *must* take place in reality. Therefore, in application to the natural world, every step of inference, which is 100% reliable in terms of logic, acquires a probabilistic sense. An inference of the form $A \rightarrow A$ is absolutely reliable, because it denotes an event that actually takes place (i.e. 'we think what we see'). However, the more steps are required to 'reach' the deduced formula, the less probable it becomes that a succession of events takes place in reality exactly as inferred. Hence, we can introduce a *conceptual probability* of a statement as a measure of *conceptual uncertainty in application to the natural world*. Its first and rough estimation can be defined as (minimum required number of steps of inference)$^{-1}$. For instance, in the example given above (see Section 4), each of the conclusions of Fisher was deduced from the corresponding assumptions by 11 steps, so their conceptual probability can be estimated as 1/11, or 0.09. Further development of logical assessments would lead to better evaluation of conceptual probability. However, even in its present form, in the authors' view, this probability can be used in the Bayes formula as the probability of hypothesis and in the event trees as the probability of each branch. We do admit that this is not the best possible solution but, hopefully, it is a better option than ascribing numerical values of probability by intuition, which is what scientists are forced to do at present (Aspinall and Cooke, 1998; Newhall and Hoblitt, 2002).

It should be noted that the chains of inference may be based solely on observational evidence, which, contrary to concepts arising from theoretical considerations and modeling, is: (1) absolutely reliable; (2) independent of premises accepted by one and not accepted by another scientist; and (3) easily understandable by non-professionals. Hence, not only the application of logic to observational knowledge would optimize the research work and evaluation of the work of the experts, but also help scientists communicate with civil authorities, journalists and population.

At the same time, compact domains of knowledge, which describe particular eruptive phenomena (extrusive dome growth and collapse, pyroclastic flows, lahars, etc.) and have proved self-consistent, can be subject to *logical modeling*, i.e. thorough application of logical calculi to any of its statements. Any question formulated in terms of these domains (or *finite worlds*, as logicians say) will make sense and have a correct answer. These will be the first self-consistent 'islands' of strict knowledge, the first 'crystallized' fragments in the 'hot and viscous stuff' of volcanological observations. Maybe some commonly accepted statements will require re-formulation to become deducible formulae. Widening of these domains would eventually lead to an increased formalization of volcanology.

6. Conclusions

Propositional logic may be used as an assessment tool and thus have wide application to the knowledge derived from volcanological observations. The assessments of individual statements and rationales for truth and satisfiability can be done by truth tables and reduction to normal forms. Assessments of a set of statements for self-consistency and deducibility of a statement from the set are accomplished by various logical calculi, the simplest of which is reduction to normal forms. These procedures allow us to find contradictions, build self-consistent domains of observational knowledge and obtain strict inference within these domains. The latter will eventually lead to the logical modeling of volcanic objects.

Logical processing of knowledge may be useful in field volcanological studies and in hazard assessment for optimization of research work, of recommendations by experts and of communication with non-professionals. It would especially benefit formalized expert judgement procedures and may be useful for optimization of event trees and Bayesian approach. The results of this study show that strict reasoning is possible in the area of volcanology that has been considered least certain and most intuitive – its observational knowledge.

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Appendix. Basic tautologies and equivalences of propositional logic

Basic tautologies

1. $A \supset (B \supset A))$
2. $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$
3. $A \supset (B \supset (A \& B))$
4. $(A \& B) \supset A$
5. $(A \& B) \supset B$
6. $A \supset (A \vee B)$
7. $B \supset (A \vee B)$
8. $(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$
9. $(A \supset B) \supset ((A \supset \neg B) \supset \neg A)$
10. $A \supset (\neg A \supset B)$
11. $\neg \neg A \supset A$

Basic equivalences

1. $A \& B$ is equivalent to $\neg(\neg A \vee \neg B)$
2. $A \& B$ is equivalent to $\neg(A \supset \neg B)$
3. $A \vee B$ is equivalent to $\neg(\neg A \& \neg B)$
4. $A \supset B$ is equivalent to $\neg A \vee B$
5. $A \equiv B$ is equivalent to $(A \supset B) \& (B \supset A)$
6. $A \equiv B$ is equivalent to $(\neg A \vee B) \& (\neg B \vee A)$
7. $\neg \neg A$ is equivalent to A
8. $A \& B$ is equivalent to $B \& A$
9. $A \& (B \& C)$ is equivalent to $(A \& B) \& C$
10. $A \vee B$ is equivalent to $B \vee A$
11. $A \vee (B \vee C)$ is equivalent to $(A \vee B) \vee C$
12. $A \vee (B \& C)$ is equivalent to $(A \vee B) \& (A \vee C)$
13. $A \& (B \vee C)$ is equivalent to $(A \& B) \vee (A \& C)$
14. $A \& A$ is equivalent to A
15. $A \vee A$ is equivalent to A
16. $\neg(A \& B)$ is equivalent to $\neg A \vee \neg B$
17. $\neg(A \vee B)$ is equivalent to $\neg A \& \neg B$
18. $(A \vee B) \& (\neg A \vee B)$ is equivalent to B
19. $A \& (A \vee B)$ is equivalent to A
20. $A \vee (A \& B)$ is equivalent to A
21. $(A \vee C) \& (B \vee \neg C)$ is equivalent to $(A \vee C) \& (B \vee \neg C) \& (A \vee B)$
22. $(A \& C) \vee (B \& \neg C)$ is equivalent to $(A \& C) \vee (B \& \neg C) \vee (A \& B)$

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