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Limitations on gyration constants for main symmetry systems of rocks

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Abstract

In the case of propagation of plane elastic waves in anisotropic gyrotropic media, Christoffel tensor is complex; its real part contains stiffnesses and an imaginary part includes components of the fifth-rank gyration tensor. Inequalities relating stiffnesses and gyration constants are derived from the conditions for potential energy to be positive. The necessary and sufficient conditions for the positive definiteness of the complex matrix of stiffnesses and gyration constants are used. Sets of inequalities are obtained for two types of rocks belonging to acentric limit groups $\infty\infty$ and ∞. These inequalities provide a possibility to carry out modelling of elastic wave propagation in the media considered, setting the values of gyration constants not arbitrarily but in accordance with physical laws.

Keywords: Seismic-wave propagation; Stability conditions; Anisotropy; Gyrotropy

1. Introduction

Gyrotropy is known as an exhibition of the first-order spatial dispersion. Optical gyrotropy is known since 1811 (F. Arago), acoustical gyrotropy since the 1960s (Andronov, 1960; Kluge, 1966; Portigal and Burstein, 1968; Pine, 1970). Recently, the concept of seismic gyrotropy was introduced (Obolentseva, 1992, 1996) and developed (Obolentseva and Chichinina, 1997; Chichinina and Obolentseva, 1997, 1998; Chichinina, 1998, 2000; Obolentseva et al., 2000).

In a gyrotropic elastic medium, Hooke's law and wave equation are (Sirotin and Shascolskaya, 1979)

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl} + b_{ijklm}\frac{\partial \varepsilon_{kl}}{\partial x_m}, \tag{1}$$

$$c_{ijkl}\frac{\partial^2 u_k}{\partial x_j \partial x_l} + b_{ijklm}\frac{\partial^3 u_k}{\partial x_j \partial x_l \partial x_m} = \rho\frac{\partial^2 u_i}{\partial t^2}, \quad i = 1,2,3, \tag{2}$$

respectively.

For plane harmonic waves of circular frequency ω

$$\boldsymbol{u}(\boldsymbol{r},t) = u_0\boldsymbol{A}\exp[i2\pi/\lambda(\boldsymbol{nr} - Vt)] \ (2\pi/\lambda = \omega/V),$$

propagating with phase velocity V in the direction of the wave normal $\boldsymbol{n}$ and polarized along a unit vector $\boldsymbol{A}$, wave equations (2) become

$$(c_{ijkl}n_jn_l + i\omega/Vb_{ijklm}n_jn_ln_m)A_k = \rho V^2A_i, \quad i = 1,2,3. \tag{3}$$

These are Christoffel equations for an anisotropic gyrotropic medium which show that tensor $\boldsymbol{c}$ has

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become complex: $c_{ijkl} \rightarrow c_{ijkl} + i(\omega/V)b_{ijklm}n_m$. Denote it by $\boldsymbol{a}$:

$$a_{ijkl} = c_{ijkl} + ikb_{ijklm}n_m, \tag{4}$$

where $k = \omega/V$ is a wave number.

Properties of inner symmetry of tensor $\boldsymbol{a}$ are

$$a_{ijkl} = \bar{a}_{klij}, \ a_{ijkl} = a_{jikl} = a_{ijlk} = a_{jilk}. \tag{5}$$

The first equation in Eq. (5) implies that tensor $\boldsymbol{a}$ is Hermitian. This property is due to the symmetry of tensor $\boldsymbol{c}$ and antisymmetry of tensor $\boldsymbol{b}$ to the permutation of the first and the second pairs of indices. (For more details, the symmetry of tensor $\boldsymbol{b}$ is covered in Obolentseva, 1993, 1996.) The symmetry of tensor $\boldsymbol{a}$ in permutations of indices in the first and the second pairs (the rest of the equations in Eq. (5)) is apparent from this property of tensors $\boldsymbol{c}$ and $\boldsymbol{b}$.

To study elastic-wave propagation in anisotropic gyrotropic media by means of mathematical modelling, one needs to know both the values of elastic moduli (c_{ijkl}) and the values of gyration constants (b_{ijklm}). The present work represents the limitations on constants (and, hence, gyration constants b_{ijklm}) derived in the same way as limitations on stiffnesses c_{ijkl} (e.g., Fedorov, 1968), i.e., from the requirement for elastic energy to be positive. This reduces to the requirement for quadratic form with respect to small strains to be positive definite. The coefficients of this form are, in the case considered, the moduli a_{ijkl}.

2. Lagrangian and potential energy in a gyrotropic medium

Within the framework of continuum elastic theory, the Lagrangian density for an anisotropic gyrotropic medium is (Goldstein, 1950)

$$L = \frac{1}{2}\rho\dot{u}_t\dot{u}_t - \frac{1}{2}c_{ijkl}\frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial x_l} - b_{ijklm}\frac{\partial u_i}{\partial x_j}\frac{\partial^2 u_k}{\partial x_l \partial x_m}.$$

The second and the third terms are potential energy, the last one therewith is due to gyrotropic effects. The terms with $\boldsymbol{b}$ in L differ from the terms with $\boldsymbol{c}$ by one more differentiation in respect to spatial coordinate. Therefore, the potential energy can be presented as

$$W = (c_{ijkl} + ikb_{ijklm}n_m)\frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial x_l} = a_{ijkl}\frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial x_l}. \tag{6}$$

The potential energy W will be positive if the quadratic form with the a_{ijkl} coefficients will be positive definite. The necessary and sufficient conditions for positive definiteness of the Hermitian form in Eq. (6) and of the Hermitian matrix a_{pq} ($a_{ijkl} \leftrightarrow a_{pq}$) are the following inequalities for determinants of orders 1, 2,. . ., 6 (Korn and Korn, 1968):

$$a_{11} > 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} > 0, \tag{7}$$

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{14} \\ a_{21} & a_{22} & \dots & a_{24} \\ a_{31} & a_{32} & \dots & a_{34} \\ a_{41} & a_{42} & \dots & a_{44} \end{vmatrix} > 0, \begin{vmatrix} a_{11} & a_2 & \dots & a_{15} \\ a_{21} & a_2 & \dots & a_{25} \\ \dots & \dots & \dots & \dots \\ a_{51} & a_2 & \dots & a_{55} \end{vmatrix} > 0, \begin{vmatrix} a_{11} & a_{12} & \dots & a_{16} \\ a_{21} & a_{22} & \dots & a_{26} \\ \dots & \dots & \dots & \dots \\ a_{61} & a_{62} & \dots & a_{66} \end{vmatrix} > 0, \tag{8}$$

which are formed from the initial matrix (a_{pq}) of the sixth order. Matrix presentation instead of tensor one is made in keeping with the known rule for indices: $11 \rightarrow 1$, $22 \rightarrow 2$, $33 \rightarrow 3$, $23 \rightarrow 4$, $13 \rightarrow 5$, $12 \rightarrow 6$.

The system of inequalities (Eqs. (7) and (8)) looks like in the case of an anisotropic medium without gyration differing by complexity of elements $a_{pq} = \Re a_{pq} + i\,\Im a_{pq}$, where $\Re a_{pq} = c_{pq}$, $\Im a_{pq} = kb_{pqm}n_m$.

3. Limitations on gyration constants for anisotropic gyrotropic media

3.1. On acentric limit groups

To derive inequalities relating gyration constants $b_{pqm}n_m$ and stiffnesses c_{pq}, it is necessary to develop the determinants in inequalities (7) and (8). In a general case (36 elastic moduli c_{pq} and 90 gyration constants $b_{pqm}n_m$), when developing the determinants, one obtains cumbersome expressions resisting any analysis. Therefore, generally, conditions (7) and (8) ought to be checked numerically for given sets of constants c_{pq}, $b_{pqm}n_m$. Such a means is inconvenient to apply; however, in practice, it is of no importance, because seismic models of the lowest symmetry (triclinic system) are not usually in use.

In this work, the inequalities of the form $f_i(c_{ij}, b_{ijm})>0$ for propagation in the directions $\boldsymbol{n}=(1, 0, 0)$, (0, 1, 0), (0, 0, 1) are derived for two limit acentric groups (gyrotropy may exist only for acentric groups): $\infty\infty$ and ∞. Limit groups, or Curie groups, are those point groups which include symmetry axes of order ∞ (on these groups, see, e.g., Sirotin and Shascolskaya, 1979). The groups $\infty\infty$ and ∞ describe more often the symmetry properties of geological media.

A medium $\infty\infty$ has an infinite number of axes ∞; it is called a rotation group. A geometrical figure symbolizing this group is a rotating sphere, left and right. The non-gyrotropic analog of a medium $\infty\infty$ is a medium $\infty\infty m$. It has an infinite number of axes ∞ and an infinite number of planes m passing through these axes. A geometrical symbol of this group is a sphere at rest. A medium of the group symmetry $\infty\infty$ is called gyrotropic, whereas a medium of the group symmetry $\infty\infty m$ is called isotropic. The media $\infty\infty$ are acoustically active: they rotate polarization plane of shear waves in all directions of wave propagation.

A medium ∞ has one symmetry axis of infinite order. A geometrical image of the group ∞ is a rotating cone, left and right. An example of non-gyrotropic medium with an axis of infinite order is a medium of group symmetry ∞/m having an axis ∞ and a symmetry plane normal to this axis. A geometrical image of the group ∞/m is a cylinder at rest. Three groups (out of five) with an axis ∞ are acentric: ∞, $\infty 2$, ∞m, the two groups have a symmetry center: ∞/m, ∞/mm. Hence, only the media of symmetry groups ∞, $\infty 2$, ∞m may be gyrotropic.

The causes for a geological medium with an axis ∞ to be gyrotropic lie in the dissymmetry of its microstructure (Obolentseva and Chichinina, 1997; Chichinina and Obolentseva, 1997, 1998; Chichinina, 1998, 2000). Rotation of shear-wave polarization plane occurs only during propagation along symmetry axis, for all other directions polarizations of the two shear waves are elliptical, clockwise and counter-clockwise.

The media of the group symmetry $\infty\infty$ are characterized by the two elastic moduli (Lame constants λ, μ) and one gyration constant (denote it v). The media of the group symmetry $\infty\infty$ are described by the five elastic moduli (transversely isotropic media) and nine gyration constants.

3.2. Inequalities for symmetry group $\infty\infty$

In this group, nonzero are nine components of tensor $\boldsymbol{c}$:

$$
\begin{aligned}
c_{11} &= c_{22} = c_{33} = \lambda + 2\mu, \\
c_{44} &= c_{55} = c_{66} = \mu, \\
c_{12} &= c_{13} = c_{23} = \lambda,
\end{aligned} \tag{9}
$$

and their isomers $c_{21}=c_{31}=c_{32}=\lambda$. The rest, 24 components out of 36, are equal to zero.

The $\boldsymbol{b}$ tensor has the following nonzero components (Obolentseva, 1993):

$$
\begin{aligned}
b_{12131} &= b_{23212} = b_{31323} = v, \\
b_{11123} &= b_{22231} = b_{33312} = 2v, \\
b_{11132} &= b_{22213} = b_{33321} = -2v,
\end{aligned}
$$

or, in matrix presentation,

$$
\begin{aligned}
b_{651} &= b_{462} = b_{543} = v, \\
b_{163} &= b_{241} = b_{352} = 2v, \\
b_{152} &= b_{263} = b_{341} = -2v,
\end{aligned} \tag{10}
$$

and their isomers, in all 18 components; v is a pseudoscalar.

Matrices (c_{pq}) and (b_{pqm}), $m=1, 2, 3$, are of the following form:

$$\begin{vmatrix} c_{33} & c_{13} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{33} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{44} \end{vmatrix},$$

$c_{13}=c_{33}-2c_{44}$,

$$\begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_{241} & 0 & 0 \\ 0 & 0 & 0 & b_{341} & 0 & 0 \\ 0 & -b_{241} & -b_{341} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -b_{651} \\ 0 & 0 & 0 & 0 & b_{651} & 0 \end{vmatrix},$$

$$\begin{vmatrix} 0 & 0 & 0 & 0 & b_{152} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{352} & 0 \\ 0 & 0 & 0 & 0 & 0 & b_{462} \\ -b_{152} & 0 & -b_{352} & 0 & 0 & 0 \\ 0 & 0 & 0 & -b_{462} & 0 & 0 \end{vmatrix},$$

$$\begin{vmatrix} 0 & 0 & 0 & 0 & 0 & b_{163} \\ 0 & 0 & 0 & 0 & 0 & b_{263} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -b_{543} & 0 \\ 0 & 0 & 0 & b_{543} & 0 & 0 \\ -b_{163} & -b_{263} & 0 & 0 & 0 & 0 \end{vmatrix}.$$

Their components, c_{pq} and b_{pqm}, are equal to the constants from Eqs. (9) and (10).

Substitution of $a_{pq}=c_{pq}+ikb_{pqm}$ in the determinants (Eqs. (7) and (8)) gives a set of inequalities. The inequalities for determinants of orders 1, 2, 3 are free from gyration constants and relate stiffnesses c_{33}, c_{44} to each other:

$$c_{33} > 0,\ 4c_{44}(c_{33}+c_{44}) > 0,\ 4c_{44}^2(3c_{33}-4c_{44}) > 0.$$

Development of the determinants of orders 4, 5, 6 furnishes the desired inequalities relating components of tensors ***c*** and ***b***.

The inequalities for the determinant of *the fourth order* written for propagation along directions $n=(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ are as follows:

$$4(3c_{33}-4c_{44})c_{44}(c_{44}^2-4k^2v^2) > 0,$$
$$4(3c_{33}-4c_{44})c_{44}^3 > 0,\ 4(3c_{33}-4c_{44})c_{44}^3 > 0.$$

The second and third inequalities contain only stiffnesses and give the relation $c_{44}/c_{33}<3/4$. An inequality with gyration constant is the first, it contains as a factor the inequality $(c_{44}^2-4k^2v^2)>0$ relating the c_{44} constant with the gyration constant v. Thus, the limitation on the v constant is very simple:

$$k^2v^2 < \frac{1}{4}c_{44}^2. \quad (11)$$

It follows also from this inequality that v may be positive as well as negative; since v is a pseudoscalar, its sign indicates the direction of rotation of shear-wave polarization plane (clockwise or counter-clockwise).

Development of the determinant of *the fifth order* yields the following inequalities for $\boldsymbol{n}=(1, 0, 0), (0, 1, 0), (0, 0, 1)$:

$$4(3c_{33}-4c_{44})c_{44}^2(c_{44}^2-4k^2v^2)>0,$$
$$4(3c_{33}-4c_{44})c_{44}^2(c_{44}^2-4k^2v^2)>0,$$
$$(3c_{33}-4c_{44})c_{44}^2(4c_{44}^2-4k^2v^2)>0.$$

The first two inequalities repeat expression (11) and the third inequality gives $v^2<1/(k^2c_{44}^2)$. Then the solution of inequalities $(c_{44}^2-4k^2v^2)>0$, $(4c_{44}^2-4k^2v^2)>0$ is again inequality (11).

At last, development of the determinant of *the sixth order* produces for all directions $\boldsymbol{n}=(1, 0, 0), (0, 1, 0), (0, 0, 1)$ the inequality with biquadratic polynomial in its left-hand side:

$$4(3c_{33}-4c_{44})c_{44}(4k^4v^4-5c_{44}^2k^2v^2+c_{44}^4)>0.$$

The equation $4k^4v^4-5c_{44}^2k^2v^2+c_{44}^4=0$ has the roots $v_1^2=c_{44}^2/(4k^2)$, $v_2^2=c_{44}^2/k^2$. In the interval (v_1^2, v_2^2) the polynomial has a minimum at the point $v_{\min}^2=(5/8)c_{44}^2/k^2$ equal to $(-9/4)(3c_{33}-4c_{44})c_{44}^5$. This means that positive values of the polynomial are in the intervals $(0, v_1^2)$, (v_2^2, ∞). Because the gyration constant ought to be less than the stiffness, i.e., $k^2v^2 \ll c_{44}^2$, we choose for v^2 the interval $(0, v_1^2)$. Hence, again we can write for v^2 the former inequality (11): $k^2v^2<1/4c_{44}^2$.

3.3. Inequalities for symmetry group ∞

Nonzero are nine components of tensor $\boldsymbol{c}$:

$$\begin{aligned} &c_{11}=c_{22}, \quad c_{33}, \quad c_{44}=c_{55}, \quad c_{66} \\ &c_{12}=c_{11}-2c_{66}, \quad c_{13}, \quad c_{23}=c_{13}, \end{aligned} \tag{12}$$

and their isomers $c_{21}=c_{12}$, $c_{31}=c_{13}$, $c_{32}=c_{23}$. Independent components are five. As in the case of group symmetry $\infty\infty$, the rest, 24 components out of 36, are equal to zero.

Nonzero components of tensor $\boldsymbol{b}$ are

$$b_{11123}=\alpha_2,\ b_{22123}=-\alpha_2,\ b_{13332}=-\beta_2,$$
$$b_{23331}=\beta_2,\ b_{13233}=-\gamma_2,\ b_{12131}=-\delta_2,$$
$$b_{12232}=\delta_2,\ b_{11231}=\eta_2,\ b_{22132}=-\eta_2,$$
$$b_{11132}=2\delta_2-\eta_2,\ b_{22231}=\eta_2-2\delta_2,$$
$$b_{11333}=\alpha_m,\ b_{22323}=\alpha_m,\ b_{13331}=-\beta_m,$$
$$b_{23332}=-\beta_m,\ b_{12132}=-\delta_m,\ b_{12231}=-\delta_m,$$
$$b_{11232}=\eta_m,\ b_{22131}=\eta_m,\ b_{11131}=\eta_m-2\delta_m,$$
$$b_{22232}=\eta_m-2\delta_m$$

and isomers in accordance with the symmetry properties (Eq. (5)) of tensor $\boldsymbol{a}$ and, hence, $\boldsymbol{b}$. The above components of the $\boldsymbol{b}$ tensor are valid for the point group symmetry 6 (Kumaraswamy and Krishnamurthy, 1980). According to Hermann's theorem, they are also valid for the fifth rank $\boldsymbol{b}$ tensor of the symmetry group ∞.

In matrix presentation the above expressions are written as follows

$$\begin{aligned}
&b_{163}=\alpha_2 && b_{263}=-\alpha_2 \\
&b_{532}=-\beta_2 && b_{431}=\beta_2 \\
&b_{543}=-\gamma_2 && \\
&b_{651}=-\delta_2 && b_{642}=\delta_2 \\
&b_{141}=\eta_2 && b_{252}=-\eta_2 \\
&b_{152}=2\delta_2-\eta_2 && b_{241}=\eta_2-2\delta_2 \\
&b_{133}=\alpha_m && b_{233}=\alpha_m \\
&b_{531}=-\beta_m && b_{432}=-\beta_m \\
&b_{652}=-\delta_m && b_{641}=-\delta_m \\
&b_{142}=\eta_m && b_{251}=\eta_m \\
&b_{151}=\eta_m-2\delta_m && b_{242}=\eta_m-2\delta_m.
\end{aligned} \tag{13}$$

Correspondingly, matrices (c_{pq}) and(b_{pqm}), $m=1$, 2, 3, appear as

$$\begin{vmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{vmatrix},$$

$c_{12}=c_{11}-2c_{66}$,

$$\begin{vmatrix} 0 & 0 & 0 & b_{141} & b_{151} & 0 \\ 0 & 0 & 0 & b_{241} & b_{251} & 0 \\ 0 & 0 & 0 & -b_{431} & -b_{531} & 0 \\ -b_{141} & -b_{241} & b_{431} & 0 & 0 & -b_{641} \\ -b_{151} & b_{521} & b_{531} & 0 & 0 & -b_{651} \\ 0 & 0 & 0 & b_{641} & b_{651} & 0 \end{vmatrix},$$

$$\begin{vmatrix} 0 & 0 & 0 & b_{142} & b_{152} & 0 \\ 0 & 0 & 0 & b_{242} & b_{252} & 0 \\ 0 & 0 & 0 & -b_{432} & -b_{532} & 0 \\ -b_{142} & -b_{242} & b_{432} & 0 & 0 & -b_{642} \\ -b_{152} & -b_{252} & b_{532} & 0 & 0 & -b_{652} \\ 0 & 0 & 0 & b_{642} & b_{652} & 0 \end{vmatrix},$$

$$\begin{vmatrix} 0 & 0 & b_{133} & 0 & 0 & b_{163} \\ 0 & 0 & b_{233} & 0 & 0 & b_{263} \\ -b_{133} & -b_{233} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -b_{543} & 0 \\ 0 & 0 & 0 & b_{543} & 0 & 0 \\ -b_{163} & -b_{263} & 0 & 0 & 0 & 0 \end{vmatrix}.$$

The components c_{pq} and b_{pqm} are equal to the constants from Eqs. (12) and (13).

Now let the matrices $\boldsymbol{c}$, $\boldsymbol{b}$ be substituted in conditions (7) and (8) for the positiveness of the potential energy of the medium to be considered. Unlike the case of the medium $\infty\infty$, relations between stiffnesses c_{ij} result only from the determinants of orders 1, 2. They are

$$c_{11}>0,\ c_{11}^2-c_{12}^2>0.$$

The determinants of the orders 3,. . .,6 relate stiffnesses and gyration constants.

The requirement for the determinant of *the third order* to be positive gives in the case $\boldsymbol{n}$=(0, 0, 1) the inequality

$$4c_{66}[-k^2\alpha_m^2-c_{13}^2+c_{33}(c_{11}-c_{66})]>0,$$

from which it follows the inequality for the gyration constant α_m:

$$k^2\alpha_m^2<c_{33}(c_{11}-c_{66})-c_{13}^2. \tag{14}$$

The inequalities for the determinant of *the fourth order* provide the relations between the gyration constants β_2, δ_2, η_2, β_m, δ_m, η_m, α_m and stiffnesses c_{ij}.

If $\boldsymbol{n}$=(1, 0, 0) then

$$\begin{aligned}4[&k^2\beta_2^2c_{66}(c_{66}-c_{11})+k^2\delta_2^2(c_{13}^2-c_{11}c_{33})-k^2\eta_2^2c_{33}c_{66}\\&+2k^2\beta_2\delta_2c_{13}c_{66}-2k^2\beta_2\eta_2c_{13}c_{66}+2k^2\delta_2\eta_2c_{33}c_{66}\\&+c_{44}c_{66}(c_{11}c_{33}-c_{33}c_{66}-c_{13}^2)]>0.\end{aligned} \tag{15}$$

If $\boldsymbol{n}$=(0, 1, 0), the inequality looks like in the case $\boldsymbol{n}$=(1, 0, 0) with replacement β_2, δ_2, η_2 by $-\beta_m$, δ_m, η_m:

$$4[k^2\beta_m^2c_{66}(c_{66}-c_{11})+k^2\delta_m^2(c_{13}^2-c_{11}c_{33})-k^2\eta_m^2c_{33}c_{66} -2k^2\beta_m\delta_mc_{13}c_{66}+2k^2\beta_m\eta_mc_{13}c_{66}+2k^2\delta_m\eta_mc_{33}c_{66} + c_{44}c_{66}(c_{11}c_{33}-c_{33}c_{66}-c_{13}^2)] > 0. \quad (16)$$

The left-hand side of the inequality for the direction $\boldsymbol{n}$=(1, 0, 0) is a quadratic form with respect to variables $k\beta_2$, $k\delta_2$, $k\eta_2$; correspondingly, the left-hand side of the inequality for $\boldsymbol{n}$=(0, 1, 0) is a quadratic form with respect to variables $k\beta_m$, $k\delta_m$, $k\eta_m$.

For the direction of the symmetry axis $\boldsymbol{n}$=(0, 0, 1), the inequality for α_m arises which is the same as for the determinant of the third order.

Development of the determinant *of the fifth order* for $\boldsymbol{n}$=(1, 0, 0) and $\boldsymbol{n}$=(0, 1, 0) gives the following inequality:

$$\begin{aligned}4(&k^4\beta_m^2\delta_2^2c_{11}-2k^4\beta_2\beta_m\delta_2\delta_mc_{11}+4k^4\beta_2\beta_m\delta_2\delta_mc_{66}\\&+4k^4\beta_m\delta_2^2\delta_mc_{13}+k^4\beta_2^2\delta_m^2c_{11}-4k^4\beta_2\delta_2\delta_m^2c_{13}\\&+4k^4\delta_2^2\delta_m^2c_{33}-2k^4\beta_m^2\delta_2\eta_2c_{66}-2k^4\beta_2\beta_m\delta_m\eta_2c_{66}\\&-2k^4\beta_m\delta_2\delta_m\eta_2c_{13}+2k^4\beta_2\delta_m^2\eta_2c_{13}-4k^4\delta_2\delta_m^2\eta_2c_{33}\\&+k^4\beta_m^2\eta_2^2c_{66}+k^4\delta_m^2\eta_2^2c_{33}-2k^4\beta_2\beta_m\delta_2\eta_mc_{66}\\&-2k^4\beta_m\delta_2^2\eta_mc_{13}-2k^4\beta_2^2\delta_m\eta_mc_{66}+2k^4\beta_2\delta_2\delta_m\eta_mc_{13}\\&-4k^4\delta_2^2\delta_m\eta_mc_{33}+2k^4\beta_2\beta_m\eta_2\eta_mc_{66}\\&+2k^4\delta_2\delta_m\eta_2\eta_mc_{33}+k^4\beta_2^2\eta_m^2c_{66}+k^4\delta_2^2\eta_m^2c_{33}\\&-k^2\beta_2^2c_{11}c_{44}c_{66}-k^2\beta_m^2c_{11}c_{44}c_{66}+k^2\beta_2^2c_{44}c_{66}^2\\&+k^2\beta_m^2c_{44}c_{66}^2+2k^2\beta_2\delta_2c_{13}c_{44}c_{66}+k^2\delta_2^2c_{13}^2c_{44}\\&-k^2\delta_2^2c_{11}c_{33}c_{44}-2k^2\beta_m\delta_mc_{13}c_{44}c_{66}+k^2\delta_m^2c_{13}^2c_{44}\\&-k^2\delta_m^2c_{11}c_{33}c_{44}-2k^2\beta_2\eta_2c_{13}c_{44}c_{66}\\&+2k^2\delta_2\eta_2c_{33}c_{44}c_{66}-k^2\eta_2^2c_{33}c_{44}c_{66}\\&+2k^2\beta_m\eta_mc_{13}c_{44}c_{66}+2k^2\delta_m\eta_mc_{33}c_{44}c_{66}\\&-k^2\eta_m^2c_{33}c_{44}c_{66}-c_{13}^2c_{44}^2c_{66}+c_{11}c_{33}c_{44}^2c_{66}\\&-c_{33}c_{44}^2c_{66}^2)>0.\end{aligned} \quad (17)$$

For propagation along the symmetry axis $\boldsymbol{n}$=(0, 0, 1), the inequality is very simple:

$$4c_{66}[-k^2\alpha_m^2-c_{13}^2+c_{33}(c_{11}-c_{66})](c_{44}^2-k^2\gamma_2^2)>0. \quad (18)$$

The new inequality $c_{44}^2-k^2\gamma_2^2>0$ gives for the gyration constant γ_2 the limitation

$$k^2\gamma_2^2 < c_{44}^2. \quad (19)$$

Similar expressions appear in the result of developing the determinants of *the sixth order*. The inequalities for directions $\boldsymbol{n}$=(1, 0, 0) and $\boldsymbol{n}$=(0, 1, 0) are again the same, i.e.,

$$\begin{aligned}4\big(&k^4\beta_2^2\delta_2^2c_{11}c_{66}+k^4\beta_m^2\delta_2^2c_{11}c_{66}-k^4\beta_2^2\delta_2^2c_{66}^2\\&-2k^4\beta_2\delta_2^3c_{13}c_{66}-k^4\delta_2^4c_{13}^2+k^4\delta_2^4c_{11}c_{33}\\&+2k^4\beta_2\beta_m\delta_2\delta_mc_{66}^2+2k^4\beta_m\delta_2^2\delta_mc_{13}c_{66}\\&+k^4\beta_2^2\delta_m^2c_{11}c_{66}+k^4\beta_m^2\delta_m^2c_{11}c_{66}-k^4\beta_m^2\delta_m^2c_{66}^2\\&-2k^4\beta_2\delta_2\delta_m^2c_{13}c_{66}-2k^4\delta_2^2\delta_m^2c_{13}^2+2k^4\delta_2^2\delta_m^2c_{11}c_{33}\\&+2k^4\beta_m\delta_m^3c_{13}c_{66}-k^4\delta_m^4c_{13}^2+k^4\delta_m^4c_{11}c_{33}\\&-2k^4\beta_m^2\delta_2\eta_2c_{66}^2+2k^4\beta_2\delta_2^2\eta_2c_{13}c_{66}\\&-2k^4\delta_2^3\eta_2c_{33}c_{66}-2k^4\beta_2\beta_m\delta_m\eta_2c_{66}^2\\&+2k^4\beta_2\delta_m^2\eta_2c_{13}c_{66}-2k^4\delta_2\delta_m^2\eta_2c_{33}c_{66}\\&+k^4\beta_m^2\eta_2^2c_{66}^2+k^4\delta_2^2\eta_2^2c_{33}c_{66}+k^4\delta_m^2\eta_2^2c_{33}c_{66}\\&-2k^4\beta_2\beta_m\delta_2\eta_mc_{66}^2-2k^4\beta_m\delta_2^2\eta_mc_{13}c_{66}\\&-2k^4\beta_2^2\delta_m\eta_mc_{66}^2-2k^4\delta_2^2\delta_m\eta_mc_{33}c_{66}\\&-2k^4\beta_m\delta_m^2\eta_mc_{13}c_{66}-2k^4\delta_m^3\eta_mc_{33}c_{66}\\&+2k^4\beta_2\beta_m\eta_2\eta_mc_{66}^2+k^4\beta_2^2\eta_m^2c^2+k^4\delta_2^2\eta_m^2c_{33}c_{66}\\&+k^4\delta_m^2\eta_m^2c_{33}c_{66}-k^2\beta_2^2c_{11}c_{44}c_{66}^2-k^2\beta_m^2c_{11}c_{44}c_{66}^2\\&+k^2\beta_2^2c_{44}c_{66}^3+k^2\beta_m^2c_{44}c_{66}^3+2k^2\beta_2\delta_2c_{13}c_{44}c_{66}^2\\&+2k^2\delta_2^2c_{13}^2c_{44}c_{66}-2k^2\delta_2^2c_{11}c_{33}c_{44}c_{66}\\&+k^2\delta_2^2c_{33}c_{44}c_{66}^2-2k^2\beta_m\delta_mc_{13}c_{44}c_{66}^2\\&+2k^2\delta_m^2c_{13}^2c_{44}c_{66}-2k^2\delta_m^2c_{11}c_{33}c_{44}c_{66}\\&+k^2\delta_m^2c_{33}c_{44}c_{66}^2-2k^2\beta_2\eta_2c_{13}c_{44}c_{66}^2\\&+2k^2\delta_2\eta_2c_{33}c_{44}c_{66}^2-k^2\eta_2^2c_{33}c_{44}c_{66}^2\\&+2k^2\beta_m\eta_mc_{13}c_{44}c_{66}^2+2k^2\delta_m\eta_mc_{33}c_{44}c_{66}^2\\&-k^2\eta_m^2c_{33}c_{44}c_{66}^2-c_{13}^2c_{44}^2c_{66}^2+c_{11}c_{33}c_{44}^2c_{66}^2\\&-c_{33}c_{44}^2c_{66}^3\big)>0.\end{aligned} \quad (20)$$

The inequality for $\boldsymbol{n}$=(0, 0, 1) direction adds one more inequality to Eq. (18):

$$4[-k^2\alpha_m^2+c_{33}(c_{11}-c_{66})-c_{13}^2](-k^2\gamma_2^2+c_{44}^2) \times(-k^2\alpha_2^2+c_{66}^2)>0.$$

It contains the limitation for gyration constant α_2:

$$k^2\alpha_2^2 < c_{66}^2. \tag{21}$$

Thus, the limitations sought are given by the inequalities (14), (21) and (19) for gyration constants α_m, α_2, γ_2, (15) for the gyration constants β_2, δ_2, η_2, (16) for the gyration constants β_m, δ_m, η_m, and (17) and (20) for the gyration constants β_2, δ_2, η_2, β_m, δ_m, η_m.

4. Conclusions

The derived inequalities for relations between stiffnesses and gyration constants are supposed to provide a more appropriate way to carry out computations for anisotropic gyrotropic media than to choose these constants arbitrarily in the belief that gyration constants, by analogy with optics and acoustics, ought to be a hundred and a thousand times less than stiffnesses. The inequality for a gyrotropic medium with identical properties in all directions (symmetry group $\infty\infty$) and the inequality for $\boldsymbol{n}$=(0, 0, 1) direction in a gyrotropic medium of the symmetry group ∞ are simple. The inequalities for $\boldsymbol{n}$=(1, 0, 0) and $\boldsymbol{n}$=(0, 1, 0) directions in a medium of the symmetry group ∞ need to be investigated and, if possible, transformed.

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