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# Formation of Concentric Rings Around Sources[1]

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*Zones of increased concentration formed by a solvent flowing from a source are considered. A matehmatical model for forming such zones is proposed. It takes into account that such a zone is composed of a set of independent particles. Hence the distribution of a substance around the source can be explained by movement of an individual particle. In the model this movement is a continuous semi-Markov process with terminal stopping at some random point in space. Parameters of the process depend on the velocity field of the flow. Forward and backward partial differential equations for the distribution density of a random stopping point of the process are derived. The forward equation is investigated for the centrally symmetric case. Solutions of the equation demonstrate either a maximum or a local minimum at the source location. In the latter case a concentric ring around the source is formed. If different substances vary in their absorption rates, they can form separable concentration zones as a family of concentric rings.*

**KEY WORDS:** continuous semi-Markov process, absorption, terminal stopping, partial elliptical differential equation, concentration zone.

## INTRODUCTION

In Harlamov (1978), a model of accumulation of accessory minerals in sediment layers was proposed on the basis of the theory of diffusion Markov processes. In this work, an exponential stopping time independent of the process was considered. To obtain the distribution of the process at such a random time the above accumulation problem was reduced to Laplace transformation of the forward Kolmogorov equation. Examples of diffusion processes with terminal stopping considered in this work demonstrate increased concentration zones at points where either shift or diffusion coefficients are discontinuous. Moreover, Harlamov (2000b) proposed a simple model for formation of zones of increased concentration in the neighborhood of a source on basis of the one-dimensional semi-Markov theory. In this model, a particle of the substance moves along a straight line. The law of this movement is a monotone continuous semi-Markov process. It is well known that trajectories of such a process can have finite intervals of constancy and a terminal

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stopping time. Finite intervals of constancy are not essential from a geological point of view. On the other hand the infinite interval begun at the terminal stopping time determines the permanent position of a particle on the line. If the line is a ray in some centrally symmetric field, the set of these positions form a zone of increased concentration around the source.

In what follows, the term "stopping" refers to the terminal stopping, i.e., the infinite interval of constancy. It can be interpreted as sedimentation of the particle due to physical or chemical interaction with the substance of the filter.

In the present paper we propose a more general semi-Markov model of accumulation than that of Harlamov (2000b). We investigate two types of processes with smooth and diffusive character of transportation. Such processes are interesting from a geological point of view (see Korzhinskii, 1982). On the geological timescale the observed distribution of substance can be interpreted as a limit distribution of the intensity measure, i.e., an expectation of some random measure (see Harlamov, 1978).

## SEMI-MARKOV PROCESSES WITH STOPPING

A random process in a metric space of states $X$ and with trajectories $\xi : \xi(t) \in X$ $(t \geq 0)$ is said to be semi-Markov if it has the Markov property with respect to the first exit time $\sigma_\Delta$ from any open set $\Delta \subset X$, where $\sigma_\Delta(\xi) = \inf\{t : \xi(t) \notin \Delta\}$. This process can be determined by a consistent family of probability measures $(P_x)$ on the set of all functions $\xi$ continuous from the right and having limits from the left (Skorokhod space $D$). Let $\pi_t(\xi)$ be the value of the process at time $t \geq 0$. We say that the subprobability distribution depending on the point $x \in X$ of the form

$$F_\tau(A \times S|x) = P_x(\tau \in A, \pi_\tau \in S) \quad (A \subset R_+, S \subset X), \tag{1}$$

$(F_\tau(R_+ \times X|x) \leq 1)$is the semi-Markov transition function corresponding to the Markov time $\tau$. The family $(P_x)$ is uniquely determined by the set of all semi-Markov transition functions $(F_\tau)$ or by the set of all semi-Markov transition generating functions $(f_\tau)$. The latter is the Laplace transform of $F_\tau$ in the first argument:

$$f_\tau(\lambda, S|x) = \int_o^\infty e^{-\lambda t} F_\tau(dt \times S|x) \equiv P_x(e^{-\lambda\tau}; \pi_\tau \in S, \tau < \infty). \tag{2}$$

To construct the semi-Markov process it is sufficient to use these families for all $\tau = \sigma_\Delta$, where $\Delta$ is an open set.

Here and in what follows we use denotation $\mu(\varphi; S)$ for the integral of the function $\varphi(x)I_S(x)$ with respect to the measure $\mu(dx)$, omitting the first argument if $\varphi = 1$, and omitting the second argument if $S$ is the whole space on which the measure is determined.

We are interested in continuous processes stopping permanently at a random time

$$\varsigma = \varsigma(\xi) = \inf\{t \geq 0 : \zeta = \zeta(\xi) = \inf\{t \geq 0 : \text{ such that for all } \\ s \geq t \quad \xi(s) = \xi(t)\}, \tag{3}$$

i.e., the time when an infinite interval of constancy begins. Generally this time depends on the future, i.e., the event $\{\varsigma \leq t\}$ does not only depend on values of the trajectory before the time $t$. In other words this moment is not a "stopping time" or "Markov time" in the sense of the theory of stochastic processes. However, for semi-Markov processes we can express the distribution of this time and the value of the process on the infinite interval of constancy in terms of semi-Markov transition functions. It can be found as a limit of step functions associated with a continuous function. Let $o_r$ be the first exit time from the open spherical neighborhood (of radius $r$) of the initial point of the trajectory; $o_r^n$ be the sequence of iterated times $\sigma_r^1 = \sigma_r$; $\sigma_r^{n+1} = \sigma_r^n + \sigma_r \circ \theta_{\sigma_r^n}$, $(n \geq 1)$, where $\theta_t$ is a shift operator: $\theta_t(\xi)(s) = \xi(t+s)$; $L_r\xi$ be the step function corresponding to this sequence: $L_r(\xi)(t) = \pi_{\sigma_r^n}\xi$ if and only if $\sigma_r^n(\xi) \leq t < \sigma_r^{n+1}(\xi)$. Evidently $\varsigma(L_r(\xi)) \leq \varsigma(\xi)$ and $\varsigma(L_r(\xi)) \to \varsigma(\xi)$ as $r \to 0$. Also $\pi(\varsigma(L_r(\xi))) \to \pi(\varsigma(\xi))$. This follows from the definition of continuity if $\varsigma$ is a point of continuity of the function $\xi$. But it is true even if $\varsigma$ is a point of discontinuity of the function $\xi$ since in this situation there exists $r_0$ such that $\varsigma(\xi) = \varsigma(L_r(\xi))$ for all $r < r_0$. Denote

$$F_\varsigma(A \times S|x) = P_x(\varsigma \in A, \pi_\varsigma \in S), \quad H_\varsigma(S|x) = F_\varsigma(R_+ \times S|x). \tag{4}$$

So the measure $H_\varsigma(S|x)$ is a weak limit of the sequence of measures $H_\varsigma^r(S|x) \equiv P_x(\varsigma(L_r\xi) < \infty, \pi(\varsigma(L_r\xi)) \in S)$ as $r \to 0$. Let us find the distribution of the couple $(\varsigma(L_r), \pi(\varsigma(L_r)))$:

$$\begin{aligned} P_x(\varsigma(L_r) < t, \pi(\varsigma(L_r)) \in S) &= \sum_{n=0}^{\infty} P_x(\sigma_r^n < t, \sigma_r \circ \theta_{\sigma_r^n} = \infty, \pi(\sigma_r^n) \in S) \\ &= \sum_{n=0}^{\infty} \int_S F_{\sigma_r^n}([0,t) \times dx_1|x)(1 - H_r(X|x_1)) \\ &= \int_S U_r([0,t) \times dx_1|x)(1 - H_r(X|x_1)), \end{aligned} \tag{5}$$

where $H_r(S|x) = F_{\sigma_r}(R_+ \times S|x)$ and $U_r([0,t) \times S|x) = \sum_{n=0}^{\infty} F_{\sigma_r^n}([0,t) \times S|x)$.

This is a measure on $R_+ \times X$ that expresses the intensity measure of the locally finite integer-valued measure $N([0,t\} \times S)$, counting points of the two-dimensional marked point process $(\sigma_r^n, \pi(\sigma_r^n))\quad (0 \le n < \infty)$ belonging to the set $[0, t) \times S$. In particular

$$H_\varsigma^r(S|x) = \int_S Z_r(dx_1|x)(1 - H_r(X|x)), \tag{6}$$

where $ZZ_r(S|x) = U_r(R_+ \times S|x)$. Generally the measure under the integral sign in (6) tends to infinity and the integrated function tends to zero as $r \to 0$. In some cases it is possible to find asymptotics of these functions.

Note that the right-hand side of formula (6) determines the distribution of the limit point of the trajectory in both cases: $\varsigma < \infty$ and $\varsigma = \infty$. The latter case is if $\lim \xi(t)$ (as $t \to \infty$) almost certainly exists $P_x$. In what follows we suppose $P_x(\varsigma < \infty) = 1$. Then the limit point is always a stopping point and for all $x$ the measure $H_\varsigma(S|x)$ is a probability measure. It is not difficult to give and to justify sufficient conditions for this property to hold.

## DIFFUSION AND SMOOTH TYPES OF PARTICLE MOVEMENT

We consider two basic subclasses of semi-Markov processes: diffusion and smooth types. For a time homogeneous process its distribution “on the whole” is determined by the set of all “local” distributions of processes that begin from different points of the state space, i.e., all “germs” of processes considered from the initial time until the first exit time from a small neighborhood of its initial point. A difference between diffusion and smooth semi-Markov process types becomes apparent when analyzing distributions of the first exit point from a small neighborhood of the initial point. The one-dimensional continuous process is the simplest case. Let $a < x < b$ and

$$g_{(a,b)}(x) = P_x(\sigma_{(a,b)} < \infty, \pi_{\sigma_{(a,b)}} = a), \qquad h_{(a,b)}(x) = P_x(\sigma_{(a,b)} < \infty, \pi_{\sigma_{(a,b)}} = b). \tag{7}$$

For a diffusion process we have

$$g_{(x-r,x+r)}(x) = 1/2 - b(x)r - (1/2)c(x)r^2 + o(r^2), \tag{8}$$

$$h_{(x-r,x+r)}(x) = 1/2 + b(x)r - (1/2)c(x)r^2 + o(r^2), \tag{9}$$

where $b(x)$ is the shift parameter determining tendency of displacement to the right (if $b(x)$ is positive) or to the left (if $b(x)$ is negative); $c(x) > 0$ is the parameter of stopping. In this connection

$$P_x(\varsigma < \sigma_{(a,b)}) \le P_x(\sigma_{(a,b)} = \infty) \tag{10}$$

and

$$P_x(\sigma_{(a,b)} < \infty) = g_{(a,b)}(x) + h_{(a,b)}(x) = 1 - c(x)r^2 + o(r^2). \tag{11}$$

For a smooth process we have

$$g_{(x-r,x+r)}(x) = 1 - c(x)r + o(r), \qquad h_{(x-r,x+r)}(x) = o(r), \tag{12}$$

or

$$h_{(x-r,x+r)}(x) = 1 - c(x)r + o(r), \qquad g_{(x-r,x+r)}(x) = o(r). \tag{13}$$

The parameter $c(x)$ also determines stopping of movement, although in a different way than in the diffusion case. For $d$-dimensional space ($d \ge 0$) the difference between diffusion and nondiffusion characters of movement in principle is as shown above. For a diffusion process starting at point $x$ there exists a nondegenerate linear mapping of the space $R^d$ on itself that preserves motionless the point $x$ such that in the new space the distribution of the first exit point from a small ball with center $x$ is uniform on the surface of the ball (in zero approach). In the first and second approach shift and stopping members are added to the uniform member. For a nondiffusion process the distribution of the first exit point from a small ball with center $x$ is concentrated (in zero approach) on intersection of the surface of the ball with some hyperplane. For a smooth process this “hyperplane” is a line and the first exit point in zero approach is unique. The point of exit determines the unique trace of the process going across the point $x$. In the first approach a member determining stopping inside the ball is added. Note that in Harlamov (2000a) another definition of the diffusion process was proposed. This is a process such that its semi-Markov transition generating function (as a function of an initial point) satisfies a partial differential equation of the second order of elliptical type with corresponding boundary conditions, on boundaries of corresponding regions. Practically in Harlamov (2000a) equivalence of the two definitions of diffusion process was established: on the basis of a differential equation and on the basis of distribution of the first exit point from a small neighborhood of the initial point.

## ACCUMULATION EQUATIONS FOR DIFFUSION-TYPE PROCESSES

Let us consider a continuous semi-Markov process of diffusion type and its “accumulation kernel” $H_\varsigma(S|x)$ the probability measure depending on initial point of the process. We are going to derive the forward and backward accumulation differential equations with respect to the output ($S$) and input ($x$) arguments correspondingly. Let us begin with derivation of the backward equation. As shown in Harlamov (2000a) for any twice differentiable function $\varphi$ there exists the limit

$$\begin{aligned} a_0(\varphi|x) &\equiv \lim_{r\to 0}\frac{1}{r^2}(H_r(\varphi|x)-\varphi(x)) \\ &= \frac{1}{2}\sum_{ij} a^{ij}(x)\varphi''_{ij}(x)+\sum_{i=1}^{d} b^i(x)\varphi'_i(x)-c(x)\varphi(x), \end{aligned} \tag{14}$$

where $d \geq 1$ is the dimension of the Euclidean space $X$; $(a^{ij}(x))$ is some symmetric positive definite matrix of coefficients, depending on a point of the state space and also the trace of this matrix (the sum of its diagonal members) is equal to unity; $(b^i(x))(1 \leq i \leq d)$is some vector field of coefficients; $c(x)$is some positive function; $\varphi'_i$, $\varphi''_{ij}$ are partial derivatives of $\varphi$ by coordinates identified as $i$ and $j$ respectively. Let us consider the function $L^R\xi$ for the arbitrary function $\xi \in \boldsymbol{D}$ and $\mathrm{R} > 0$ such that $L^R\xi(\mathrm{t}) = \xi(0)$ if $t < \sigma_R(\xi)$, and $L^R\xi(t) = \xi(t)$ if $t \geq \sigma_R(\xi)$. Let

$$H^R_\varsigma(S|x) = P_x(\pi_\varsigma \circ L^R \in S, \varsigma \circ L^R < \infty). \tag{15}$$

Then

$$H^R_\varsigma(\varphi|x) = \varphi(x)(1-H_R(X|x)) + \int_X H_R(dy|x)H_\varsigma(\varphi|y). \tag{16}$$

Let us subtract $H_\varsigma(\varphi|x)$ from both sides and divide them by $R^2$. Then, under suppositions that $H_\varsigma(\varphi|x)$ is continuous and twice differentiable (this condition can be justified) and $R \to 0$, we obtain on the right-hand side

$$\frac{1}{2d}\varphi(x)c(x) + \frac{1}{2d}a_0(H_\varsigma(\varphi)|x) \tag{17}$$

and on the left the limit of the expression

$$\begin{aligned} &\big(H^R_\varsigma(\varphi|x) - H_\varsigma(\varphi|x)\big)/R^2 = \frac{1}{R^2}P_x(\varphi(x)-\varphi(\pi_\varsigma); \varsigma < \sigma_R) \\ &\leq \max\{|\varphi(x)-\varphi(x_1)| : |x-x_1| \leq R\}\frac{1}{R^2}(1-H_R(X|x)), \end{aligned} \tag{18}$$

that is equal to zero. This yields that $H_\varsigma(\varphi) \equiv H_\varsigma(\varphi|.)$ satisfies the equation

$$\frac{1}{2}\sum_{ij} a^{ij}(H_\varsigma(\varphi))''_{ij} \quad + \sum_{i=1}^{d} b^i(H_\varsigma(\varphi))'_i - cH_\varsigma(\varphi) + c\varphi = 0. \tag{19}$$

This equation is said to be *backward accumulation equation*. The solution we are interested in tends to zero at infinity. Uniqueness of this solution follows from the maximum principle. Let us derive the forward accumulation equation. In this case instead of the kernel $H_\varsigma(S|x)$ it is more convenient to analyze its average with respect to some measure. Let $\mu(x)$ be a probability distribution density on $X$ and

$$H_\varsigma(S|\mu) = \int_X H_\varsigma(S|x)\mu(x)dx. \tag{20}$$

The average kernels $H^r_\varsigma(S|\mu)$ and $Z_r(S|\mu)$ and the measure $P_\mu$ have similar sense. According to eq. (6) we have

$$H^r_\varsigma(\varphi|\mu) = \int_X Z_r(dx|\mu)\varphi(x)(1 - H_r(X|x)). \tag{21}$$

For any continuous function $\varphi$ the left side of this equation tends to a limit $H_\varsigma(\varphi|\mu)$. On the right side we have $(1 - H_r(X|x))/r^2 \to c(x)$. Assume the function $c(x)$ to be continuous. Then as $r \to 0$ there exists a weak limit $W(S|\mu)$ of the measure $r^2 Z_r(S|\mu)$, and consequently

$$H_\varsigma(\varphi|\mu) = \int_X W(dx|\mu)\varphi(x)c(x). \tag{22}$$

Assume that there exists the density $h_\varsigma(x|\mu)$ of the measure $H_\varsigma(S|\mu)$. Then there exists the density $w(x|\mu)$ of the measure $W(S|\mu)$ and in this case $h_\varsigma(x|\mu) = w(x|\mu)c(x)$. According to definition of the function $Z_r$ (see (1)) we have

$$\begin{aligned} Z_r(\varphi|\mu) &= \sum_{k=0}^{\infty} P_\mu(\varphi(\pi_{\sigma_r^k}); \sigma_r^k < \infty) \\ &= \varphi(\mu) + \sum_{k=1}^{\infty} P_\mu(\varphi(\pi_{\sigma_r}) \circ \theta_{\sigma_r^{k-1}}; \sigma_r^{k-1} < \infty, \sigma_r \circ \theta_{\sigma_r^{k-1}} < \infty) \end{aligned}$$

$$= \varphi(\mu) + \sum_{k=1}^{\infty} P_\mu(P_{\sigma_r^{k-1}}(\varphi(\pi_{\sigma_r}); \sigma_r < \infty); \quad \sigma_r^{k-1} < \infty)$$

$$= \varphi(\mu) + \int_X Z_r(dx|\mu) P_x(\varphi(\pi_{\sigma_r}); \; \sigma_r < \infty), \tag{23}$$

where $\varphi(\mu) = \int_X \mu(x)\varphi(x)dx$. We get the equation

$$\varphi(\mu) + \int_X Z_r(dx|\mu) \left( P_x(\varphi(\pi_{\sigma_r}); \sigma_r < \infty) - \varphi(x) \right) = 0. \tag{24}$$

Suppose the function $a_0(\varphi|x)$ is continuous on $X$. Then multiplying the measure $Z_r$ by $r^2$, dividing the integrated difference into $r^2$, and going to the limit as $r \to 0$, we obtain the equation

$$\varphi(\mu) + \int_X W(dx|\mu) a_0(\varphi|x) = 0. \tag{25}$$

The integral in this expression can be represented in the form

$$\int_X W(dx|\mu) \leq \left( \frac{1}{2} \sum_{ij} a^{ij}(x)\varphi''_{ij}(x) + \sum_{i=1}^{d} b^i(x)\varphi'_i(x) - c(x)\varphi(x) \right) dx = 0. \tag{26}$$

Assume that the function $\varphi$ and all its first-order partial derivatives tend to zero at infinity. Then using integration by parts, we obtain another representation of the integral

$$\int_X \varphi(x) \leq \left( \frac{1}{2} \sum_{ij} (a^{ij}(x)w(x|\mu))''_{ij} - \sum_{i=1}^{d} (b^i(x)w(x|\mu))'_i \right.$$
$$\left. - c(x)w(x|\mu) \right) dx. \tag{27}$$

Choosing an arbitrary function $\varphi$, we imply that where $w(\mu) = w(\cdot|\mu)$ and $w(x|\mu) = \int_X w(x|x_1)\mu(x_1)dx_1$. We obtain *the forward accumulation equation* if we substitute $w(x|\mu) = h_\varsigma(x|\mu)c(x)$ in the latter equation. Finally we obtain

$$\frac{1}{2} \sum_{ij} (h_\varsigma(\mu)a^{ij}/c)''_{ij} - \sum_{i=1}^{d} (h_\varsigma(\mu)b^i/c)'_i - h_\varsigma(\mu) + \mu = 0. \tag{28}$$

The only restriction is that for any initial probability density $\mu$ the solution $h_\varsigma(\mu)$ we are interested in is also a probability density, i.e., nonnegative, integrable and its integral equals unity. Such a solution is unique. In differential equation theory Eq. (28) is called conjugate to Eq. (19).

Note that in the theory of Markov processes with break (the break time $\varsigma$ is identified with the first entrance time to infinity) the kernel $H_\varsigma(S|x)$ is interpreted as the distribution of the point of the process just before the break (depending on the initial point). Therefore Eqs. (19) and (28) can be derived by methods of diffusion Markov processes. We demonstrate a "semi-Markov method" of deducing them on the basis of semi-Markov interpretation.

## ACCUMULATION EQUATIONS FOR SMOOTH-TYPE PROCESSES

Semi-Markov processes of smooth type can be defined by the following property of distribution of the first exit point from a small ball neighborhood of the initial point of the process:

$$H_r(\varphi|x) = \varphi(x + r\overline{b})(1 - c(x)r + o(r)) \qquad (r \to 0), \tag{29}$$

where $\varphi$ is a continuous function; $\overline{b} = \overline{b}(x) = (b^i(x))$ is a point on the surface of the unit sphere. The family $(\overline{b}(x)) \quad (x \in X)$ determines a field of directions in the space $X$. We suppose the vector function $\overline{b}(x)$ to be continuous and the function $c(x)$ to be continuos and positive. Under such assumptions for all $\varphi$, there exists the limit

$$\alpha_0(\varphi|x) \equiv \lim_{r \to 0} \frac{1}{r}(H_r(\varphi|x) - \varphi(x)) = \sum_{i=1}^{d} b^i(x)\varphi_i'(x) - c(x)\varphi(x). \tag{30}$$

To derive the backward accumulation equation, we use Eq. (16) with another normalizing factor. From this follows the identity $\alpha_0(H_\varsigma(\varphi|x) + c(x)\varphi(x) = 0$ that implies the equation

$$\sum_{i=1}^{d} b^i(H_\varsigma(\varphi))_i' - cH_\varsigma(\varphi) + c\varphi = 0. \tag{31}$$

This is *the backward accumulation equation* for processes of smooth type. We are interested in bounded nonnegative solutions tending to zero at infinity if the parametric function is the same. In this connection $\max H_\varsigma(\varphi) \le \max \varphi$.

To derive the forward accumulation equation, we use Eq. (24). In this case the measure $r\,Z_r(S|\mu)$ tends weakly to some finite measure $Y(S|\mu)$. Its density

$y(x|\mu)$ (if it exists) is connected with the density of the measure $H_\varsigma(S|\mu)$ by the equality: $h_\varsigma(x|\mu) = y(x|\mu)c(x)$. From the identity (4), we obtain the equation

$$\varphi(\mu) + \int_X y(x|\mu)\alpha_0(\varphi|x)dx = 0. \tag{32}$$

Substituting the value of the operator $\alpha_0$ in the left part and integrating by parts, we take the identity

$$\int_X \varphi(x)\left( -\sum_{i=1}^{d} (y(x|\mu)b^i(x)\right)_i^{\prime} - c(x)y(x|\mu) + \mu(x))dx = 0, \tag{33}$$

from which we obtain the equation

$$-\sum_{i=1}^{d} (y(\mu)b^i)_i^{\prime} - cy(\mu) + \mu = 0. \tag{34}$$

Replacing the function $y$, we get *the forward accumulation equation* for processes of smooth type:

$$\sum_{i=1}^{d} (h_\varsigma(\mu)b^i/c)_i^{\prime} + h_\varsigma(\mu) - \mu = 0. \tag{35}$$

The required solution is a probability density if the function $\mu$ is the same.

## EQUATIONS WITH REGARD TO CENTRAL SYMMETRY

Backward accumulation equations relate to a problem of reconstruction of the initial distribution of matter on the basis of a defined result of its transport. In this paper we are not going to solve this problem. In what follows we investigate forward accumulation equations with rather simple fields of coefficients. They answer the question of how the matter around a source is distributed.

Assume the set of coefficients of the differential equations to satisfy the principle of central symmetry with respect to the origin of the coordinates. We consider a particular case of such a set of coefficients, namely

$$a^{ij}(x) = \frac{a(r)}{d}\delta^{ij}, \quad b^i(x) = b(r)\frac{x^i}{r}, \quad c = c(r), \tag{36}$$

where $x^i$ is the $i$-th coordinate of the vector $x$; $r = \sqrt{\sum_{i=1}^{d}(x^i)^2}$ is the length of $x$; $\delta^{ij}$ is the Kronecker symbol ($\delta^{ij} = 0$ if $i \neq j$ and $\delta^{ii} = 1$); $a(r)$, $b(r)$, $c(r)$ are continuous positive functions of $r \geq 0$ (for a smooth-type process $b(r) = 1$). Hence the vector field at any point but the origin is directed along the ray going from the origin through this point (*radial vector field*). Such a field represents an idealized picture of velocity directions inside a laminar liquid stream flowing from a point source in $d$-dimensional space ($d \geq 1$).

Let us consider the forward accumulation equation for a process of Diffusion type with the function $\mu$ of degenerate form. Let $\mu$ be the Unit loading at the origin of coordinates that we denote as $\overline{0}$. Hence $\int_X \varphi(x)\mu(x)\,dx = \varphi(\overline{0})$ where $\varphi$ is any continuous bounded function. Evidently in this case the density $h = h_\varsigma(\cdot|\mu)$ represents a centrally symmetric function $h = h(r)$. We have

$$(ha^{ij}/c)'_i = \frac{1}{d}(h/c)'\frac{\partial r}{\partial x^i} = \frac{1}{d}(h/c)'\frac{x^i}{r}, \tag{37}$$

$$(ha^{ij}/c)''_{ij} = \frac{1}{d}\frac{(h/c)'}{r^2} + \frac{1}{d}\left(\frac{(h/c)''}{r} - \frac{(h/c)'}{r^2}\right)\frac{(x^i)^2}{r}, \tag{38}$$

$$hx^i b/(rc))'_i = \frac{hb/c}{r} + \left(\frac{(hb/c)'}{r} - \frac{(hb/c)}{r^2}\right)\frac{(x^i)^2}{r}. \tag{39}$$

Here and in what follows denotations $f'$, $f''$ (without fixing arguments the functions are differentiated by) relate to derivatives of $f$ by $r$. Therefore we obtain

$$\begin{aligned}
&\sum_{ij}(ha^{ij}/c)''_{ij}\sum_{i=1}^{d}(ha^{ii}/c)''_{ii}\\
&\quad = \frac{(h/c)'}{r} + \frac{1}{d} \leq \left(\frac{(h/c)''}{r} - \frac{(h/c)'}{r^2}\right)r = \frac{1}{d}(h/c)'' + \frac{d-1}{d}\frac{(h/c)'}{r},
\end{aligned} \tag{40}$$

$$\begin{aligned}
&\sum_{i=1}^{d}(hx^i b/(rc))'_i = dhb/(rc) + \leq \left(\frac{(hb/c)'}{r} - \frac{(hb/c)}{r^2}\right)r\\
&\quad = (hb/c)' + (d-1)\frac{hb/c}{r}.
\end{aligned} \tag{41}$$

Equation (5) is being transformed into the equation

$$\frac{1}{2d}(h/c)'' + \frac{d-1}{2d}\frac{(h/c)'}{r} - (hb/c)' - (d-1)\frac{hb/c}{r} - h = 0 \qquad (r > 0). \tag{42}$$

Correspondingly Eq. (35) for a smooth process is transformed into the equation

$$(h/c)' + (d-1)\frac{h/c}{r} + h = 0 \qquad (r > 0). \tag{43}$$

The unit loading at the origin of the coordinates affects properties of solutions of Eqs. (42) and (43). Under our suppositions, Eq. (28) can be represented in the form $\nabla^2(ha/c)-$ div $(h\overline{b}/c) - h + \mu = 0$ where $\nabla^2 u$ is the Laplacian applied to the function $u$; div $\overline{v}$ is the divergence of the vector field $\overline{v}$; $\overline{b}$ is the vector field with coordinates $b^i$. Let us integrate all the members of this equation over a small ball neighborhood (of radius $R$) of the origin of coordinates. Under our supposition the integral of the last term is equal to unit for any $R > 0$. The integral of the third term tends to zero (if $h$ is bounded or tends to infinity not very quickly as its argument tends to $\overline{0}$). The first and second terms remain. From the theory of differential equations it follows that the integral of the divergence of the given vector field over a ball is equal to the integral of the function $hb/c$ over the surface of the ball, i.e., it is equal to $R^{d-1}\omega_d h(R)b(R)/c(R)$, where $\omega_d$ is the area of the unit ball surface in $d$-dimensional space. The integral of $\nabla^2(h/c)$ over the ball is equal to the integral of the function $-(h/c)'$ over the surface of the ball, i.e., it is equal to $-R^{d-1}\omega_d(h/c)'(R)$. Therefore the following condition must hold

$$R^{d-1}\omega_d(h/c)'(R) + R^{d-1}\omega_d h(R)b(R)/c(R) \to 1. \tag{44}$$

It is true if the function $b$ is bounded in a neighborhood of the origin and the function $h/c$ has a pole in the point $\overline{0}$ of the corresponding order. Namely

$$h(r)/c(r) \sim \begin{cases} (-\log r)/(2\pi), & d = 2 \\ 1/(r^{d-2}(d-2)\omega_d), & d \geq 3. \end{cases} \tag{45}$$

Note that near such a pole the function tends to infinity not very quickly. In this case the integral of the divergence tends to zero as $R \to 0$. Thus the desired centrally symmetric solution of the forward accumulation equation must be of order (45) and consequently it depends on the rate of $c(r)$ at the zero point. If the function $b$ is not bounded in a neighborhood of zero, it can happen that the second term of the equation determines the order of the solution. This term is determining for Eq. (43) when the second derivatives are absent. To assign the order of solutions at the origin means in fact to assign initial conditions for Eqs. (42) and (43). However it is not convenient to use these conditions for drawing graphs of solutions because of their instability. We seek positive and integrable solutions on the positive semiaxis:

$$\int_0^\infty h(r) r^{d-1} dr < \infty, \tag{46}$$

and therefore, $h(r) \to 0 \quad (r \to \infty)$.
Also, under "time inversion," solutions of the equations become stable. Hence one can find solutions of Eqs. (8) and (9) with the help of the computer on any finite interval $(\varepsilon, T)\,(0 < \varepsilon < T)$ replacing $r \mapsto T - r$. Below we bring some examples of choice of parameters for Eqs. (8) and (9), having physical interpretation, and also their solutions in analytical (if possible) or graphical form (by computer).

## CHOICE OF COEFFICIENTS

Although solutions of accumulation equations depend on the ratio $a^{ij}/c$ and $b^i/c$, the interpretation of these ratios is more natural if one considers the fields $(a^{ij})$, $(b^i)$, and $c$ separately because each of them has specific physical sense. They may be mutually dependent.

In nature, generally liquids and gases play the role of carrier for a particle of matter. As a rule, liquids determine a smooth type of a process, and gases a diffusion type. We take the simplest supposition with respect to diffusion. We consider it to be constant over all space. The reason for such a choice follows from the interpretation of diffusion as a result of chaotic movement of molecules in the stable thermal field. It is possible to give other interpretations of diffusion, for example, turbulence. However taking into account turbulence would change our model and the form of related differential equations. In the present paper we consider only laminar streams of liquids and gases.

In the diffusion case, the velocity $\overline{v}$ of the carrier matter determines the tendency of movement of a particle but not its actual displacement. In this case we take the field of coefficients $\overline{b}$ to be equal to the field of velocities. In the smooth case the velocity field plays the main role in transportation of a particle. In this case we suppose $\overline{b} = \overline{v}/|\overline{v}|$. It means these fields coincide in directions but differ in values of vectors. In both types of processes velocity values can affect the coefficient of absorption $c$. In the centrally symmetric case we suppose the velocity field is as follows $\overline{v}(x) = v(r)x/r$, where $v(r)$ depends on the dimensionality of the space and on loss of carrier matter during transportation of the particle. The choice is based on the principle of mass balance under stationary activity of the point source.

For incompressible liquid and without loss of liquid from the system (for example, by change of state) the mass balance equation has the form:

$$\operatorname{div} \overline{v} = 0, \tag{47}$$

hence $v' = -(d-1)v/r$ and consequently, $v(r) = v(1)/r^{d-1}$. If the incompressible liquid flows on a plane and leaves the system as vapor with the rate $v_1$ per unit square the velocity on the surface decreases faster. The balance of carrier matter has the form: $v_0 = 2\pi r\, v(r) + v_1 \pi r^2$, where $v_0$ is the activity of the source at point $\overline{0}$, hence $v(r) = (v_0 - v_1 \pi r^2)/(2\pi r)$.

In this case (in contrast to the previous one when the liquid completely covers the plane) the circle determined by the condition $v(r) > 0$ is the domain of the carrier liquid. Hence $r_{\max} = \sqrt{v_0/(\pi v_1)}$ is the radius of this circle.

In three-dimensional space the velocity of the particle transported by incompressible liquid varies in inverse proportion to the squared distance from the source: $v(r) = v(1)/r^2$. In this case it is difficult to give a reasonable interpretation to loss of the carrier matter, and we do not consider it.

For a gaseous carrier the velocity of the particle depends on pressure $p = p(r)$ that depends on resistance of the medium and decreases with distance from the source. The mass balance equation, when passing through a homogeneous porous medium, has the form:

$$\operatorname{div}(p\overline{v}) = 0, \tag{48}$$

which follows from the Boyle-Marriott law. Hence in the centrally symmetric case we obtain the equation:$(pv)' + (d-1)pv/r = 0$ and consequently, $pv = p(1)v(1)/r^{d-1}$. On the other hand from the Hagen-Poiseuille law for laminar flow of gas the velocity of the stream is proportional to gradient of pressure (see Golbert and Vigdergauz, 1974):

$$\overline{v} = -k\nabla p, \tag{49}$$

where $k > 0$ is a coefficient depending on properties of gas and the porous medium; $\nabla p$ is a vector with coordinates $p_i'$. Using the value of the product $pv$, we obtain the differential equation with respect to $p : pp' = -k_1/r^{d-1}$, where $k_1 = p(1)v(1)/k$. Hence $p^2 = p_\infty^2 + 2k_1/((d-2)r^{d-2}) \quad (d \geq 3)$ where $p_\infty$ is the pressure of gas at a point at infinite distance, for example, atmospheric pressure. Therefore for space $(d = 3)$ we obtain

$$v = \frac{m}{r^{3/2}\sqrt{r+n}},$$

where $m = p(1)v(1)/p_\infty$, $n = 2p(1)v(1)/(p_\infty k)$.

We consider the field of absorption $c(r)$ to be of two forms. First, this is a constant field $c(r) \equiv c_1 =$const. We call such a field "strong." In this case probability of absorption only depends on the passed distance (possibly, on the number of collisions of the transported particle with molecules of immovable phase). Second, this is a field varying in inverse proportion to the velocity of a carrier: $c(r) = c_1/b(r)$.

We call such a field "weak." In a weak field the probability of absorption in some interval of immovable phase depends not only on the length of this interval but on the time it takes the particle to interact with the immobile phase. The second dependence seems to be more plausible. It is verified indirectly in the theory of chromatography (see Harlamov, 2000b). There can be also intermediate laws of interaction, but we do not consider them in the present paper.

## TRANSPORT WITH LIQUID CARRIER

The liquid carrier can flow out from the source and flood over a horizontal surface. In this case we deal with a two-dimensional accumulation problem. If the carrier is not being lost as a result of evaporation, its radial velocity decreases because of geometry of plane. If the carrier is being lost by evaporation, its radial velocity decreases faster and reaches value zero at a finite distance from the source. The liquid carrier can penetrate into a three-dimensional volume filled with porous matter. In this case the carrier is not lost; the radial velocity decreases only because of the geometry of the space. We consider also two forms of absorption fields for each dimension: strong and weak. We do not take into account diffusion for a liquid carrier. Therefore in this case we deal with differential equations of the first order. For dimension $d = 2$ three types of coefficients are investigated:

1. $c(r) = c_1$, flood over without evaporation; strong absorption field. In this case the solution of Eq. (43) has the form

$$h = C\frac{1}{r}\exp(-c_1 r). \tag{50}$$

   The distribution density of the accumulated matter has an acute maximum at the place of the source.

2. $c(r) = c_1 r,$ flood over without evaporation; weak absorption field.

$$h = C\exp\left(-\frac{c_1 r^2}{2}\right) \tag{51}$$

   This is the unique case when the accumulated matter has a normal distribution as a result of transporting matter by a liquid carrier (see Harlamov, 2000b).

3. $c = \frac{2c_1\pi r}{v_0 - v_1\pi r^2} \quad \leq \left(0 < r < \sqrt{\frac{v_0}{\pi v_1}}\right)$, flood over with evaporation; strong absorption field. In this case the solution of equation (43) has the form

$$h = C\left(v_0 - v_1\pi r^2\right)^{c_1/v_1 - 1}. \tag{52}$$

There is clearly expressed dependence of the form of distribution on the ratio of the absorption and evaporation coefficients. If $c_1/v_1 > 1$ the matter is being accumulated in the form of a hill above the source. If $c_1/v_1 < 1$ the matter is concentrated near the boundary of a circular domain forming a ring with a sharp border.

For dimensionality $d = 3$ two types of coefficients are investigated:

1. $c = c_1$, penetration of liquid into a three-dimensional volume; strong absorption field. The solution of Eq. (43) has the form:

$$h = C \frac{1}{r^2} \exp(-c_1 r) \tag{53}$$

   There is an acute maximum at the source location.
2. $c = c_1 r^2$, penetrating of liquid into three-dimensional volume; weak absorption field. The solution of Eq. (43) has the form:

$$h = C \exp\left(-\frac{c_1 r^3}{3}\right). \tag{54}$$

   This distribution is similar to normal but it has a more clear expressed boundary between large and small values of the density than that of the normal distribution (see Harlamov, 2000b).

## TRANSPORT WITH GASEOUS CARRIER

We consider an inflow of a gaseous carrier into a porous medium, where the gas decelerates rather less than would a liquid. Accumulation of matter corresponds to two types of fields of absorption coefficients:

1. $b = \frac{m}{r^{3/2}\sqrt{r+n}}$, $c = c_1$, strong absorption field. In this case Eq. (8) has the form

$$h'' + h' \leq \left(\frac{2}{r} - 6b\right) - h\left(\frac{3nb}{r(r+n)} + 6c_1\right) = 0. \tag{55}$$

   Its analytical solution is not known. Graphs of solutions are shown (Figs. 1 and 2). There is an acute maximum at the source; the rate of decrease of density depends on the value $c_1$.

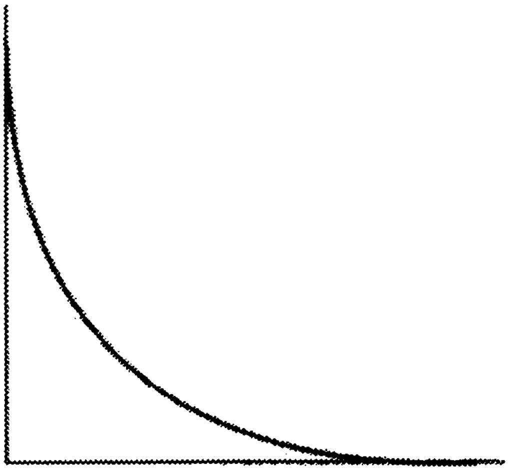

**Figure 1.** The solution of Eq. (55) with $c_1 = 1$.

2. $b = \frac{m}{r^{3/2}\sqrt{r+n}}$, $c = c_1/b$, weak absorption field. In this case Eq. (42) has the form

$$h'' - Fh' + Gh = 0, \tag{56}$$

where

$$F = \frac{6r + 4n}{r(r+n)}, \qquad G = -\frac{5r^2 + 6rn + (9/4)n^2}{r^2(r+n)^2} + \frac{b(36r + 24n)}{r(r+n)} - \frac{6c_1}{b}.$$

Its analytical solution is not known. Graphs of solutions are shown (Figs. 3 and 4). The distribution has a “crater” at the source location. The radius of the ring of maximum values depends on $c_1$.

To obtain graphics we use a standard algorithm for approximate solution of systems of first-order differential equations, solved with respect to derivatives. This algorithm is implemented in computer program “Stend” by Prof V. A. Proursin. In all the figures, profiles of distribution densities are shown as functions of the distance from a source at the origin (on the left).

## CONCLUSION

In the present work in the context of the theory of semi-Markov processes we obtained forward and backward accumulation equations. The density of the measure of accumulated matter is proportional to the distribution density of the

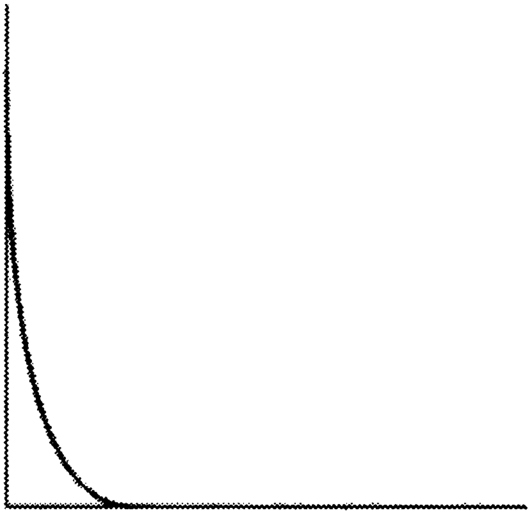

**Figure 2.** The solution of Eq. (55) with $c_1 = 10$.

process at time of stopping. When analyzing the process of accumulation two problems arise: forward and backward. The backward problem is to reconstruct a source on the base of an observable distribution. For this aim the backward equation can be used. We do not consider this problem in the present paper. The main

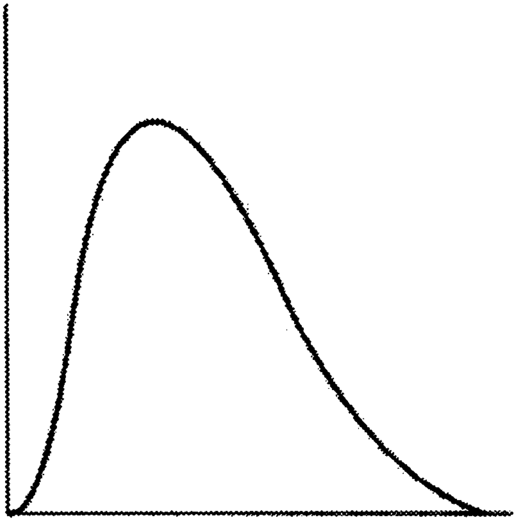

**Figure 3.** The solution of Eq. (56) with $c_1 = 1$.

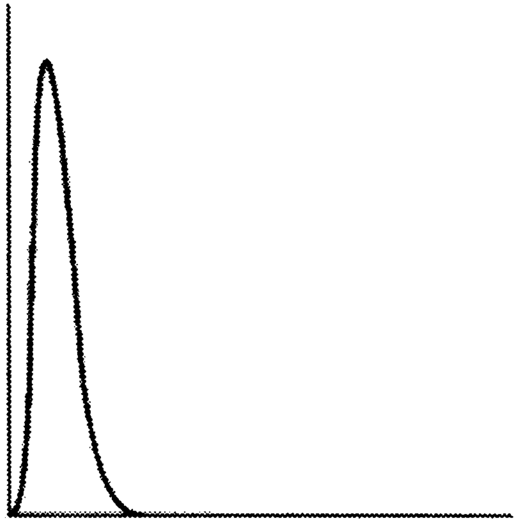

**Figure 4.** The solution of Eq. (56) with $c_1 = 10$.

content of this paper is to derive forward accumulation equations for diffusion and smooth types of processes and to solve them under a simplifying assumption of circular symmetry. Thus we consider a point source in two- and three-dimensional homogeneous and isotropic mediums. It is shown that under some combinations of the model parameters increased concentration zones of accumulated matter can have either maximum or local minimum at the source. In the latter case increased concentration zones form concentric rings or spheres around the source. The radii of these rings and spheres depend on the rate and character of absorption of moving particles by the substance of the filter. Hence this property can imply separation of components of a mixture which differ in their absorption rates.

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