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Power-law attenuation in acoustic and isotropic anelastic media

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SUMMARY

Power attenuation laws were originally introduced for phenomenological descriptions of attenuation in a large variety of materials. More recently they have been applied in viscoelastic inversion. Power-law models can be readily expressed in terms of scalar acoustic partial integro-differential equations. The restrictions on the exponent of the power law imposed by the assumptions of causality and dissipativity are discussed. An extension of power-law attenuation models from acoustic to isotropic anelastic media is proposed. The guiding principle is that the attenuation of longitudinal and transverse waves satisfies the power law. Restrictions imposed by thermodynamics and well-posedness considerations are discussed.

Key words: anelasticity, attenuation, wave propagation.

1 INTRODUCTION

In view of a recent successful application of power-law models of wave attenuation in inversion of scattering data (Ribodetti & Hanyga 2003) it has become necessary to develop a systematic theory of viscoelastic media consistent with this property. Even though some aspects of power laws such as causality were tentatively explored before (Szabo 1994, 1995; Szabo & Wu 2000), a complete mathematical model, including the equations of motion, has not been developed before. The main appeal of power laws resides in their simplicity and consistency with the basic requirements of physics. As a result they have found applications in phenomenological acoustics (Szabo 1994, 1995), inversion (Ribodetti & Hanyga 2003) and simplified formulations of wave propagation in microheterogeneous layered media (Hanyga & Rok 2000).

Power-law models belong to the family of fractional-derivative viscoelastic models commonly used in polymer rheology and engineering. In comparison with the Zener, Maxwell and Kelvin–Voigt models their fractional counterparts offer a considerably more economic parametrization over a very wide range of frequencies (Bagley & Torvik 1986; Soula & Chevalier 1998). Viscous models based on the assumption of power-law attenuation provide an ulterior simplification of the fractional counterpart of the Zener model, known as the Cole–Cole or Bagley–Torvik model, which is of some importance for inverse problems. Prony series representations of viscoelastic kernels, which owe their popularity in seismology to their computational advantages (Day & Minster 1984; Emmerich & Korn 1987; Carcione *et al.* 1988), are totally inappropriate for parameter estimation due to an arbitrary choice of relaxation times and a relatively large number of parameters required for modelling the full frequency range.

Power-law attenuation is the simplest causal model of intrinsic attenuation consistent with the requirements of causality and dissipativity. According to the power-law model the attenuation of a wavefield is represented in the frequency domain by a factor of $\exp(-ap^{\alpha})$, with $p = -i\omega$, $a \geq 0$ and $0 < \alpha < 1$. It should be emphasized that our definition of a power-law model differs significantly from that of Bagley (1989) and Bagley & Torvik (1983a). Bagley and Torvik's power-law model is physically unacceptable since it involves an unbounded stress relaxation function and leads to an infinite speed of wave propagation (Renardy 1982). According to the definition of a power law assumed here only the stress relaxation modulus is unbounded for small delay times. In terms of creep and stress relaxation tests, a step-like input strain causes an infinite jump of stress in the power-law model of Bagley and Torvik. In the power-law model adopted here the stress jump is finite, but the initial rate of stress relaxation (and creep) is infinite, which is consistent with experimental evidence (Nutting 1943; Rabotnov 1969). The limiting value $\alpha = 1$ corresponds in the time domain to a time-shifted delta function, while for $\alpha > 1$ the inverse Laplace transform does not exist. On the other hand, for $0 < \alpha < 1$ the inverse Laplace transform does exist (Pollard 1946) and is the solution of a scalar partial integro-differential equation belonging to a class investigated by Buchen & Mainardi (1975), Lokshin & Rok (1978), Hanyga & Seredyńska (1999b, 2002) and Hanyga & Rok (2000). For background material on fractional calculus and fractional-order equations in mechanics, see Mainardi (1997). The same equation can be expressed in terms of fractional time derivatives (Podlubny 1998) of an order <2:

$$\left[\mathrm{D}^2 + \sum_k b_k \mathrm{D}^{\alpha_k} + c\right] u = v^2 \nabla^2 u, \qquad 0 < \alpha_k < 2,$$

where $\mathrm{D} = \partial/\partial t$. Such equations have become a commonplace in engineering, structural mechanics and acoustics and several numerical methods have been developed for dealing with them (Padovan 1987; Schmidt *et al.* 2000; Schmidt & Gaul 2003). Recent numerical experiments with inversion of laboratory and field seismic viscoelastic data (Ribodetti & Hanyga 2003) indicate that power laws with $0 < \alpha < 1$ provide an excellent parametrization of attenuative properties of some real media.

The formulation adopted in Hanyga & Seredyńska (Hanyga & Seredyńska 1999a,b, 2002) is not consistent with the usual formulation of viscoelasticity because in the above-mentioned papers the memory effects appear in the inertial part of the differential operator. The two formulations are nevertheless equivalent since the convolution operator, including the elastic response, is invertible and its inverse can be applied to the spatial part. There is, however, a price to be paid for shifting the memory terms to the spatial part of the differential operator: the memory kernel is an algebraic function in the first case and it involves a derivative of the generalized exponential function (essentially the Mittag–Leffler function) in the second case. Accordingly, the stress relaxation function involves the Mittag–Leffler function but the creep function is algebraic (Engler 1997).

It has been shown that a more sophisticated model, known as the Cole–Cole or Bagley–Torvik model of viscoelastic response, known in polymer rheology (Rouse 1953; Ferry *et al.* 1955; Bagley & Torvik 1983b; Torvik & Bagley 1983; Bagley & Torvik 1986; Friedrich & Braun 1992), rock mechanics (Batzle *et al.* 2001), seismic wave propagation (Hanyga 2003a) and in the context of dielectric properties of geological materials (Cole & Cole 1941), is very successful in problems of creep and wave attenuation. It is, however, easy to see that power-law models provide a high-frequency approximation for this model. Consequently, assuming the validity of the Cole–Cole model, a power law can be considered as a way of dealing with problems involving frequencies above the transition frequency range.

In an anelastic medium the power law is expected to apply to every bulk wave mode. In general it is impossible to construct an anelastic model with these properties. It is, however, reasonable to expect that the homogeneous isotropic case is an exception since it can be reduced to two scalar equations, which can be cast in a form consistent with the power-law attenuation. In this paper we shall discuss an isotropic anelastic medium in which the longitudinal and transverse waves exhibit power-law attenuation.

Power-law models belong to the class of singular memory models. In the context of viscoelasticity the singular memory property is equivalent to the statement that the time derivative of the stress relaxation function (i.e. the stress relaxation kernel) has an integrable singularity at 0. An important consequence of singular memory is smoothness of the wavefield at the wave fronts (Renardy *et al.* 1987; Hanyga & Seredyńska 2002). This property of the wavefields implies that the signals build-up with some delay after the passage of the wave front. In view of the dependence of the signal delay on the propagation distance the traveltime concept necessitates a revision. Signal delay has important consequences for positioning of scatterers in inversion (Hanyga & Seredyńska 1999a). The implications for seismic inversion methods of singular memory in general and of power laws in particular are discussed in a forthcoming paper by Ribodetti & Hanyga (2003).

2 SCALAR PARTIAL INTEGRO-DIFFERENTIAL EQUATIONS FOR POWER-LAW ATTENUATION

A scalar model of a single-mode wave propagation with a power-law attenuation can be represented by a frequency-domain expression:

$$\hat{u}(\omega, \mathbf{x}) \propto f(\omega) \exp[i\omega T(\mathbf{x}) - (-i\omega)^{\alpha} A(\mathbf{x})] \tag{1}$$

with $A \geq 0$, $0 < \alpha < 1$, and a source spectrum $f(\omega)$. It is easy to see that the inverse Fourier transform $u(t, \mathbf{x})$ of $\hat{u}(\omega, \mathbf{x})$ vanishes for $t > T$ provided the function $f(\omega)$ is analytic in the upper complex half-plane (Hanyga & Seredyńska 2002). In fact, with $f = 1$ and $T = 0$ it is a totally skewed α-stable probability distribution, vanishing identically on the negative real axis (Kreis & Pipkin 1986). Following the method of Appendix D in Hanyga & Seredyńska (2002) one can easily show that eq. (1) is the dominating term of the frequency domain solution

$$\tilde{u}(p, \mathbf{x}) = \frac{1 + \tilde{K}(p)}{4\pi r} \mathrm{e}^{-rp\phi(p)/c} \tag{2}$$

of the initial-value problem (IVP)

$$u_{tt} + K(t) * u_{tt} = c^2 \nabla^2 u + \delta(t)\delta(\mathbf{x}); \qquad u(0, \mathbf{x}) = u_t(0, \mathbf{x}) = 0, \tag{3}$$

where $s = -i\omega$ is the Laplace-transform variable conjugate for t,

$$\phi(p) = 1 + (\tau p)^{\alpha - 1}$$

and

$$1 + \tilde{K}(p) = \phi(s)^2 \equiv 1 + 2(\tau p)^{\alpha-1} + (\tau p)^{2\alpha-2} \tag{4}$$

with a constant $\tau > 0$ of dimension $[T]$. The kernel $K(t)$, given by the formula

$$\left[2(t/\tau)^{-\alpha} / \Gamma(1-\alpha) + (t/\tau)^{1-2\alpha} \Gamma(2 - 2\alpha)\right] / \tau \tag{5}$$

is singular at $t = 0$ and locally integrable.

The complex-valued phase velocity is given by the formula

$$v(\omega) = c/\phi(-i\omega). \tag{6}$$

The squared complex-valued phase velocity $v(\omega)^2$ can be identified with $\kappa(\omega)/\rho(\omega)$, where κ denotes the complex bulk modulus and ρ is the density. Its real part is plotted for $\alpha = 0.3$, 0.5, 0.66 in Fig. 1. Even though the velocity vanishes for zero frequency the

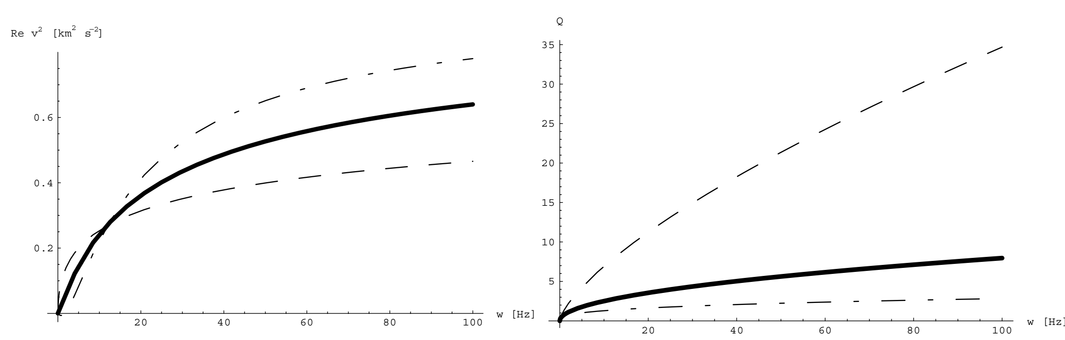

Figure 1. The functions $\Re[v^2](\omega)$, $Q(\omega)$ for $c = 1$ km s^{-1}, $\tau = 2\pi/50$ Hz, $\alpha = 0.5$ (solid line), 0.3 (dashed), 0.66 (dot-dashed).

initial-boundary value problems for eq. (3) are well-posed (Engler 1997). The quality factor Q can be determined from eq. (2):

$$Q(\omega)^{-1} = 4\pi \cos(\pi\alpha/2)(\omega\tau)^{\alpha-1}. \tag{7}$$

Since the kernel $\delta(t) + K(t)$ is invertible in the convolution algebra, eq. (3) is equivalent to a viscoacoustic equation

$$u_{tt} = c^2\nabla^2 u + c^2 f(t) * \nabla^2 u + [1 + f(t)]\delta(\mathbf{x}) \tag{8}$$

$$u(0, \mathbf{x}) = u_t(0, \mathbf{x}) = 0, \tag{9}$$

where $f(t)$ is the inverse Fourier transform of $v(\omega)/c^2 - 1$. The stress response kernel $f(t)$ can be expressed in terms of the generalized Mittag–Leffler functions (Engler 1997). The creep function of the corresponding viscoelastic material is, however, quite simple (Engler 1997)

$$J(t) = 1 + 2\frac{(t/\tau)^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{(t/\tau)^{2(1-\alpha)}}{\Gamma(3-2\alpha)}.$$

The inverse Laplace transform of $\tilde{u}(p, \mathbf{x})$ can be expressed in terms of Wright functions as defined in Podlubny (1998). For $\alpha = \frac{1}{2}, \frac{1}{3}, \frac{2}{3}$ the time-domain solution can be expressed in terms of easily computable functions (elementary functions, the complementary error function and the Airy functions Hanyga & Seredyńska 2002). For slightly more general kernels

$$K(t) = \left[2a(\mathbf{x})(t/\tau)^{-\alpha}/\,\Gamma(1-\alpha) + b(\mathbf{x})(t/\tau)^{1-2\alpha}/\,\Gamma(2-2\alpha)\right]/\tau \tag{10}$$

the initial-value problems for eq. (3) can be solved approximately by the Born approximation (Hanyga & Seredyńska 1999a) or, for $\alpha = \frac{1}{2}, \frac{1}{3}, \frac{2}{3}$, by ray-asymptotic methods (Hanyga & Seredyńska 2002). Regularity and local behaviour of the solutions are studied in much detail in the paper of Engler (1997).

Numerical schemes for integration of the IVP (3) can be obtained by combining a spatial discretization method (pseudospectral (Boyd 1989; Fornberg 1996) or finite-element (FE) method (Zienkiewicz & Taylor 1989–1991)) with a time-stepping algorithm. The spatial discretization scheme reduces the original problem to a system of ordinary differential equations involving fractional derivatives (Podlubny 1998). FE methods combined with a variety of time-stepping algorithms involving fractional derivatives have frequently been used in engineering (Padovan 1987; Enelund & Josefson 1997; Enelund & Lesieutre 1999; Yuan & Agrawal 1998; Schmidt *et al.* 2000). More rigorous stable second-order methods for fractional-order ordinary differential equation have been recently developed by Diethelm (1997) and Diethelm *et al.* (2002). This algorithm, subject to stability requirements, can be used in connection with the Galerkin method.

In seismology fractional derivatives have been used in connection with what is commonly known as Kjartansson's model of constant Q (Kjartansson 1979). The same model was originally formulated in the time domain using fractional derivatives by Caputo (1967). A more recent analysis of the Caputo equation can be found in Hanyga (2002), while a numerical implementation was developed by Carcione *et al.* (2002). The Caputo–Kjartansson constant-Q model

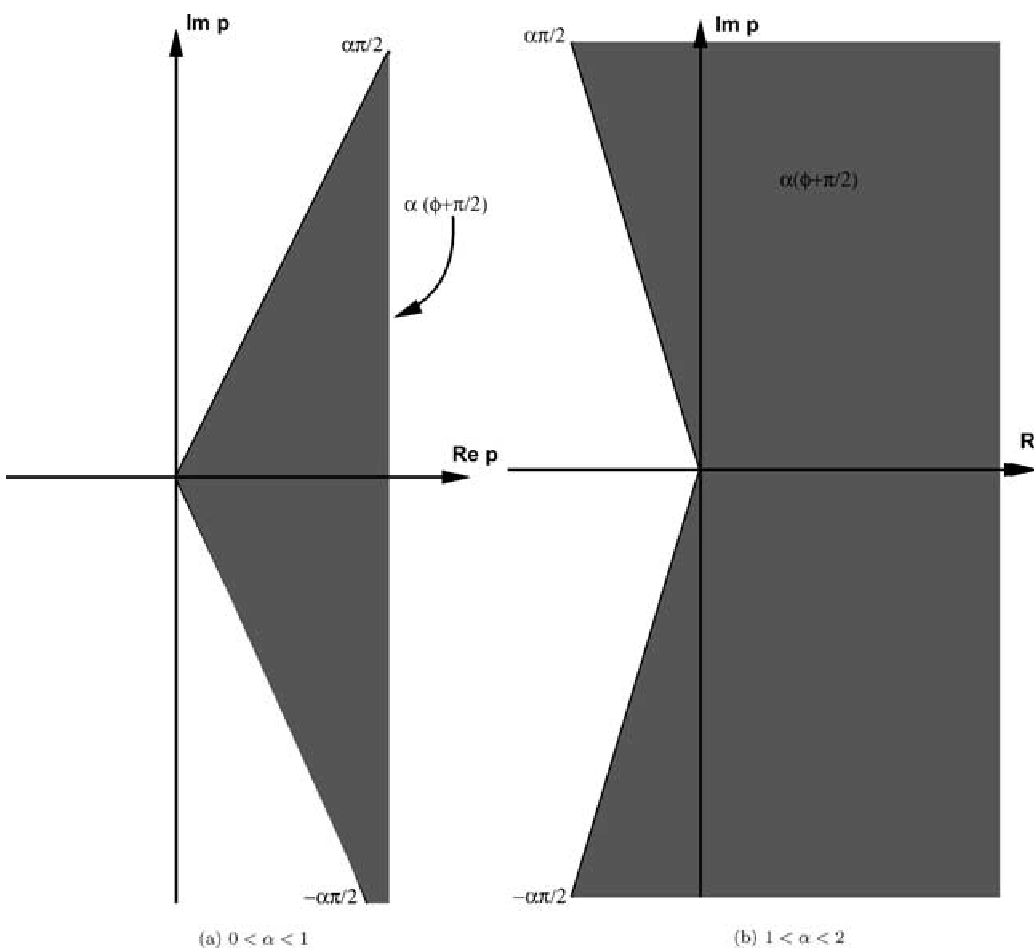

Figure 2. Causality criterion $\cos[\alpha(\phi + \pi/2)] > 0$ for $f_\alpha \in \mathcal{S}'(\mathbb{R})$.

is causal, but in view of an unbounded phase velocity the disturbances propagate with infinite speed. On the other hand, hyperbolic constant-Q models are known to violate causality.

3 LIMITS FOR THE POWER LAW IMPLIED BY CAUSALITY

Power laws have been studied in many papers because of their simplicity, most recently by Szabo and by Ochmann and Makarov. The papers of Szabo (1994, 1995) and Ochmann & Makarov (1993) were largely motivated by experimental evidence for exponents α exceeding 1. Szabo realized that $\alpha > 1$ was inconsistent with causality but he mistakenly assumed that the limitation had its source in excessive limitations on the regularity of the functions for which the Laplace transform is $\exp(-ap^\alpha)$. Szabo guessed that the problem could be worked around by allowing for distributions instead of functions. He was, however, wrong: it is precisely causality rather than regularity of the inverse Laplace transform of a power law that is in question for $\alpha > 1$. It is fairly obvious that higher values of α improve convergence of the inverse Laplace transform and consequently regularity can only improve while causality is lost for $\alpha \geq 1$. More precisely, the inverse Laplace transform

$$f_\alpha(t) = \frac{1}{2\pi i} \int_{-i\infty+c}^{i\infty+c} \mathrm{e}^{pt-ap^\alpha}\, dp \tag{11}$$

converges provided

$$a \cos(\alpha\pi/2) > 0. \tag{12}$$

Assuming that inequality (12) is satisfied, the rate of convergence obviously improves with increasing α. In order to prove that $f_\alpha(t) = 0$ for $t < 0$ one might attempt to close the contour by a half-circle at infinity in the right half of the complex p-plane. Assuming as usual that the complex p-plane is cut along the negative real axis and $-\pi < \phi := \arg p < \pi$, the integrand has no singularities in the right-hand half-plane and it suffices to prove that the contribution of the half-circle tends to 0 as the half-circle recedes to infinity. The absolute value of the integrand is $\exp[-a\, \cos(\alpha\phi) R^\alpha r]$, where $R := |p|$, $r = |\mathbf{x}|$, with $-\pi/2 \leq \phi \leq \pi/2$. For $\alpha > 1$ the absolute value of the integrand does not decrease in some sectors and the argument fails. We thus failed to obtain a sufficient condition for causality. In order to obtain a necessary and sufficient condition it is necessary to turn to a Paley–Wiener theorem (Paley & Wiener 1934, Theorem XII).

Following Szabo we may ask concerning the existence of a real- or complex-valued *function* $f_\alpha(t)$, vanishing for $t < T$, for some real number T of which $\exp(-ap^\alpha)$ is the Laplace transform. The function $\exp(-ap^\alpha) \equiv \exp[-a(-i\omega)^\alpha]$ of a real variable ω is square-integrable function provided eq. (12) holds. By the above Paley–Wiener theorem existence of a square-integrable function f_α vanishing on a half-line is equivalent to the inequality

$$\int_{-\infty}^{\infty} \omega^\alpha (1+\omega^2)^{-1}\, d\omega < \infty, \tag{13}$$

which reduces to $\alpha < 1$.

Szabo (1995) and Szabo & Wu (2000) guess that higher values of α can be accommodated if f_α can be allowed to be a distribution. Theorem 7.4.3 of Hörmander (1983) can be considered as the Paley–Wiener theorem for causal distributions in the tempered distribution space $\mathcal{S}'$ (*cf.* also Dautray & Lions 1992). Without going into details, causality for $f_\alpha \in \mathcal{S}'$ is now equivalent to the possibility of bounding $\exp[-a \cos\{\alpha(\phi - \pi/2)\}\rho^\alpha\}$, in the half-plane $\Im\omega > 0$, with $\omega = \rho \exp(i\phi)$, by a polynomial in ρ. That this condition is satisfied for $0 < \alpha < 1$ and violated for $\alpha \geq 1$, can be seen from Fig. 2. The

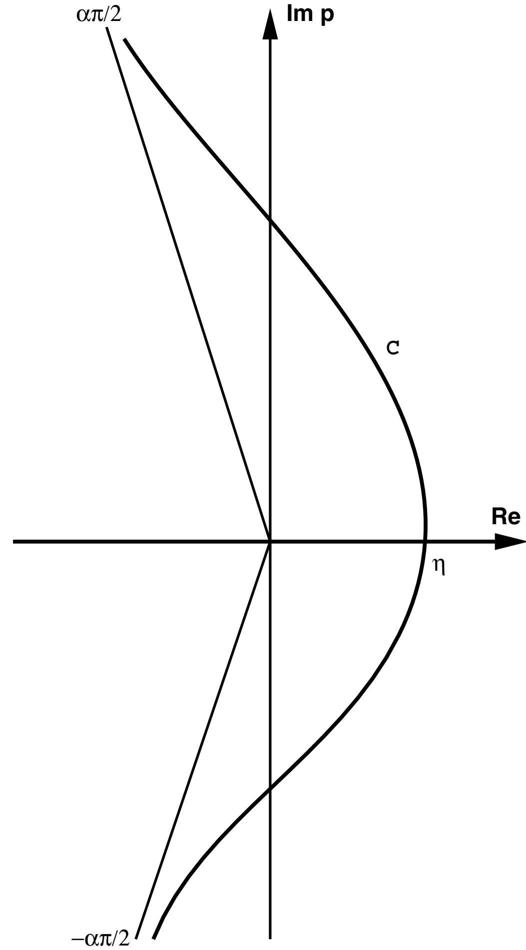

Figure 3. Causality criterion $\cos[\alpha(\phi + \pi/2)] > 0$ for $\exp(-\eta t) f_\alpha(t) \in \mathcal{S}'(\mathbb{R})$, $1 < \alpha < 2$. The complex numbers z with $\arg z = \alpha\,(\phi + \pi/2)$ lie to the right of curve C.

argument of the cosine lies in the shaded part of the complex p-plane and it is obvious that the cosine changes sign in the case of $1 < \alpha < 2$. The growth limitations on f_α can be weakened even further so that $\exp(-\eta t) f_\alpha(t)$, $\eta \geq 0$, belongs to $\mathcal{S}'$. The result is the same (Fig. 3). Consequently causality, rather than regularity or growth restrictions, of f_α excludes $\alpha \geq 1$. This leads to a challenging problem: what is the mathematical model for power-law attenuation for $\alpha \geq 1$? As Ursin & Toverud (2002) surmised one has to look for more complicated laws with a different power law for very small and very large frequencies.

The above considerations can be generalized to more general phase factors $\exp[\Phi(p)r]$. A closer look at the arguments used above shows that the restriction applies to the asymptotic high-frequency behaviour of an arbitrary phase function $\Phi(p)$ provided its asymptotic behaviour is given by a power law: $\Phi(p) \sim \text{constant}\, p^\alpha$.

Experimental evidence in favour of higher values of the exponent in some materials indicates that one must look beyond the power laws. Experiments involve in general a limited frequency band, although validity of the power law has been reported for frequencies up to 100 MHz. It follows that power laws with exponents exceeding 1 can only hold for a finite frequency band. As an example, the Cole–Cole law (Hanyga 2003a), known to satisfy the causality requirement, has a low-frequency asymptotic behaviour corresponding to a power law with $1 < \alpha < 2$, while for high frequencies it behaves like a power law with $0 < \alpha < 1$:

$$v(\omega)^2 = c^2 \frac{1 + a(\tau p)^{-\alpha}}{1 + (\tau p)^{-\alpha}}. \tag{14}$$

Thermodynamic conditions restrict the parameters to the ranges $0 < \alpha < 1$ and $a \leq 1$ (Hanyga 2003a). Subtracting the linear part of the phase representing the time-shift (traveltime) pr/c, it is easy to see that the remainder of the phase satisfies the Paley–Wiener theorem, i.e. the time-domain solution vanishes for $t > r/c$. In the high-frequency limit the phase

$$pr/v(\omega) \sim pr/c - \frac{1 - a}{2c\tau} r(\tau p)^{1-\alpha}$$

is approximated by a power law with the exponent $1 - \alpha < 1$ as expected. For low frequencies, however, the exponent appears to exceed 1:

$$pr/v(\omega) \sim pr/(ac) - (1/a - 1) r(\tau p)^{1+\alpha}/(c\tau).$$

Assuming the Cole–Cole law, we can expect a power law with an exponent exceeding 1 provided $1/\tau$ is larger than the upper frequency limit in the experiments. More flexibility can be obtained by recourse to a generalized Cole–Cole law, for example

$$v^2 = c^2 \frac{1 + a(\tau p)^{-\alpha} + b(\tau p)^{-\beta}}{1 + (\tau p)^{-\alpha}}$$

with $\alpha < \beta$ with appropriate restrictions on the parameters (Rossikhin & Shitikova 1997).

4 LIMITS FOR THE POWER-LAW EXPONENT IMPLIED BY DISSIPATIVITY

We shall discuss dissipation in the context of pure viscoelasticity, with thermal variables eliminated by some procedure. In this context the energy represents the sum of the kinetic energy and the stored elastic energy. Since thermal energy is not accounted for, dissipation results in a non-conservation of energy.

Thermodynamic analysis of dissipativity in hereditary viscoelastic media is complicated by the non-uniqueness of the stored energy. We shall consider the scalar power-law model in the formulation (8). Multiplying eq. (8) by u_t and rearranging the resulting equation leads to the following energy balance:

$$\frac{d}{dt} \frac{u_t^2}{2} + \frac{1}{2} \nabla u_t \cdot c^2 F * \nabla u_t = \operatorname{div} \left[u_t c^2 f * \nabla u \right], \tag{15}$$

where $F(t)$ is defined by its Laplace transform

$$\tilde{F}(p) = \frac{1}{p\,[1 + ap^{\alpha-1}]^2}. \tag{16}$$

The second term on the left-hand side represents energy dissipation while the term on the right-hand side is the divergence of an energy flux.

Assume that either $\Omega = \mathbb{R}^d$ or

$$\partial\Omega = \Sigma_1 \cup \Sigma_2, \quad \Sigma_1 \cap \Sigma_2 = \emptyset$$

and u satisfies homogeneous Dirichlet and Neumann boundary conditions:

$$u|_{\Sigma_1} = 0 \tag{17}$$

$$\mathbf{n} \cdot \nabla u|_{\Sigma_2} = 0. \tag{18}$$

Integrating eq. (15) over Ω and over a time interval $[T_1, T_2]$ we have

$$\int_\Omega \left[u_t^2/2\right]_{T_1}^{T_2} dx + \int_\Omega \int_{T_1}^{T_2} D\, dx\, dt = 0, \tag{19}$$

where

$$D := c^2 \nabla u_t \cdot F * \nabla u_t. \tag{20}$$

The second term of eq. (19) is non-negative provided the kernel F is of positive type (Gripenberg *et al.* 1990). By a version of the Bochner theorem adapted for causal kernels (Gripenberg *et al.* 1990) this condition is satisfied provided the necessary and sufficient condition

$$\Re \frac{1}{p\,[1 + ap^{\alpha-1}]^2} \geq 0 \quad \text{for} \quad \Re p > 0$$

holds. Working out the last expression results in the condition

$$\cos(\phi) + 2ar^{-\alpha} \cos[(1 - \alpha)\phi] + a^2 r^{-2\alpha} \cos[(1 - 2\alpha)\phi] \geq 0$$

for $-\pi/2 < \phi < \pi/2$, $0 < r < \infty$ where $p = r\exp(i\phi)$. It is clear that the above condition is satisfied for $0 < \alpha \leq 1$. For $\alpha > 1$ the last term is negative and dominates for small r. Consequently, dissipativity confirms the restriction $0 < \alpha < 1$, already obtained from the causality considerations.

A kernel of positive type is non-negative and non-increasing. Since $\tilde{F}(0+) = \int_0^\infty F(t)\, dt = \infty$, the kernel F is not integrable. Using appropriate Tauberian theorems (Widder 1946; Doetsch 1958), the kernel $f(t) \sim \delta(t) - (2\alpha/\tau)(t/\tau)^{-\alpha}/\Gamma(1 - \alpha)$ at $t = 0$ and $f(t) \sim a^{-2}(t/\tau)^{2\alpha-3}/\Gamma(2\alpha - 2)$ at infinity.

5 COMPLETELY MONOTONIC STRESS RELAXATION FUNCTIONS

In the acoustic model thermodynamic restrictions imply some rather obvious conditions on the coefficients of the complex moduli. In an anelastic medium thermodynamic restrictions of non-negative dissipation impose some requirements that apply to each bulk wave mode separately and to the relations between the constitutive parameters controlling the two waves. Such conditions are formulated in Hanyga (2003c). These conditions have to be sharpened further in order to prove the existence and uniqueness of solutions (Hanyga 2003a,b). The second reference involves a significant relaxation of sufficient conditions for well-posedness of initial-value problems of viscoelasticity.

Earlier treatments of the existence and uniqueness problems were either restricted to scalar equations (Renardy *et al.* 1987), to quasistatic and harmonic solutions (Christensen 1971; Fabrizio & Morro 1992) or to media satisfying some separability conditions which decouple a scalar relaxation function (Prüss 1993; Desch & Grimmer 1989) from an anisotropic elastic spatial operator. Such a separability assumption is inappropriate on physical grounds. For example, in the context of viscoelasticity the equations considered in the book of Prüss (1993) would apply when the viscoelastic stress relaxation function

$$G_{klmn}(t) = M(t) c^0_{klmn}$$

with a frequency-independent stiffness tensor c^0_{klmn}. Consequently, the paper of Hanyga (2003b) marks a significant progress in the general formulation of linear viscoelasticity.

For a general viscoelastic medium with the stress–strain constitutive relation

$$\sigma_{kl}(t) = G_{klmn}(0) e_{mn}(t) + \int_0^\infty G'_{klmn}(s) e_{mn}(t - s)\, ds$$
$$\equiv \int_0^\infty G_{klmn}(s) e'_{mn}(t - s)\, ds, \tag{21}$$

where G_{klmn} denotes the stress relaxation function, the thermodynamic conditions are

$$(-1)^j \frac{d^j}{dt^j} G_{klmn}(t) v_{kl} v_{mn} \geq 0 \tag{22}$$

for all symmetric tensors v_{kl} and for $j = 0, 1, 2$. A sufficient condition for well-posedness of initial-value problems is the same set of inequalities extended to all positive integer values of j (Hanyga 2003b). A tensor-valued function with the last property is called completely monotonic (Gripenberg *et al.* 1990). The less restrictive condition, eq. (22) for $j = 0, 1, 2$, is related to the stress relaxation function being of positive type:

$$\int_{-\infty}^{T} dt \int_0^{\infty} G_{klmn}(s) v_{kl}(t-s) v_{mn}(t)\, ds \geq 0 \tag{23}$$

for every continuous symmetric tensor-valued function $v_{kl}(t)$. In particular, with $v_{kl}(t) = e'_{kl}(t)$, eq. (23) implies non-negative energy dissipation

$$\int_{-\infty}^{T} \sigma_{kl}(t) e'_{kl}(t)\, dt \geq 0. \tag{24}$$

Day (1970) tried to give an interpretation of complete monotonicity of stress relaxation functions in terms of work. Molinari (1975) noted that complete monotonicity is equivalent to the existence of a relaxation spectrum and to the existence of a creep spectrum expressible as a Stieltjes measure. He also studied regularity of the relaxation and creep functions and relations between them under the assumption that the former is completely monotonic. By a theorem in Del Piero & Deseri (1995) the relaxation function $G_{klmn}(t)$ is completely monotonic if: (1) G_{klmn} is of positive type and (2) G'_{klmn} is integrable, $G' \in \mathcal{L}^1(\mathbb{R}_+)$. This seems to give a thermodynamic support for complete monotonicity. Unfortunately some weakly singular stress relaxation kernels G'_{klmn} exhibit algebraic decay at large t, $G'_{klmn}(t) \sim \text{constant} \times t^{-\gamma}$ for $t \to \infty$, $0 < \gamma < 1$ and hence they are not integrable (two examples of completely monotonic weakly singular non-integrable functions can be found in Hanyga 2003c).

A scalar function $g(t)$, defined for $0 < t < \infty$, with a possible singularity at 0, is said to be completely monotonic if it is non-negative and its derivatives satisfy the inequalities $(-1)^n g^{(n)}(t) \geq 0$, $n = 0, 1, 2, \ldots$. Since the stress relaxation kernels are often defined in terms of their Fourier transforms, a spectral criterion of complete monotonicity is often more useful. According to a theorem in Gripenberg *et al.* (1990) a function $g(t)$ is completely monotonic if and only if its Laplace transform satisfies the inequality $\Im[p\tilde{g}(p)] > 0$ for all p in the right half of the complex p-plane $\Re p > 0$. Using this criterion it is easy to show that the stress relaxation function of the power-law model, defined by its Fourier transform $c^2/[p\phi(p)^2]$, is completely monotonic. Indeed, for $p = r\exp(i\psi)$, $-\pi/2 \leq \psi \leq \pi/2$,

$$\Im\left[c^2(1 + ap^{-\alpha})^{-2}\right] = 2ar^{-\alpha} \sin(\alpha\psi)[1 + a\cos(\alpha\psi)] \geq 0.$$

A special case of a completely monotonic tensor function is

$$G_{klmn}(t) = \sum_q f_q(t) B^{(q)}_{klmn} \tag{25}$$

with time-independent non-negative tensors $B^{(q)}_{klmn}$ and completely monotonic scalar functions $f_q(t)$. This class of anisotropic viscoelastic media was investigated by Hanyga (2003a,c).

An example of eq. (25) can be derived from the Kelvin decomposition of an elastic stiffness tensor

$$c_{klmn} = \sum_{q=1}^{6} g_q e^{(q)}_{kl} e^{(q)}_{mn}$$

with the eigenstrains $e^{(q)}_{kl}$ essentially determined by the material symmetry of the medium (Helbig 1994), and eigenvalues $g_q \geq 0$, by setting

$$B^{(q)}_{klmn} = e^{(q)}_{kl} e^{(q)}_{mn}, \qquad q = 1, \ldots, 6.$$

Anelastic relaxation can be grafted on an elastic anisotropic model by replacing the constants g_q with the scalar stress relaxation functions $f_q(t)$ (Carcione *et al.* 1996; Carcione 2001).

For an isotropic elastic medium the normalized eigenstrains consist of a pure dilatation $e^{(1)}_{kl} = (1/\sqrt{3})\delta_{kl}$ and five linearly independent isochoric strains. The corresponding eigenvalues g_q expressed in terms of the Lamé coefficients are

$$g_1 = 3\lambda + 2\mu \tag{26}$$

$$g_q = 2\mu \quad \text{for} \quad 2 \leq q \leq 6. \tag{27}$$

6 POWER-LAW ATTENUATION IN ISOTROPIC ANELASTICITY

Such properties as power-law attenuation or constant Q (Hanyga 2001) pertain to bulk wave modes. Attenuation can be traced back to relaxation processes that can be formulated at the level of constitutive relations. Specific viscous relaxation mechanisms and relaxation times can be associated with special deformation types. In an anisotropic anelastic medium the latter are the six eigenstrains associated with the stiffness coefficients (Helbig 1994). The stiffnesses of several eigenstrains affect each bulk wave mode. Consequently, there is no way of matching the power-law attenuation of the three bulk wave modes by the six relaxation mechanisms. Isotropic anelastic media are, however, an exception. For isotropic media it is possible to match power-law attenuation of the two bulk wave modes by appropriate relaxation mechanisms.

The potentials ϕ, $\mathbf{Y}$ of an elastic field in a homogeneous isotropic medium satisfy the equations

$$v_1^{-2}\phi_{,tt} = \nabla^2\phi \tag{28}$$

$$v_2^{-2}\mathbf{Y}_{,tt} = \nabla^2\mathbf{Y}. \tag{29}$$

Comparison with the scalar equations discussed in Hanyga & Seredyńska (2002) suggests that the power-law behaviour for the longitudinal and transverse waves can be achieved by setting

$$v_1^2 \equiv (\lambda + 2\mu)/\rho = M_1^{-2}\left(1 + a_1 p^{-\alpha_1}\right)^{-2} \tag{30}$$

$$v_2^2 \equiv 2\mu/\rho = M_2^{-2}\left(1 + a_2 p^{-\alpha_2}\right)^{-2} \tag{31}$$

with $0 < \alpha_1, \alpha_2 < 1$ and, for definiteness $M_1, M_2 > 0$.

In order to verify thermodynamic restrictions we must examine the eigenvalues $f_1 = 3\lambda + 2\mu$ and $f_2 = 2\mu$. In the elastic case they must be positive, while in the viscoelastic case they should become completely monotonic stress relaxation functions (eigenvalues of $G_{klmn}(t)$). Furthermore, the tensor $G_{klmn}(0)$ responsible for immediate response should be greater or equal than the relaxed tensor

$$G^{\infty}_{klmn} := \lim_{t\to\infty} G_{klmn}(t)$$

that is

$$G_{klmn}(0) v_{kl} v_{mn} \geq G^{\infty}_{klmn} v_{kl} v_{mn} \tag{32}$$

for every symmetric matrix v_{kl}. For the special case (25) this condition boils down to

$$f_q(0) \geq f_q^{\infty} := \lim_{t\to\infty} f_q(t), \quad \text{for all} \quad q.$$

The limits $t \to 0, \infty$ of $f_q(t)$ are equal to the limits of the Laplace transforms $\widetilde{f}_q(p)$ for $p \to \infty$, 0, respectively. In order to ensure actual stress relaxation for deviatoric strains $q \geq 2$ we require that $a_2 \geq 1$.

Supplementary conditions are obtained by considering the limits $p \to 0, \infty$ of the Laplace transform of the eigenvalue

$$f_1 = 3\lambda + 2\mu \equiv 3(\lambda + 2\mu) - 4\mu \equiv \rho\left(3v_1^2 - 4v_2^2\right)$$

with v_1, v_2 given by eqs (30) and (31). We recall from the previous section that the real part of the Laplace transform of f_1 must be positive. For $q = 1$ we must examine the function

$$(3\lambda + 2\mu)/\rho = 3M_1^{-2}\left(1 + a_1 p^{-\alpha_1}\right)^{-2} - 4M_2^{-2}\left(1 + a_2 p^{-\alpha_2}\right)^{-2}.$$

For $p \to \infty$ we obtain the inequality

$$M_1 < \sqrt{\frac{3}{4}} M_2 \tag{33}$$

familiar from elasticity. For $p \to 0$

$$(3\lambda + 2\mu)/\rho \sim 3M_1^{-2} a_1^{-2} p^{2\alpha_1} - 4M_2^{-2} a_2^{-2} p^{2\alpha_2}.$$

Since this expression must remain non-negative for arbitrary positive p, the first term on the right-hand side must dominate for small p, which implies that

$$\alpha_2 \geq \alpha_1. \tag{34}$$

For $\alpha_1 = \alpha_2$ we must additionally require that

$$0 < 3M_1^{-2} a_1^{-2} - 4M_2^{-2} a_2^{-2} \leq 3M_1^{-2} - 4M_2^{-2}.$$

The second inequality is implied by eq. (32). In particular,

$$1 - a_1^{-2} \geq \frac{4}{3}\frac{M_1^2}{M_2^2}\left(1 - a_2^{-2}\right). \tag{35}$$

The above conditions are necessary but not sufficient.

7 FUNDAMENTAL SOLUTIONS AND WAVE ATTENUATION

The results of Hanyga & Seredyńska (2002) yield the solutions of the initial-value problem (IVP)

$$v_1^{-2} * \phi_{,tt} = \nabla^2 \phi$$
$$\phi(0, \mathbf{x}) = 0, \qquad \phi_{,t}(0, \mathbf{x}) = \delta(\mathbf{x}), \tag{36}$$

where v_1^{-2} is considered as a time convolution kernel (the inverse Laplace transform of $v_1^{-2}(p)$ and

$$f * g(t) := \int_0^t f(s) g(t - s)\, ds$$

is the Volterra convolution.

The frequency-domain solution of the IVP (36) is given by the formula

$$\phi(p, \mathbf{x}) = \frac{1}{4\pi r} v_1^{-2} \exp(-pr/v_1) \tag{37}$$

(Hanyga & Seredyńska 2002). The exponential can be expressed in a more explicit way

$$\phi(p, \mathbf{x}) = \frac{1}{4\pi r} v_1^{-2} \exp\left(-M_1 pr - M_1 a_1 r p^{1-\alpha_1}\right). \tag{38}$$

The first term of the exponent represents the traveltime of the longitudinal wave $T_1 = M_1 r$. The second term represents an attenuation with the quality factor $Q_1 = 1/(4\pi v_1 M_1 a_1 \omega^{\alpha_1})$. The frequency dependence of Q is consistent with the high-frequency behaviour of the Cole–Cole model.

The results for the vector potential are analogous.

For $\alpha_1 = \frac{1}{2}, \frac{1}{3}, \frac{2}{3}$ the fundamental solution can be calculated in an explicit time-domain form (Hanyga & Seredyńska 2002). This has some importance for practical inversion of seismic data.

Eq. (36) can be rewritten in the form

$$\phi_{,tt} = v_1^2 * \nabla^2 \phi,$$

where v_1^2 is now the inverse Laplace transform of $v_1^2(p)$. As we know from Section 4, the inverse Laplace transform of $v_1^2(p)/p$ is completely monotonic. This implies well-posedness for the scalar equations governing the potentials.

For an inhomogeneous isotropic medium the elastic field cannot be represented in terms of the potentials. In order to demonstrate the well-posedness one has to prove that the eigenvalue $f_1(t)$ is completely monotonic. Whether this condition is satisfied depends on numerical relations between M_1, M_2, a_1, a_2, which are difficult to obtain in an explicit form. Asymptotic solutions of the anelastic equations of motion can, however, be obtained along the lines laid out by Hanyga & Seredyńska (2002). The phases of the longitudinal and transverse waves have the same form as the potentials.

8 CONCLUSIONS

Power attenuation laws have been used in phenomenological acoustics because of their extreme simplicity as well as their conformity with the physical requirements of causality and dissipativity. Power laws can be alternatively viewed as high-frequency asymptotic approximations of a wide class of phase functions, for which the phase function considered as a function of p has regular variation. In particular, the power law is a high-frequency form of the Cole–Cole law, known for its flexibility and covering a wide range of frequencies. Additional parameters in the Cole–Cole law make it, however, less amenable to inversion based on a restricted data set. In view of economic parametrization the class of power-law materials is convenient in inversion. Power laws contain enough information concerning the wave front and signal propagation for wave inversion problems.

Causality and dissipativity restricts the frequency dependence of attenuation in a power-law medium A = constant $\times\ \omega^\alpha$ to $0 < \alpha < 1$. The same inequality applies to high-frequency limits of more complicated attenuation models. This frequency dependence is consistent with high-frequency behaviour of the Cole–Cole model (Hanyga 2003a).

Power-law models of attenuation can also be applied to longitudinal and transverse waves in an isotropic anelastic medium. We either have $\alpha_2 > \alpha_1$ and the transverse quality Q_2 decays faster than Q_1 with increasing frequency, or else $\alpha_1 = \alpha_2$ and a relation between the quality factors is implied by eq. (35).

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