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Research Note

Summation and decomposition of principal stresses in the crust*

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In the compilation of *World Stress Map*, 9% of data comes from overcoring and hydraulic fracturing measurement, 23% from borehole breaking off, 63% from earthquake focal mechanism, and 5% from young geological investigation (Zoback, *et al*, 1989). Only overcoring and hydraulic fracturing can provide both the orientation and magnitude of the horizontal stress, all other methods can only provide the orientation, but no the magnitude of the stresses. Although some researchers tried to estimate magnitudes of stresses in earthquake mechanism research based on some additional assumptions (CHEN, Duda, 1996; ZHAO, *et al*, 2002). This method, however, has not been widely applied. What kind of analysis can be done with orientation-only data? What kind of incorrect operations should be avoided? These are basic important problems. However, some confusions and misunderstandings exist. For example, a simple operation is to use the average of measured orientation to represent the principal stress orientation in a specific area; or decompose a stress into a summation of long wavelength and short wave length components. Is it correct to do in this way? Some fundamental ideas are hidden in these seemingly simple problems. We will discuss these questions in this note.

1 Description of stress state and algebraic operation of stress tensor

Stress tensor is used to describe the stress state at a location (Timshenko, Goodier, 1970)

$$\begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix} \tag{1}$$

where $\tau_{xy}=\tau_{yx}$, $\tau_{xz}=\tau_{zx}$, $\tau_{xz}=\tau_{zx}$, therefore, only six components are independent. If the vertical direction is chosen to be the z-axis, then $\sigma_z=0$ at the surface. In dealing with geological problems close to the surface, the stress state can usually be assumed as a two-dimensional plane stress. The stress tensor can be reduced as:

$$\begin{pmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{pmatrix} \tag{2}$$

in this case, there are four components, but only three of them are independent. The stress state is determined if the three independent components are known. From these components, normal and shear stress on any plane at this location can be calculated. Addition, decomposition, calculation of the mean and other algebraic operations must be done with respect to these components.

Although tensor expression provides the basics for stress state description and operations, for geological

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analysis this expression is not easily visualized. It is more convenient for geologists to describe the stress state by orientation and magnitude of principal stresses, which can be obtained from calculation of the eigenvalues and eigenvectors of the stress tensor. In two-dimensional case, the magnitude and orientation of the two principal stresses are:

$$\sigma_{1,2} = \frac{\sigma_1 + \sigma_2}{2} \pm \sqrt{\frac{(\sigma_x - \sigma_y)^2}{4} + \tau_{xy}^2} \tag{3}$$

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \tag{4}$$

Because the maximum and minimum stresses are always perpendicular to each other, therefore, it is also three quantities that are needed to describe the stress state. It is noted that to describe the stress state at a point, tensor, instead vector, must be used. From three components of stress tensor, principal stresses can be calculated; and vice versa: components of stress tensor can be calculated from principal stresses. No matter which kind of description, or a mixture of two expressions, such as knowing σ_x, σ_y and direction of the principal stress, three independent parameters are always necessary.

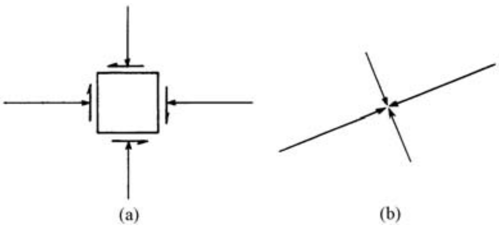

Figure 1 (a) To describe the stress state by stress tensor, algebraic operations can be done to the comp onents; (b) To describe the stress state by the magnitudes and orientation of principal stresses, it is visualized, but algebraic operations can not be done directly to the principal stresses.

Plane stress state can be shown graphically as Figure 1. If stress tensor components are used, the corresponding graphic representation is shown in Figure 1a; if the principal stresses are used, the corresponding graphic representation is shown in Figure 1b. Vectors can be used to represent the displacement and velocity of a particle, or the traction (including normal and shear stress) on a given plane (the normal of the plane is given). As traction is concerned, the precondition is the plane must be assigned. In geological literatures, it is popular to state that Indian plate collides with Chinese mainland and to show figures of NNE arrows in the Indian side. It is correct if the arrows represent the velocity of motion, however, it is incorrect if the arrows are attempted to represent stresses and the location and strike of the boundary are not given.

2 Azimuth angle can not be added, decomposed or averaged in simple ways

If stress at a point is measured, the first measurement yield a result:

$$\begin{pmatrix} \sigma_{x_1} & \tau_{xy_1} \\ \tau_{xy_1} & \sigma_{y_1} \end{pmatrix} \tag{5}$$

the orientation of the principal stress is determined by:

$$\tan 2\theta_1 = \frac{2\tau_{xy_1}}{\sigma_{x_1} - \sigma_{y_1}} \tag{6}$$

the second measurement yield a result:

$$\begin{pmatrix} \sigma_{x_2} & \tau_{xy_2} \\ \tau_{xy_2} & \sigma_{y_2} \end{pmatrix} \tag{7}$$

the corresponding orientation of principal stress is:

$$\tan 2\theta_2 = \frac{2\tau_{xy_2}}{\sigma_{x_2} - \sigma_{y_2}} \tag{8}$$

then, their summation is:

$$\begin{pmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{pmatrix} = \begin{pmatrix} \sigma_{x_1} & \tau_{xy_1} \\ \tau_{xy_1} & \sigma_{y_1} \end{pmatrix} + \begin{pmatrix} \sigma_{x_2} & \tau_{xy_2} \\ \tau_{xy_2} & \sigma_{y_2} \end{pmatrix} = \begin{pmatrix} \sigma_{x_1}+\sigma_{x_2} & \tau_{xy_1}+\tau_{xy_2} \\ \tau_{xy_1}+\tau_{xy_2} & \sigma_{y_1}+\sigma_{y_2} \end{pmatrix} \tag{9}$$

the orientation of the summation of two stress tensors can be calculated as:

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2(\tau_{xy_1}+\tau_{xy_2})}{\sigma_{x_1}+\sigma_{x_2}-\sigma_{y_1}-\sigma_{y_2}} \tag{10}$$

however, if the mean of θ_1 and θ_2 is calculated simply as $\theta' = (\theta_1 + \theta_2)/2$, it is obtained that

$$\tan 2\theta' = \frac{\tan\theta_1 + \tan\theta_2}{1-\tan\theta_1 \tan\theta_2} \tag{11}$$

where

$$\tan\theta_1 = \frac{\sqrt{1+\tan^2 2\theta_1} - 1}{\tan 2\theta_1} \tag{12}$$

$$\tan\theta_2 = \frac{\sqrt{1+\tan^2 2\theta_2} - 1}{\tan 2\theta_2} \tag{13}$$

obviously, in ordinary cases, θ calculated from (10) does not equal to θ', which is calculated from (11), (12), (13), (6) and (8). Two examples below bring this point out clearly.

Example 1. If stress tensor 1 has $\sigma_{x_1}=2$, $\sigma_{y_1}=0$, $\tau_{xy_1}=0$; and stress tensor 2 has $\sigma_{x_2}=0$, $\sigma_{y_2}=1$, $\tau_{xy_2}=0$, then stress tensor 1 has a maximum principal stress $\sigma_1^{(1)}=2$, a minimum principal stress $\sigma_2^{(1)}=0$, and the angle between the maximum principal stress and the x-axis $\theta_1=0$. Stress tensor 2 has a maximum principal stress $\sigma_1^{(2)}=2$, a minimum principal stress $\sigma_2^{(2)}=0$, and the angle between the maximum principal stress and x-axis $\theta_2=90°$. If one calculates the orientation of the stress summation simply by using the mean of the two orientation angles, it would lead to a wrong result that the principal orientation of the stress tensor summation is at an angle of 45° with the x-axis. Actually, since the summation has components as $\sigma_x=2$, $\sigma_y=1$, $\tau_{xy_1}=0$; the maximum principal stress $\sigma_1=2$, the minimum principal stress $\sigma_2=1$, and the orientation of the maximum principal stress is at an angle of $\theta=0$ with the x-axis (Figure 2).

Example 2. If stress tensor 1 has $\sigma_{x_1}=2$, $\sigma_{x_2}=3$, $\tau_{xy_1}=0$; and stress tensor 2 has $\sigma_{x_2}=0$, $\sigma_{x_2}=0$, $\tau_{xy_2}=1$, then stress tensor 1 has the maximum principal stress $\sigma_1^{(1)}=3$, the minimum principal stress $\sigma_2^{(1)}=2$, and the angle between the maximum principal stress and x axis $\theta_1=90°$; stress tensor 2has the maximum principal stress $\sigma_1^{(2)}=1$, the minimum principal stress $\sigma_2^{(2)}=-1$, and the angle between the maximum principal stress and the x-axis $\theta_2=45°$. However, the simple average would give a wrong conclusion that the angle between maximum principal stress and the x-axis is 67.5°, while the actual orientation of summation stress tensor is $\theta=58.3°$, and

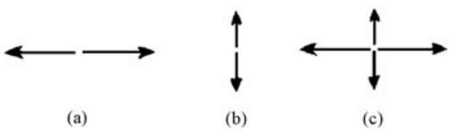

Figure 2 (a) Stress tensor 1; (b) Stress tensor 2; (c) Summation of two stress tensors

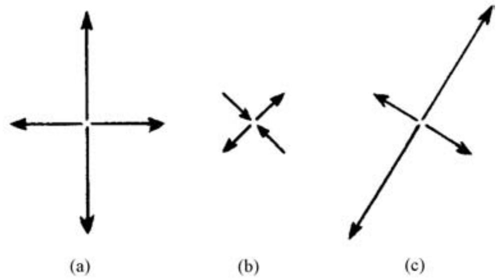

Figure 3 (a) Stress tensor 1; (b) Stress tensor 2; (c) Summation of two stress tensors maximum principal stress σ_1=3.61, minimum principal stress σ_2=1.39, as calculated from summation of components σ_x=2, σ_y=3, and τ_{xy_1}=1(Figure 3).

In summary, during summation or decomposition of two-dimensional stress tensors (by equation 9), there are 3 tensors (9 independent components) concerned, and in order to calculate 3 unknown components, 6 independent components are required to be known, either in the form of stress tensor components or in the form of principal stresses.

3 Conditions for calculating the principal orientation by simple mean

As we discussed, generally orientation of the principal stress cannot be simply averaged from multi-measurements, are there special cases that the average can be directly done?

It is known that in the coordinate system which use the principal axes as *x-y* axes, stress tensor can be expressed as:

$$\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} = \begin{pmatrix} \frac{\sigma_1+\sigma_2}{2} & 0 \\ 0 & \frac{\sigma_1+\sigma_2}{2} \end{pmatrix} + \begin{pmatrix} \frac{\sigma_1-\sigma_2}{2} & 0 \\ 0 & -\frac{\sigma_1-\sigma_2}{2} \end{pmatrix} = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} + \begin{pmatrix} \Delta\sigma & 0 \\ 0 & -\Delta\sigma \end{pmatrix} \tag{14}$$

where, $p=(\sigma_1+\sigma_2)/2$, and $\Delta\sigma=(\sigma_1-\sigma_2)/2$, they are the mean hydrostatic pressure and the maximum shear stress respectively. If the EW direction is chosen to be the *x*-axis and the NS direction is chosen to be the *y*-axis in geological study, the angle between maximum principal stress and *x*-axis is denoted by θ, the stress tensor in the coordinate system is expressed as:

$$\begin{pmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{pmatrix} = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} + \begin{pmatrix} \Delta\sigma\cos 2\theta & \Delta\sigma\sin 2\theta \\ \Delta\sigma\sin 2\theta & -\Delta\sigma\cos 2\theta \end{pmatrix} \tag{15}$$

the orientation of the principal stress is solely determined by the second term in equation (15)

If the first measurement has a result:

$$\begin{pmatrix} \sigma_{x_1} & \tau_{xy_1} \\ \tau_{xy_1} & \sigma_{y_1} \end{pmatrix} = \begin{pmatrix} p_1 & 0 \\ 0 & p_1 \end{pmatrix} + \begin{pmatrix} \Delta\sigma^{(1)}\cos 2\theta_1 & \Delta\sigma^{(1)}\sin 2\theta_1 \\ \Delta\sigma^{(1)}\sin 2\theta_1 & -\Delta\sigma^{(1)}\cos 2\theta_1 \end{pmatrix} \tag{16}$$

the second measurement has:

$$\begin{pmatrix} \sigma_{x_2} & \tau_{xy_2} \\ \tau_{xy_2} & \sigma_{y_2} \end{pmatrix} = \begin{pmatrix} p_2 & 0 \\ 0 & p_2 \end{pmatrix} + \begin{pmatrix} \Delta\sigma^{(2)}\cos 2\theta_2 & \Delta\sigma^{(2)}\sin 2\theta_2 \\ \Delta\sigma^{(2)}\sin 2\theta_2 & -\Delta\sigma^{(2)}\cos 2\theta_2 \end{pmatrix} \tag{17}$$

when $\Delta\sigma^{(1)}=\Delta\sigma^{(2)}=\Delta S$, their summation is:

$$\begin{aligned} \begin{pmatrix} \sigma_{x_1} & \tau_{xy_1} \\ \tau_{xy_1} & \sigma_{y_1} \end{pmatrix} &= \begin{pmatrix} p_1+p_2 & 0 \\ 0 & p_1+p_2 \end{pmatrix} \\ &+ \begin{pmatrix} 2\Delta S\cos(\theta_1+\theta_2)\cos(\theta_1-\theta_2) & 2\Delta S\sin(\theta_1+\theta_2)\cos(\theta_1-\theta_2) \\ 2\Delta S\sin(\theta_1+\theta_2)\cos(\theta_1-\theta_2) & -2\Delta S\cos(\theta_1+\theta_2)\cos(\theta_1-\theta_2) \end{pmatrix} \end{aligned} \tag{18}$$

it is easy to show that in this case,

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x-\sigma_y} = \frac{\sin(\theta_1+\theta_2)\cos(\theta_1-\theta_2)}{\cos(\theta_1+\theta_2)\cos(\theta_1-\theta_2)} = \tan(\theta_1+\theta_2) \tag{19}$$

Therefore, the simple mean of θ_1 and θ_2 calculated as $\theta'=(\theta_1+\theta_2)/2$ is right equal to the orientation θ of the principal stress of the summation of the two stress tensors in the condition that maximum shear stresses of each stress tensor are equal. In calculation of the mean of *n* stress tensors, it is required for the same condition $\Delta\sigma^{(i)}=\Delta S$.

4 Discussion and Conclusion

It is summarized that: (1) Stress tensor must be used for description of the stress state, and algebraic operations of addition, decomposition, averaging, *etc*. must be done to the components of the tensor. However, this method is difficult for geologists to visualize their meanings. (2) Using the orientation and magnitude of the principal stresses provide an alternative method to describe the stress state for easy visualization, however, algebraic operation cannot be done to the principal stresses directly. Only firstly transforming the principal stresses into stress components for operation, and then transforming the results back to principal stress expression can obtain the orientation and magnitude of the resultant tensor.

As focal mechanism data is concerned, if the moment tensor of earthquakes are known, it is easy to make operations for the components of the tensor to calculate the average moment as well as the magnitudes and orientation of the principal stresses. However, if we use the initial P wave data, we can get the orientation only, but not magnitude of the stress. In principle, it is usually not feasible to get the orientation by simple average, especially if the orientation angles differ to each other significantly. Such average can be done only if the maximum shear stress of all measurements are equal, and it is difficult to satisfy such a stringent condition. However, if the orientation measurements are relatively clustered, it is acceptable as an approximation, as long as the user keeps alert on the pitfalls.

The conclusion is also valid for some other problems: such as calculating the mean value of stress measurements obtained from several times of overcoring in the same borehole; computation of stress in an stress observatory which has both absolute stress measurement at the beginning, and continuous observation of subsequent stress variations; decomposition of stress into components of long wave length and short wave length; Summation of stress from the background stress and stress changes during certain duration obtained from GPS or InSAR measurements. Great attention must be paid to their algebraic operations, in any case if only the orientation of principal stress is known, but the magnitude of stress tensor is unknown.

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