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Relations between reflection and transmission responses of three-dimensional inhomogeneous media

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SUMMARY

Relations between reflection and transmission responses of horizontally layered media were formulated by Claerbout in 1968 and by many others. In this paper we derive similar relations for 3-D inhomogeneous media. As the starting point for these derivations, we make use of two types of propagation invariants, based on one-way reciprocity theorems of the convolution type and of the correlation type. We obtain relations between reflection and transmission responses, including their codas, due to internal multiple scattering. These relations can be used for deriving the reflection response from transmission measurements (which is useful for seismic imaging of the subsurface, using passive recordings of noise sources in the subsurface, also known as acoustic daylight imaging) as well as for deriving the transmission coda from the reflection measurements (which is useful for seismic imaging schemes that take internal multiple scattering into account). Furthermore, following the same approach, we obtain mutual relations between reflection responses with and without free-surface multiples. The convolution-type relations are similar to those used by Berkhout and others for surface-related multiple elimination, whereas the correlation-type relations resemble Schuster's relations for seismic interferometry. Last, but not least, we obtain expressions for the reflection response at a boundary below an inhomogeneous medium, which may be useful for imaging the medium 'from below'. The main text of this paper deals with the acoustic situation; the Appendices provide extensions to the elastodynamic situation.

Key words: acoustic daylight imaging, coda, propagation invariant, reciprocity, seismic interferometry.

INTRODUCTION

In this paper we present a unified approach for deriving relationships between seismic reflection and transmission responses in 3-D inhomogeneous acoustic and full elastic media. In all relations, the codas due to internal multiple scattering are included. We consider situations with and without a free surface on top of the configuration; below a specific depth level we assume that the medium is homogeneous. Hence, the responses of interest are:

(1) reflection responses at the upper boundary, with and without free-surface multiples at the upper boundary;
(2) transmission responses between the upper and lower boundaries, with and without free-surface multiples at the upper boundary;
(3) reflection responses at the lower boundary, with and without free-surface multiples at the upper boundary.

Apart from relations between reflection and transmission responses we will encounter relations between reflection responses with and without free-surface multiples and relations between reflection responses at the upper and lower boundaries.

The relations between reflection and transmission responses are the basis for deriving the reflection response from the transmission response, and the coda of the transmission response from the reflection response. The former relation is relevant for 'acoustic daylight imaging', that is, for imaging the subsurface from passive recordings of the transmission responses of natural noise sources in the subsurface (Rickett and Claerbout 1999). The latter relation is useful for deriving seismic imaging schemes that take internal multiple scattering into account.

The relations between reflection responses with and without free-surface multiples are the basis for surface-related multiple elimination (Verschuur *et al.* 1992; van Borselen *et al.* 1996) as well as for interferometric imaging, that is, imaging the subsurface from cross-correlated data (Schuster 2001).

Finally, the relations between reflection responses at the upper and lower boundaries may prove to be useful for imaging the medium 'from below', in addition to conventional imaging 'from above'.

Similar relations have been obtained by many authors for acoustic as well as elastodynamic waves in horizontally layered media. We mention Claerbout (1968), Frasier (1970), Kennett *et al.* (1978), Ursin (1983) and Chapman (1994). In the main text of this paper we derive the relations for 3-D inhomogeneous acoustic media. In Appendix A we generalize the results for the full elastic situation.

PROPAGATION INVARIANTS AND RECIPROCITY THEOREMS

Haines (1988), Kennett *et al.* (1990), Koketsu *et al.* (1991) and Takenaka *et al.* (1993) used 'propagation invariants' for 3-D inhomogeneous acoustic and elastic media as the starting point for deriving symmetry properties of reflection and transmission responses as well as for designing efficient numerical modelling schemes. For the acoustic situation the propagation invariant in the space–frequency domain is given by

$$\int_{\Sigma} \{P_A V_{3,B} - V_{3,A} P_B\}\, \mathrm{d}^2\mathbf{x}, \tag{1}$$

where Σ is an arbitrary horizontal integration boundary, $P = P(\mathbf{x}, \omega)$ and $V_3 = V_3(\mathbf{x}, \omega)$ denote the acoustic pressure and the vertical component of the particle velocity, ω denotes the angular frequency, and $\mathbf{x} = (x_1, x_2, x_3)$ the Cartesian coordinate vector (throughout this paper the x_3-axis points downwards). The subscripts A and B refer to two independent acoustic states. The notion 'propagation invariant' refers to the fact that the quantity described by eq. (1) does not change when the depth level of Σ is varied. This is true for lossless as well as dissipative media; the only condition is that Σ should not encounter any sources on its journey. Hence, for a source-free 3-D inhomogeneous region enclosed by horizontal boundaries $\partial\mathcal{D}_0$ and $\partial\mathcal{D}_\mathrm{m}$, we can write

$$\int_{\partial\mathcal{D}_0} \{P_A V_{3,B} - V_{3,A} P_B\}\, \mathrm{d}^2\mathbf{x} = \int_{\partial\mathcal{D}_\mathrm{m}} \{P_A V_{3,B} - V_{3,A} P_B\}\, \mathrm{d}^2\mathbf{x}. \tag{2}$$

The procedure for deriving the symmetry properties of reflection and transmission responses employed by the above-mentioned authors can be summarized as follows.

(1) Apply plane-wave decomposition to all the wavefields in eq. (2) and use Parseval's theorem to transform eq. (2) to the horizontal wavenumber domain.

(2) Assume that the inhomogeneous region is embedded between homogeneous half-spaces above $\partial\mathcal{D}_0$ and below $\partial\mathcal{D}_\mathrm{m}$. Express the wavefields in the transformed eq. (2) in terms of downgoing and upgoing waves, using wavefield decomposition operators.

(3) Introduce reflection and transmission operators that interrelate the downgoing and upgoing wavefields in the transformed eq. (2). With these substitutions, eq. (2) leads to symmetry relations for the reflection and transmission responses defined at the boundaries $\partial\mathcal{D}_0$ and $\partial\mathcal{D}_\mathrm{m}$ of the inhomogeneous region.

(4) Optionally, when the inhomogeneous region only contains a mildly curved interface that obeys the conditions for the validity of the Rayleigh hypothesis, the integration boundaries $\partial\mathcal{D}_0$ and $\partial\mathcal{D}_\mathrm{m}$ can be moved to one and the same depth level between the boundaries, without changing the character of the downgoing and upgoing waves under the integrals. This leads to symmetry relations for the reflection and transmission responses of the curved interface, all defined at the same depth level.

It is worth noting that the propagation invariant can be seen as a special case of Rayleigh's acoustic reciprocity theorem (de Hoop 1988). Since reciprocity theorems have been derived in various forms, this enables us to formulate a number of alternatives for eq. (2) that have not been considered by the authors mentioned above.

First of all, we can distinguish between reciprocity theorems of the convolution-type and of the correlation-type (Bojarski 1983). Eq. (2) is a special case of the convolution-type reciprocity theorem (the products $P_A V_{3,B}$ etc. in the frequency domain correspond to convolutions in the time domain). From the reciprocity theorem of the correlation-type we obtain

$$\int_{\partial\mathcal{D}_0} \left\{P_A^* V_{3,B} + V_{3,A}^* P_B\right\} \mathrm{d}^2\mathbf{x} = \int_{\partial\mathcal{D}_\mathrm{m}} \left\{P_A^* V_{3,B} + V_{3,A}^* P_B\right\} \mathrm{d}^2\mathbf{x}, \tag{3}$$

where * denotes complex conjugation (the products $P_A^* V_{3,B}$ etc. in the frequency domain correspond to correlations in the time domain). Again, the medium between $\partial\mathcal{D}_0$ and $\partial\mathcal{D}_\mathrm{m}$ is arbitrarily inhomogeneous and source-free. Unlike eq. (2), which is valid for lossless as well as dissipative media, however, eq. (3) is valid only for lossless media.

Furthermore, reciprocity theorems can be formulated in terms of two-way and one-way wavefields. Eqs (2) and (3) are both expressed in terms of two-way wavefields. Reciprocity theorems for one-way (that is, downgoing and upgoing) wavefields have been derived by Wapenaar & Grimbergen (1996). For these one-way reciprocity theorems, too, we distinguish between a convolution-type and a correlation-type theorem. From the reciprocity theorem of the convolution-type for one-way wavefields we obtain, analogous to eq. (2),

$$\int_{\partial\mathcal{D}_0} \left\{P_A^+ P_B^- - P_A^- P_B^+\right\} \mathrm{d}^2\mathbf{x} = \int_{\partial\mathcal{D}_\mathrm{m}} \left\{P_A^+ P_B^- - P_A^- P_B^+\right\} \mathrm{d}^2\mathbf{x}, \tag{4}$$

where $P^+ = P^+(\mathbf{x}, \omega)$ and $P^- = P^-(\mathbf{x}, \omega)$ are flux-normalized downgoing and upgoing wavefields, respectively. Eq. (4) holds for primary and multiply reflected waves with any propagation angle (including evanescent wave modes) in 3-D lossless or dissipative inhomogeneous media. Finally, from the one-way reciprocity theorem of the correlation-type we obtain, analogous to eq. (3),

$$\int_{\partial\mathcal{D}_0} \left\{(P_A^+)^* P_B^+ - (P_A^-)^* P_B^-\right\} \mathrm{d}^2\mathbf{x} = \int_{\partial\mathcal{D}_\mathrm{m}} \left\{(P_A^+)^* P_B^+ - (P_A^-)^* P_B^-\right\} \mathrm{d}^2\mathbf{x}. \tag{5}$$

Here it is assumed that the medium is lossless and that evanescent wave modes can be neglected. As a consequence, any result obtained from eq. (5) will be spatially band-limited (we will see an effect of this in Fig. 3b). In what follows, when we speak of two-way reciprocity theorems we refer to eqs (2) and (3), whereas when we speak of one-way reciprocity theorems we refer to eqs (4) and (5).

In this paper we will use the convolution-type and correlation-type one-way reciprocity theorems with various combinations of boundary conditions to derive relations between reflection and transmission responses. The choice for the one-way theorems is mainly a matter of convenience. Since reflection and transmission responses can be seen as transfer functions between downgoing and/or upgoing one-way wavefields, the one-way reciprocity theorems lead in a straightforward manner to relations between these responses. For example, from the step-wise procedure discussed above, only step 3 is required [with eq. (2) in the wavenumber domain replaced by eq. (4) in the space domain] to establish the symmetry relations for the reflection and transmission responses. This is reviewed in the section 'source–receiver reciprocity'. Our use of correlation-type reciprocity theorems leads to results that are essentially different from those derived by the authors mentioned at the beginning of this section. Furthermore, by employing various combinations of boundary conditions (see the next section) we differentiate from previous work. As a result, we obtain a number of new relations between reflection and transmission responses that lead to a number of potential applications in seismic processing. One of the applications, acoustic daylight imaging, will be discussed in detail.

SPECIFICATION OF THE STATES FOR THE ONE-WAY RECIPROCITY THEOREMS

In this section we define the states A and B that will be used in the one-way reciprocity theorems for deriving the relations between the reflection and transmission responses. As mentioned in the Introduction, we consider responses with and without free-surface multiples. For the situation without free-surface multiples, consider the configuration denoted as State $A-1$ in Fig. 1. The boundaries $\partial\mathcal{D}_0$ and $\partial\mathcal{D}_\mathrm{m}$ are chosen at depth levels $x_{3,0}+\epsilon$ and $x_{3,\mathrm{m}}-\epsilon$, respectively (with $x_{3,\mathrm{m}} > x_{3,0}$ and ϵ a vanishing positive constant). The half-spaces above $\partial\mathcal{D}_0$ and below $\partial\mathcal{D}_\mathrm{m}$ are homogeneous; the medium in domain $\mathcal{D}$ between boundaries $\partial\mathcal{D}_0$ and $\partial\mathcal{D}_\mathrm{m}$ is 3-D inhomogeneous and source-free. We

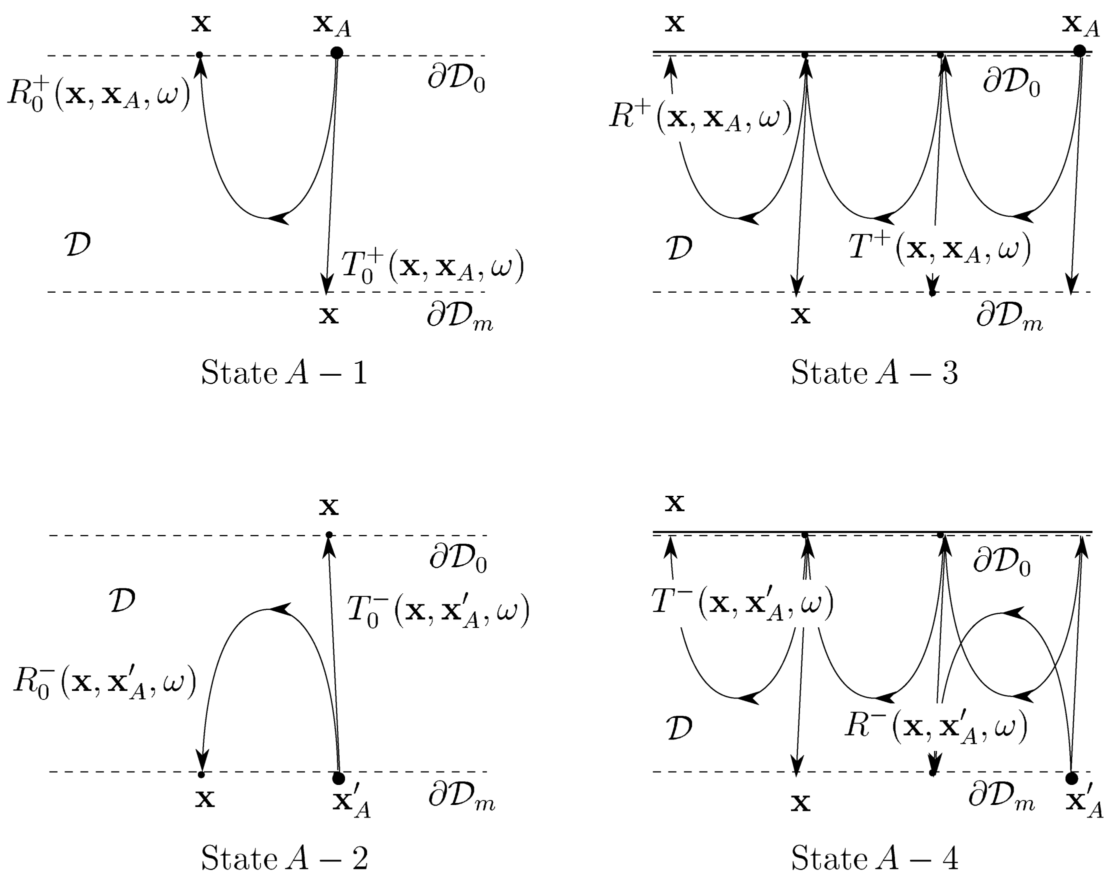


Figure 1. States for the acoustic one-way reciprocity theorems. State A-1: at $\mathbf{x}_A$ just above $\partial\mathcal{D}_0$ there is a source of downgoing waves. The half-spaces above $\partial\mathcal{D}_0$ and below $\partial\mathcal{D}_\mathrm{m}$ are homogeneous. The half-space below $\partial\mathcal{D}_\mathrm{m}$ is source-free. State A-2: as state A-1, but with a source of upgoing waves at $\mathbf{x}'_A$ just below $\partial\mathcal{D}_\mathrm{m}$. State A-3: as state A-1, but with a free surface just above $\partial\mathcal{D}_0$. State A-4: as state A-3, but with a source of upgoing waves at $\mathbf{x}'_A$ just below $\partial\mathcal{D}_\mathrm{m}$.

choose a source of downgoing waves at $\mathbf{x}_A = (\mathbf{x}_{\mathrm{H},A}, x_{3,0})$, just above the upper boundary $\partial\mathcal{D}_0$. Here we used the subscript 'H' to denote the horizontal coordinates; that is, $\mathbf{x}_{\mathrm{H}} = (x_1, x_2)$. Hence, $\mathbf{x}_{\mathrm{H},A}$ denotes the horizontal coordinates of $\mathbf{x}_A$; that is, $\mathbf{x}_{\mathrm{H},A} = (x_{1,A}, x_{2,A})$. The half-space below $\partial\mathcal{D}_{\mathrm{m}}$ is considered source-free. Hence, in the space–frequency domain, the downgoing and upgoing wavefields in state A–1 for any point $\mathbf{x}$ at the upper boundary $\partial\mathcal{D}_0$ read

$$P_A^+(\mathbf{x}, \mathbf{x}_A, \omega) = \delta(\mathbf{x}_{\mathrm{H}} - \mathbf{x}_{\mathrm{H},A}) s_A(\omega), \tag{6}$$

$$P_A^-(\mathbf{x}, \mathbf{x}_A, \omega) = R_0^+(\mathbf{x}, \mathbf{x}_A, \omega) s_A(\omega), \tag{7}$$

where $s_A(\omega)$ is the source spectrum of the source at $\mathbf{x}_A$. $R_0^+(\mathbf{x}, \mathbf{x}_A, \omega)$ is the reflection response of the inhomogeneous medium in $\mathcal{D}$, including all internal multiples, for a source at $\mathbf{x}_A$ and a receiver at $\mathbf{x}$. The subscript '0' denotes that no free-surface multiples are included; the superscript '+' denotes that this is the response to a downgoing incident wavefield. Similarly, for any $\mathbf{x}$ at the lower boundary $\partial\mathcal{D}_{\mathrm{m}}$ we have

$$P_A^+(\mathbf{x}, \mathbf{x}_A, \omega) = T_0^+(\mathbf{x}, \mathbf{x}_A, \omega) s_A(\omega), \tag{8}$$

$$P_A^-(\mathbf{x}, \mathbf{x}_A, \omega) = 0, \tag{9}$$

where $T_0^+(\mathbf{x}, \mathbf{x}_A, \omega)$ is the transmission response of the inhomogeneous medium in $\mathcal{D}$, including all internal multiples, for a source at $\mathbf{x}_A$ and a receiver at $\mathbf{x}$. Eqs (6)–(9) are summarized in Table 1 (state A–1).

Next we consider, in the same configuration, a source of upgoing waves at $\mathbf{x}'_A = (\mathbf{x}_{\mathrm{H},A'}, x_{3,\mathrm{m}})$, just below the lower boundary $\partial\mathcal{D}_{\mathrm{m}}$. The half-space above $\partial\mathcal{D}_0$ is considered source-free. This is denoted as state A-2 in Fig. 1 and Table 1. Here $T_0^-(\mathbf{x}, \mathbf{x}'_A, \omega)$ and $R_0^-(\mathbf{x}, \mathbf{x}'_A, \omega)$ are the transmission and reflection responses of the inhomogeneous medium in $\mathcal{D}$, including all internal multiples. The subscript '0' again denotes that no free-surface multiples are included; the superscript '-' denotes that these are responses to an upgoing incident wavefield.

In state A-3 we consider a situation with free-surface multiples. We define a free surface at $x_{3,0}$, that is, just above $\partial\mathcal{D}_0$, and a homogeneous source-free half-space below $\partial\mathcal{D}_{\mathrm{m}}$. We choose a source of downgoing waves at $\mathbf{x}_A = (\mathbf{x}_{\mathrm{H},A}, x_{3,0})$, that is, at the free surface. The total downgoing wavefield at $\partial\mathcal{D}_0$ consists of the superposition of a spatial delta function directly below the source and the downward-reflected upgoing wavefield due to the presence of the free surface; see Table 1, state A-3, first equation. Here r^- denotes the reflection coefficient of the free surface for upgoing waves, which is given by $r^- = -1$. $R^+(\mathbf{x}, \mathbf{x}_A, \omega)$ and $T^+(\mathbf{x}, \mathbf{x}_A, \omega)$ are the reflection and transmission responses of the inhomogeneous medium in $\mathcal{D}$, including all internal multiples as well as the free-surface multiples (denoted by the absence of subscript '0').

Finally, we consider in the same configuration (that is, with the free surface at $x_{3,0}$) a source of upgoing waves at $\mathbf{x}'_A = (\mathbf{x}'_{\mathrm{H},A}, x_{3,\mathrm{m}})$, just below the lower boundary $\partial\mathcal{D}_{\mathrm{m}}$. We denote this as state A-4 in Fig. 1 and Table 1. There are no sources at the free surface. $T^-(\mathbf{x}, \mathbf{x}'_A, \omega)$ and $R^-(\mathbf{x}, \mathbf{x}'_A, \omega)$ are the transmission and reflection responses of the inhomogeneous medium in $\mathcal{D}$, including all internal multiples as well as the free-surface multiples.

States B-1 to B-4 are defined similarly to states A-1 to A-4, except that the source points $\mathbf{x}_A$ and $\mathbf{x}'_A$ are replaced by $\mathbf{x}_B$ and $\mathbf{x}'_B$, respectively. Note that, in all the states considered, the domain $\mathcal{D}$ is source-free. This complies with the conditions for the one-way reciprocity theorems defined by eqs (4) and (5). Hence, these reciprocity theorems can be applied to find relations between any of the states A-1 to A-4 on the one hand and any of the states B-1 to B-4 on the other. In the following sections we make a selection of combinations that lead to useful relations between reflection and/or transmission responses.

Table 1. States for the acoustic one-way reciprocity theorems.

State A-1:	source point $\mathbf{x}_A$ just above $\partial\mathcal{D}_0$; homogeneous half-spaces above $\partial\mathcal{D}_0$ and below $\partial\mathcal{D}_{\mathrm{m}}$; no sources below $\partial\mathcal{D}_{\mathrm{m}}$	State A-3:	source point $\mathbf{x}_A$ just above $\partial\mathcal{D}_0$; free surface just above $\partial\mathcal{D}_0$; homogeneous source-free half-space below $\partial\mathcal{D}_{\mathrm{m}}$
$\mathbf{x} \in \partial\mathcal{D}_0$	$P_A^+(\mathbf{x}, \mathbf{x}_A, \omega) = \delta(\mathbf{x}_H - \mathbf{x}_{\mathrm{H},A}) s_A(\omega)$ $P_A^-(\mathbf{x}, \mathbf{x}_A, \omega) = R_0^+(\mathbf{x}, \mathbf{x}_A, \omega) s_A(\omega)$	$\mathbf{x} \in \partial\mathcal{D}_0$	$P_A^+(\mathbf{x}, \mathbf{x}_A, \omega) = \delta(\mathbf{x}_{\mathrm{H}} - \mathbf{x}_{\mathrm{H},A}) s_A(\omega)$ $+ r^- P_A^-(\mathbf{x}, \mathbf{x}_A, \omega)$ $P_A^-(\mathbf{x}, \mathbf{x}_A, \omega) = R^+(\mathbf{x}, \mathbf{x}_A, \omega) s_A(\omega)$
$\mathbf{x} \in \partial\mathcal{D}_{\mathrm{m}}$	$P_A^+(\mathbf{x}, \mathbf{x}_A, \omega) = T_0^+(\mathbf{x}, \mathbf{x}_A, \omega) s_A(\omega)$ $P_A^-(\mathbf{x}, \mathbf{x}_A, \omega) = 0$	$\mathbf{x} \in \partial\mathcal{D}_{\mathrm{m}}$	$P_A^+(\mathbf{x}, \mathbf{x}_A, \omega) = T^+(\mathbf{x}, \mathbf{x}_A, \omega) s_A(\omega)$ $P_A^-(\mathbf{x}, \mathbf{x}_A, \omega) = 0$
State A-2:	source point $\mathbf{x}'_A$ just below $\partial\mathcal{D}_{\mathrm{m}}$; homogeneous half-spaces above $\partial\mathcal{D}_0$ and below $\partial\mathcal{D}_{\mathrm{m}}$; no sources above $\partial\mathcal{D}_0$	State A-4:	source point $\mathbf{x}'_A$ just below $\partial\mathcal{D}_{\mathrm{m}}$; free surface, no sources, just above $\partial\mathcal{D}_0$ homogeneous half-space below $\partial\mathcal{D}_{\mathrm{m}}$
$\mathbf{x} \in \partial\mathcal{D}_0$	$P_A^+(\mathbf{x}, \mathbf{x}'_A, \omega) = 0$ $P_A^-(\mathbf{x}, \mathbf{x}'_A, \omega) = T_0^-(\mathbf{x}, \mathbf{x}'_A, \omega) s_A(\omega)$	$\mathbf{x} \in \partial\mathcal{D}_0$	$P_A^+(\mathbf{x}, \mathbf{x}'_A, \omega) = r^- P_A^-(\mathbf{x}, \mathbf{x}'_A, \omega)$ $P_A^-(\mathbf{x}, \mathbf{x}'_A, \omega) = T^-(\mathbf{x}, \mathbf{x}'_A, \omega) s_A(\omega)$
$\mathbf{x} \in \partial\mathcal{D}_{\mathrm{m}}$	$P_A^+(\mathbf{x}, \mathbf{x}'_A, \omega) = R_0^-(\mathbf{x}, \mathbf{x}'_A, \omega) s_A(\omega)$ $P_A^-(\mathbf{x}, \mathbf{x}'_A, \omega) = \delta(\mathbf{x}_{\mathrm{H}} - \mathbf{x}'_{\mathrm{H},A}) s_A(\omega)$	$\mathbf{x} \in \partial\mathcal{D}_{\mathrm{m}}$	$P_A^+(\mathbf{x}, \mathbf{x}'_A, \omega) = R^-(\mathbf{x}, \mathbf{x}'_A, \omega) s_A(\omega)$ $P_A^-(\mathbf{x}, \mathbf{x}'_A, \omega) = \delta(\mathbf{x}_{\mathrm{H}} - \mathbf{x}'_{\mathrm{H},A}) s_A(\omega)$

SOURCE–RECEIVER RECIPROCITY

In this section we derive source–receiver reciprocity relations. Although these relations are well known for full wavefields (that is, solutions of the two-way wave equation), they are less obvious for one-way wavefields (that is, solutions of the coupled one-way wave equations for downgoing and upgoing waves). Note that, if pressure-normalization were used instead of flux-normalization for the decomposition of the full wavefield into one-way wavefields in inhomogeneous media, reciprocity would not hold for the one-way wavefields. Flux-normalization was used in the derivation of the one-way reciprocity theorems (4) and (5), which explains their relatively simple form. We will now demonstrate that these theorems lead to source–receiver reciprocity relations for the reflection and transmission responses introduced in the previous section.

We substitute the expressions for the one-way wavefields of states A-1 and B-1 (Table 1) into the one-way convolution-type reciprocity theorem (eq. 4). Dividing the result by $s_A(\omega)s_B(\omega)$ we obtain

$$\int_{\partial\mathcal{D}_0} \left[\delta(\mathbf{x}_\mathrm{H} - \mathbf{x}_{\mathrm{H},A}) R_0^+(\mathbf{x}_\mathrm{H}, x_{3,0}, \mathbf{x}_B, \omega) - R_0^+(\mathbf{x}_\mathrm{H}, x_{3,0}, \mathbf{x}_A, \omega)\delta(\mathbf{x}_\mathrm{H} - \mathbf{x}_{\mathrm{H},B})\right] \mathrm{d}^2\mathbf{x} = 0, \tag{10}$$

or

$$R_0^+(\mathbf{x}_A, \mathbf{x}_B, \omega) = R_0^+(\mathbf{x}_B, \mathbf{x}_A, \omega), \tag{11}$$

for $\mathbf{x}_A$ and $\mathbf{x}_B$ just above $\partial\mathcal{D}_0$. This equation describes source–receiver reciprocity for the reflection response without the free-surface multiples, observed just above the boundary $\partial\mathcal{D}_0$.

A similar result for the reflection response observed just below the lower boundary $\partial\mathcal{D}_\mathrm{m}$ is obtained by substitution of states A-2 and B-2, yielding

$$R_0^-(\mathbf{x}'_A, \mathbf{x}'_B, \omega) = R_0^-(\mathbf{x}'_B, \mathbf{x}'_A, \omega), \tag{12}$$

for $\mathbf{x}'_A$ and $\mathbf{x}'_B$ just below $\partial\mathcal{D}_\mathrm{m}$. Substitution of states A-3 and B-3 into eq. (4) yields for the reflection response including the free-surface multiples, observed at the free surface,

$$R^+(\mathbf{x}_A, \mathbf{x}_B, \omega) = R^+(\mathbf{x}_B, \mathbf{x}_A, \omega). \tag{13}$$

Similarly, combining states A-4 and B-4 yields

$$R^-(\mathbf{x}'_A, \mathbf{x}'_B, \omega) = R^-(\mathbf{x}'_B, \mathbf{x}'_A, \omega). \tag{14}$$

Finally, source–receiver reciprocity for the transmission responses is obtained by substituting either states A-2 and B-1 or states A-4 and B-3 into eq. (4), yielding

$$T_0^+(\mathbf{x}'_A, \mathbf{x}_B, \omega) = T_0^-(\mathbf{x}_B, \mathbf{x}'_A, \omega) \tag{15}$$

and

$$T^+(\mathbf{x}'_A, \mathbf{x}_B, \omega) = T^-(\mathbf{x}_B, \mathbf{x}'_A, \omega). \tag{16}$$

Eqs (11), (12) and (15) were derived in the wavenumber domain by Haines (1988), following the stepwise procedure outlined in a previous section.

FROM TRANSMISSION TO REFLECTION

In this section we derive the first relation between reflection and transmission responses for the situation with free-surface multiples and we discuss how this relation can be used to derive the reflection response from the transmission response.

We substitute the expressions for the one-way wavefields of states A-3 and B-3 (Table 1) into the one-way correlation-type reciprocity theorem (eq. 5). Dividing the result by $s_A^*(\omega)s_B(\omega)$ we obtain

$$R^+(\mathbf{x}_A, \mathbf{x}_B, \omega) + \left\{R^+(\mathbf{x}_B, \mathbf{x}_A, \omega)\right\}^* = \delta(\mathbf{x}_{\mathrm{H},B} - \mathbf{x}_{\mathrm{H},A}) - \int_{\partial\mathcal{D}_\mathrm{m}} \left\{T^+(\mathbf{x}, \mathbf{x}_A, \omega)\right\}^* T^+(\mathbf{x}, \mathbf{x}_B, \omega)\, \mathrm{d}^2\mathbf{x}, \tag{17}$$

for $\mathbf{x}_A$ and $\mathbf{x}_B$ at the free surface, which is situated just above $\partial\mathcal{D}_0$. Note that this equation is not exact, since evanescent wave modes have been neglected in the derivation of the one-way correlation-type reciprocity theorem. Using the source–receiver reciprocity relations (13) and (16), we can rewrite eq. (17) as

$$2\Re\left[R^+(\mathbf{x}_A, \mathbf{x}_B, \omega)\right] = \delta(\mathbf{x}_{\mathrm{H},B} - \mathbf{x}_{\mathrm{H},A}) - \int_{\partial\mathcal{D}_\mathrm{m}} \{T^-(\mathbf{x}_A, \mathbf{x}, \omega)\}^* T^-(\mathbf{x}_B, \mathbf{x}, \omega)\, \mathrm{d}^2\mathbf{x}, \tag{18}$$

where $\Re\{\cdot\}$ denotes the real part. Eq. (18) is an explicit expression for the real part of the reflection response $R^+(\mathbf{x}_A, \mathbf{x}_B, \omega)$ in terms of the transmission responses $T^-(\mathbf{x}_A, \mathbf{x}, \omega)$ and $T^-(\mathbf{x}_B, \mathbf{x}, \omega)$ for a range of $\mathbf{x}$-values at $\partial\mathcal{D}_\mathrm{m}$ (which can, for example, be obtained from passive measurements of natural noise sources in the subsurface—see the discussion below). Since the reflection response in the time domain is causal, the imaginary part of $R^+(\mathbf{x}_A, \mathbf{x}_B, \omega)$ is obtained via the Hilbert transform of the real part. This is equivalent to transforming $2\Re\{R^+(\mathbf{x}_A, \mathbf{x}_B, \omega)\}$ into the time domain and setting the non-causal part to zero.

Eq. (18) is illustrated with a 2-D numerical experiment. The transmission response $T^-(\mathbf{x}_A, \mathbf{x}, \omega)$ of the syncline model in Fig. 2 is shown in the time domain in Fig. 3(a) for a fixed source at $\mathbf{x} = (0, 800)$ and a range of receiver positions $\mathbf{x}_A$ at the acquisition surface. Fig.

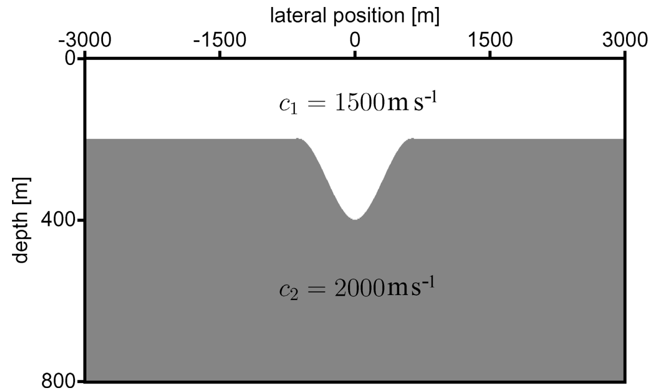


Figure 2. Syncline model, used in the numerical experiments.

3(b) shows the result (again in the time domain) of the integral on the right-hand side of eq. (18) (times -1), for $\mathbf{x}_B = (0, 0)$ and all $\mathbf{x}_A$ at the acquisition surface. The cross in the centre is a band-limited representation of the delta function (times -1) on the right-hand side of eq. (18) (bear in mind that evanescent wave modes were neglected). Fig. 3(c) shows the causal part of the data in Fig. 3(b), after muting the band-limited delta function. This is the reflection response $R^+(\mathbf{x}_A, \mathbf{x}_B, \omega)$ for a source at $\mathbf{x}_B = (0, 0)$ and receivers at all $\mathbf{x}_A$ at the acquisition surface. This result matches quite accurately the amplitude and phase of the directly modelled reflection response of the syncline model (Fig. 3d).

For the 1-D situation eq. (18) reduces to $2\Re[R^+(\omega)] = 1 - \{T^-(\omega)\}^* T^-(\omega)$. This result was previously derived (in the z-transform domain) by Claerbout (1968). Assuming that the 1-D assumption is justified, this equation implies that, when a natural noise source in the Earth's subsurface emits plane waves to the surface, passive measurements of the noise at the surface suffice to compute the reflection response of the Earth's subsurface. The seismic wavelet in this synthesized reflection response is the autocorrelation of the noise source in the subsurface. The principle of using passive noise measurements to derive the reflection response and, subsequently, to form an image of the Earth's interior, was termed 'acoustic daylight imaging' by Claerbout.

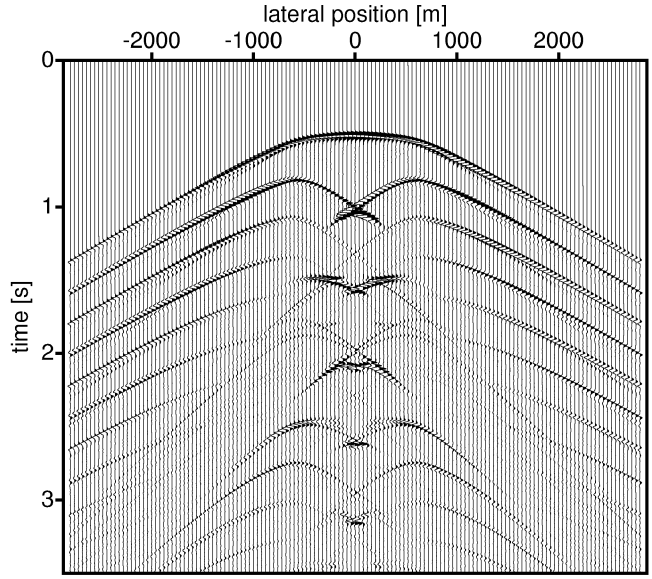
Later, Claerbout conjectured for the 3-D situation that, 'by cross-correlating noise traces recorded at two locations on the surface, we can construct the wavefield that would be recorded at one of the locations if there was a source at the other'. Rickett and Claerbout (1999) discuss several applications of this principle (including helioseismology and reservoir monitoring) as well as using numerical modelling to confirm the conjecture. Note that with eq. (18) we nearly achieved a proof of Claerbout's conjecture. The term $\{T^-(\mathbf{x}_A, \mathbf{x}, \omega)\}^* T^-(\mathbf{x}_B, \mathbf{x}, \omega)$ represents the cross-correlation of traces recorded at two locations ($\mathbf{x}_A$ and $\mathbf{x}_B$) on the surface for a source at $\mathbf{x}$ in the subsurface; the term $R^+(\mathbf{x}_A, \mathbf{x}_B, \omega)$ is the wavefield that would be recorded at one of the locations ($\mathbf{x}_A$) if there were a source at the other ($\mathbf{x}_B$). The main discrepancy between eq. (18) and Claerbout's conjecture is the integral in eq. (18) over all possible source positions $\mathbf{x}$ at surface $\partial\mathcal{D}_\mathrm{m}$. It cannot be evaluated in practice because the transmission responses are not available for all individual source positions $\mathbf{x}$. However, if we assume uncorrelated white noise sources at $\partial\mathcal{D}_\mathrm{m}$, eq. (18) can be rewritten as

$$2\Re\big[R^+(\mathbf{x}_A, \mathbf{x}_B, \omega)\big] = \delta(\mathbf{x}_{\mathrm{H},B} - \mathbf{x}_{\mathrm{H},A}) - \big\{T^-_{\mathrm{obs}}(\mathbf{x}_A, \omega)\big\}^* T^-_{\mathrm{obs}}(\mathbf{x}_B, \omega), \tag{19}$$

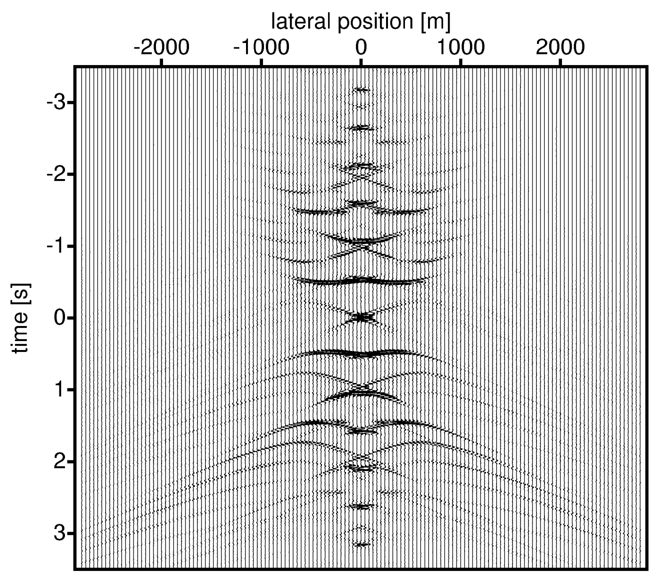
where $T^-_{\mathrm{obs}}(\mathbf{x}_A, \omega)$ and $T^-_{\mathrm{obs}}(\mathbf{x}_B, \omega)$ are the transmission responses, observed at $\mathbf{x}_A$ and $\mathbf{x}_B$ on $\partial\mathcal{D}_0$, due to a distribution of uncorrelated white noise sources along $\partial\mathcal{D}_\mathrm{m}$. Of course in reality the noise sources will not be evenly distributed along a single surface $\partial\mathcal{D}_\mathrm{m}$ (see Fig. 4). However, the actual depth of the sources is almost immaterial, since the extra traveltime between the actual source depth and $\partial\mathcal{D}_\mathrm{m}$ drops out in the correlation process. This completes the proof of Claerbout's conjecture. Eq. (19) is illustrated with a numerical experiment, again for the syncline model of Fig. 2. The transmission response $T^-_{\mathrm{obs}}(\mathbf{x}_A, \omega)$ of a distribution of noise sources below the syncline is shown in the time domain in Fig. 5(a) for a range of receiver positions $\mathbf{x}_A$ at the acquisition surface. Only the first 3.5 s of the noise recordings are shown; the actual recordings cover 66 min. Fig. 5(b) shows the result (again in the time domain) of cross-correlating the central trace of Fig. 5(a) with all other traces (times -1) and taking the causal part and muting the band-limited delta function. According to eq. (19) this again represents the reflection response $R^+(\mathbf{x}_A, \mathbf{x}_B, \omega)$ for a source at $\mathbf{x}_B = (0, 0)$ and receivers at all $\mathbf{x}_A$ at the acquisition surface (compare with Fig. 3d). Fig. 5(b) was obtained by cross-correlating only the first 10 min of the noise recordings. The experiment was repeated taking the full 66 min of the noise recordings into account. From the result, which is shown in Fig. 5(c), it is clear that using longer noise recordings leads to a better signal-to-noise ratio in the simulated reflection response.

FROM REFLECTION TO TRANSMISSION

In this section we derive the second relation between reflection and transmission responses, this time for the situation without free-surface multiples. We substitute the expressions for the one-way wavefields of states A-1 and B-1 (Table 1) into the one-way correlation-type reciprocity



(a)



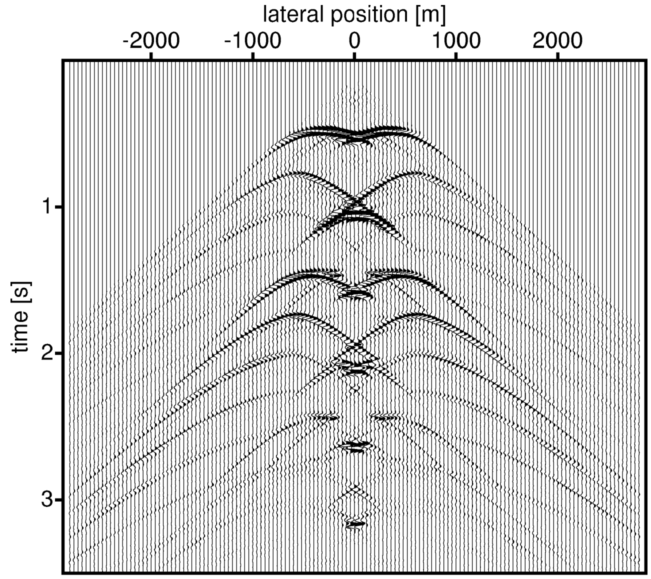
(b)

Figure 3. From transmission to reflection: illustration of eq. (18) for the 2-D medium in Fig. 2. (a) Transmission response. (b) Result of the integral on the right-hand side of eq. (18). (c) Causal part of the data in (b), after muting the band-limited delta function. This is the reflection response $R^+(\mathbf{x}_A, \mathbf{x}_B, \omega)$. (d) For comparison, the directly modelled reflection response.

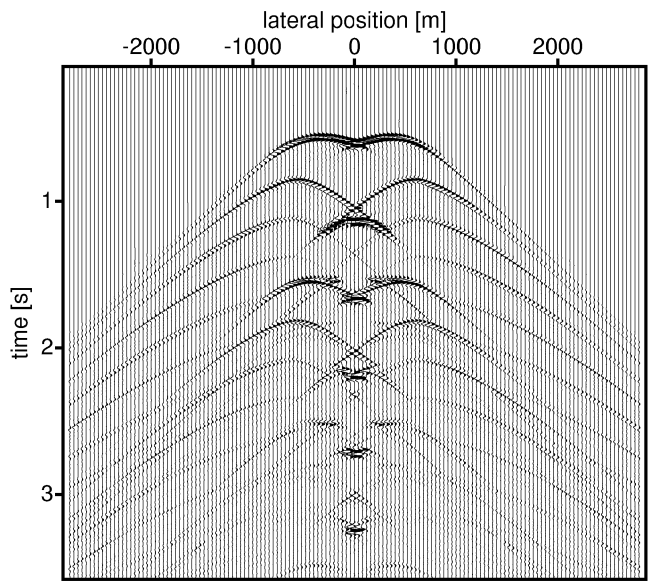
theorem (eq. 5). Dividing the result by $s_A^*(\omega)s_B(\omega)$ we obtain

$$\int_{\partial\mathcal{D}_\mathrm{m}} \left\{T_0^+(\mathbf{x}, \mathbf{x}_A, \omega)\right\}^* T_0^+(\mathbf{x}, \mathbf{x}_B, \omega)\, \mathrm{d}^2\mathbf{x} = \delta(\mathbf{x}_{\mathrm{H},B} - \mathbf{x}_{\mathrm{H},A}) - \int_{\partial\mathcal{D}_0} \left\{R_0^+(\mathbf{x}, \mathbf{x}_A, \omega)\right\}^* R_0^+(\mathbf{x}, \mathbf{x}_B, \omega)\, \mathrm{d}^2\mathbf{x}, \tag{20}$$

for $\mathbf{x}_A$ and $\mathbf{x}_B$ just above $\partial\mathcal{D}_0$. Note that this equation is not exact, since evanescent wave modes are neglected in the derivation of the one-way correlation-type reciprocity theorem. Similar expressions have been derived by Herman (1992) using the two-way reciprocity theorem (eq. 3) and by Wapenaar & Herrmann (1993) using the one-way reciprocity theorem (eq. 5). There is no unique way to resolve the transmission response $T_0^+(\mathbf{x}, \mathbf{x}_A, \omega)$ from the left-hand side of eq. (20). In Wapenaar *et al.* (2003) we discuss how to resolve the coda of the transmission response from the left-hand side of eq. (20), given the cross-correlation of the reflection response (the right-hand side of eq. 20) for a large range of source positions $\mathbf{x}_A$ and $\mathbf{x}_B$ at $x_{3,0}$. Note that the inverse of the transmission coda, in combination with an inverse primary propagator, can be used in seismic reflection imaging to obtain an image in which the internal multiple scattering effects are suppressed. In this approach, the inverse primary propagator is estimated from the traveltime information in the data (as usual), whereas the inverse transmission coda is



(c)



(d)

Figure 3. (*Continued*).

obtained from the cross-correlation of the reflection measurements. By way of comparison, we note that in the imaging scheme proposed by Weglein *et al.* (2000) the full inverse operator (primaries as well as internal multiples) is estimated directly from the reflection measurements. The advantages and disadvantages of the two methods with respect to accuracy, stability, etc. remain to be investigated.

EXPRESSIONS FOR FREE-SURFACE MULTIPLES

To find a relation between the reflection responses with and without free-surface multiples, we substitute the expressions for the one-way wavefields of states A-1 and B-3 (Table 1) into the one-way convolution-type reciprocity theorem (eq. 4). Applying the source–receiver reciprocity relation (11) to the result we obtain

$$R_0^+(\mathbf{x}_A, \mathbf{x}_B, \omega) - R^+(\mathbf{x}_A, \mathbf{x}_B, \omega) = \int_{\partial\mathcal{D}_0} R_0^+(\mathbf{x}_A, \mathbf{x}, \omega) R^+(\mathbf{x}, \mathbf{x}_B, \omega)\, \mathrm{d}^2\mathbf{x}, \tag{21}$$

for $\mathbf{x}_A$ and $\mathbf{x}_B$ just above $\partial\mathcal{D}_0$. When the reflection response $R_0^+(\mathbf{x}_A, \mathbf{x}_B, \omega)$ without free-surface multiples is known (for example as a result of numerical seismic modelling), eq. (21) is an integral equation of the second kind for the reflection response $R^+(\mathbf{x}_A, \mathbf{x}_B, \omega)$ with

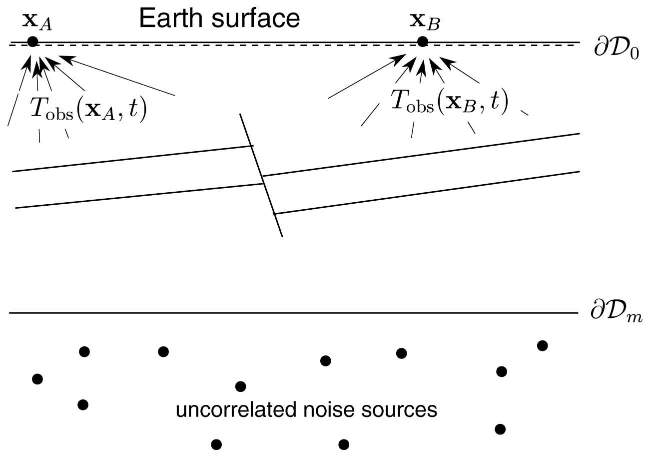


Figure 4. Transmission responses, observed at $\mathbf{x}_A$ and $\mathbf{x}_B$, due to a distribution of uncorrelated white noise sources. According to eq. (19), their cross-correlation yields the reflection response observed at $\mathbf{x}_A$, as if there were an impulsive point source at $\mathbf{x}_B$.

free-surface multiples. On the other hand, when the response $R^+(\mathbf{x}_A, \mathbf{x}_B, \omega)$ with free-surface multiples is known (from decomposed seismic measurements), eq. (21) is an integral equation of the second kind for the response $R_0^+(\mathbf{x}_A, \mathbf{x}_B, \omega)$ without free-surface multiples. A method for free-surface multiple elimination, based on a similar type of equation but derived in a different way, has been proposed by Berkhout (1982) and implemented by Verschuur *et al.* (1992). Fokkema & van den Berg (1993) used the two-way reciprocity theorem to derive a similar multiple elimination scheme, which has been implemented by van Borselen *et al.* (1996). Eq. (21) can be seen as the 'one-way counterpart' of the result of Fokkema & van den Berg (1993).

Next we substitute states *A*-2 and *B*-3 into eq. (4). Applying the source–receiver reciprocity relation (15) to the result we obtain

$$T_0^+(\mathbf{x}'_A, \mathbf{x}_B, \omega) - T^+(\mathbf{x}'_A, \mathbf{x}_B, \omega) = \int_{\partial\mathcal{D}_0} T_0^+(\mathbf{x}'_A, \mathbf{x}, \omega) R^+(\mathbf{x}, \mathbf{x}_B, \omega)\, \mathrm{d}^2\mathbf{x}, \tag{22}$$

for $\mathbf{x}_B$ just above $\partial\mathcal{D}_0$ and $\mathbf{x}'_A$ just below $\partial\mathcal{D}_\mathrm{m}$. Alternatively, substituting states *A*-4 and *B*-1 into eq. (4) and using source–receiver reciprocity relation (16) yields

$$T_0^+(\mathbf{x}'_A, \mathbf{x}_B, \omega) - T^+(\mathbf{x}'_A, \mathbf{x}_B, \omega) = \int_{\partial\mathcal{D}_0} T^+(\mathbf{x}'_A, \mathbf{x}, \omega) R_0^+(\mathbf{x}, \mathbf{x}_B, \omega)\, \mathrm{d}^2\mathbf{x}. \tag{23}$$

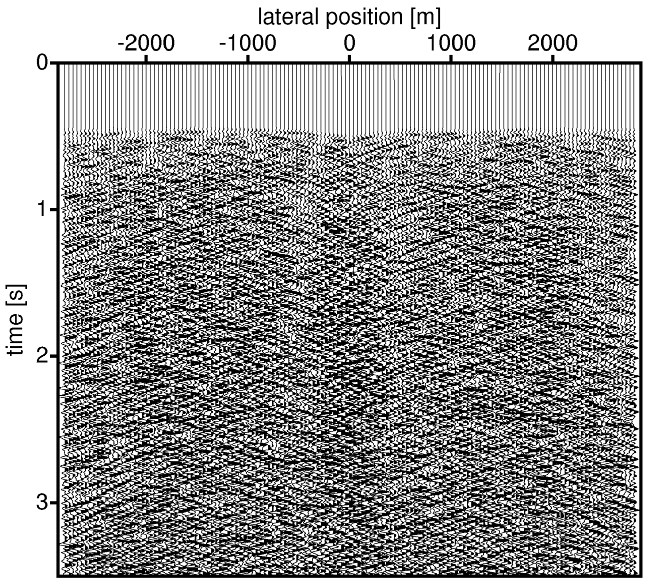
Eqs (22) and (23) interrelate the transmission responses $T_0^+(\mathbf{x}'_A, \mathbf{x}_B, \omega)$ and $T^+(\mathbf{x}'_A, \mathbf{x}_B, \omega)$ without and with free-surface multiples. Eq. (22) can be seen as an explicit expression for the transmission response $T^+(\mathbf{x}'_A, \mathbf{x}_B, \omega)$ with free-surface multiples in terms of $T_0^+(\mathbf{x}'_A, \mathbf{x}, \omega)$ and $R^+(\mathbf{x}, \mathbf{x}_B, \omega)$, whereas eq. (23) is an explicit expression for the transmission response $T_0^+(\mathbf{x}'_A, \mathbf{x}_B, \omega)$ without free-surface multiples in terms of $T^+(\mathbf{x}'_A, \mathbf{x}, \omega)$ and $R_0^+(\mathbf{x}, \mathbf{x}_B, \omega)$.

SEISMIC INTERFEROMETRY

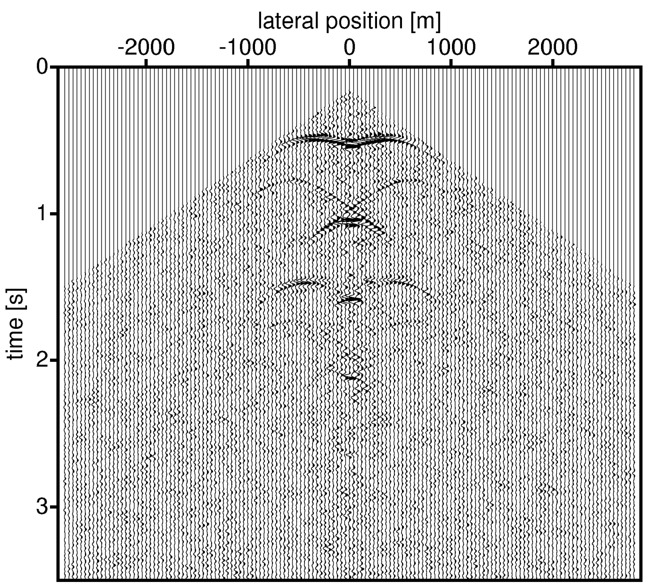
In the previous section we substituted states *A*-1 and *B*-3 into the one-way convolution-type reciprocity theorem (eq. 4) and obtained a relation between reflection responses without and with free-surface multiples (eq. 21). The right-hand side of this equation contains an integral over the product of $R_0^+(\mathbf{x}_A, \mathbf{x}, \omega)$ and $R^+(\mathbf{x}, \mathbf{x}_B, \omega)$, which corresponds to a convolution of these terms in the time domain. Schuster (2001) proposed evaluating a similar integral over the product of $\{R_0^+(\mathbf{x}_A, \mathbf{x}, \omega)\}^*$ and $R^+(\mathbf{x}_B, \mathbf{x}, \omega)$, which corresponds to a cross-correlation in the time domain of two reflection responses recorded at $\mathbf{x}_A$ and $\mathbf{x}_B$, respectively. He showed that this leads to a new reflection response observed at $\mathbf{x}_A$, as if there was a source at $\mathbf{x}_B$. Note the analogy with Claerbout's conjecture about the cross-correlation of transmission responses. Schuster (2001) used the term 'seismic interferometry' for any method that employs the cross-correlation of seismic responses recorded at different locations. We investigate the integral over the product of $\{R_0^+(\mathbf{x}_A, \mathbf{x}, \omega)\}^*$ and $R^+(\mathbf{x}_B, \mathbf{x}, \omega)$ by substituting states *A*-1 and *B*-3 into the one-way correlation-type reciprocity theorem (eq. 5) and employing the appropriate source–receiver reciprocity relations. We obtain

$$R^+(\mathbf{x}_A, \mathbf{x}_B, \omega) = \delta(\mathbf{x}_{\mathrm{H},B} - \mathbf{x}_{\mathrm{H},A}) - \int_{\partial\mathcal{D}_0} \left\{R_0^+(\mathbf{x}_A, \mathbf{x}, \omega)\right\}^* R^+(\mathbf{x}_B, \mathbf{x}, \omega)\, \mathrm{d}^2\mathbf{x} - \int_{\partial\mathcal{D}_\mathrm{m}} \{T_0^-(\mathbf{x}_A, \mathbf{x}, \omega)\}^* T^-(\mathbf{x}_B, \mathbf{x}, \omega)\, \mathrm{d}^2\mathbf{x}, \tag{24}$$

for $\mathbf{x}_A$ and $\mathbf{x}_B$ just above $\partial\mathcal{D}_0$. This equation shows that an integral over the correlation of reflection responses recorded at $\mathbf{x}_A$ and $\mathbf{x}_B$ (the first integral on the right-hand side along the common source coordinate $\mathbf{x}$ at $\partial\mathcal{D}_0$ of both responses) indeed contributes to the reflection response $R^+(\mathbf{x}_A, \mathbf{x}_B, \omega)$. This we call seismic reflection interferometry. It is illustrated with an example in Fig. 6, for the reflection data of Fig. 3(d). From Fig. 6(a), which represents the result of the first integral on the right-hand side of eq. (24), we observe that this seismic reflection interferometric result contains all events of the true reflection response $R^+(\mathbf{x}_A, \mathbf{x}_B, \omega)$ (compare with Fig. 3d), but with erroneous amplitudes.



(a)



(b)

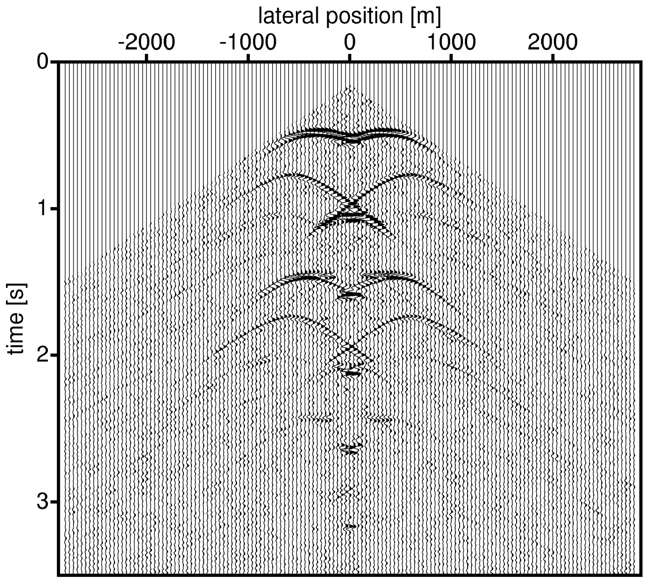
Figure 5. Acoustic daylight imaging: illustration of eq. (19) for the 2-D medium in Fig. 2. (a) Transmission response of a distribution of white noise sources below the syncline, observed at the acquisition surface (only the first 3.5 s are shown). (b) Reflection response, obtained by cross-correlating 10 min of noise (compare with Fig. 3d). (c) The same, for 66 min of noise. Note the improvement in signal-to-noise ratio.

Eq. (24) quantifies the discrepancy between this seismic reflection interferometric result and the true reflection response $R^+(\mathbf{x}_A, \mathbf{x}_B, \omega)$ (that is, the second integral on the right-hand side). In Fig. 6(b) we observe that this difference is quite significant, but since it contains the same events as the first term it can be ignored when true amplitudes are not an issue.

EXPRESSIONS FOR REFLECTIVITY 'FROM BELOW'

In this section we derive expressions for the reflection response for upgoing waves at the bottom of the configuration. We substitute the expressions for the one-way wavefields of states A-1 and B-2 (Table 1) into the one-way correlation-type reciprocity theorem (eq. 5). Dividing the result by $s_A^*(\omega)s_B(\omega)$ we obtain

$$\int_{\partial\mathcal{D}_\mathrm{m}} \left\{T_0^+(\mathbf{x}, \mathbf{x}_A, \omega)\right\}^* R_0^-(\mathbf{x}, \mathbf{x}'_B, \omega)\, \mathrm{d}^2\mathbf{x} = -\int_{\partial\mathcal{D}_0} \left\{R_0^+(\mathbf{x}, \mathbf{x}_A, \omega)\right\}^* T_0^+(\mathbf{x}'_B, \mathbf{x}, \omega)\, \mathrm{d}^2\mathbf{x}, \tag{25}$$



(c)

Figure 5. (*Continued.*)

for $\mathbf{x}_A$ just above $\partial\mathcal{D}_0$ and $\mathbf{x}'_B$ just below $\partial\mathcal{D}_\mathrm{m}$. When the reflection response $R_0^+(\mathbf{x}, \mathbf{x}_A, \omega)$ and the transmission response $T_0^+(\mathbf{x}, \mathbf{x}_A, \omega)$, both without free-surface multiples, are known, eq. (25) is an integral equation of the first kind for the reflection response $R_0^-(\mathbf{x}, \mathbf{x}'_B, \omega)$ for upgoing waves at the bottom of the configuration, just below $\partial\mathcal{D}_\mathrm{m}$, without free-surface multiples. $R_0^+(\mathbf{x}, \mathbf{x}_A, \omega)$ could be the result of free-surface multiple elimination (eq. 21), whereas $T_0^+(\mathbf{x}, \mathbf{x}_A, \omega)$ could be obtained from the transformation of the reflection response into the transmission coda (eq. 20), combined with the primary transmission response. $R_0^-(\mathbf{x}, \mathbf{x}'_B, \omega)$ can be used for imaging the configuration between the boundaries $\partial\mathcal{D}_0$ and $\partial\mathcal{D}_\mathrm{m}$ 'from below', in addition to conventional imaging 'from above'.

Next we substitute states A-3 and B-4 into eq. (5). Applying the source–receiver reciprocity relation (16) to the result, we obtain

$$\int_{\partial\mathcal{D}_\mathrm{m}} \left\{T^+(\mathbf{x}, \mathbf{x}_A, \omega)\right\}^* R^-(\mathbf{x}, \mathbf{x}'_B, \omega)\,\mathrm{d}^2\mathbf{x} = -T^+(\mathbf{x}'_B, \mathbf{x}_A, \omega), \tag{26}$$

for $\mathbf{x}_A$ at the free surface just above $\partial\mathcal{D}_0$ and $\mathbf{x}'_B$ just below $\partial\mathcal{D}_\mathrm{m}$. When the transmission response $T^+(\mathbf{x}, \mathbf{x}_A, \omega)$ with free-surface multiples is known, eq. (26) is an integral equation of the first kind for the reflection response $R^-(\mathbf{x}, \mathbf{x}'_B, \omega)$ for upgoing waves at the bottom of the configuration, just below $\partial\mathcal{D}_\mathrm{m}$, with free-surface multiples related to the free surface just above $\partial\mathcal{D}_0$.

Finally, by substituting states A-4 and B-4 into eq. (5), we obtain

$$\int_{\partial\mathcal{D}_\mathrm{m}} \left\{R^-(\mathbf{x}, \mathbf{x}'_A, \omega)\right\}^* R^-(\mathbf{x}, \mathbf{x}'_B, \omega)\,\mathrm{d}^2\mathbf{x} = \delta(\mathbf{x}'_{\mathrm{H},B} - \mathbf{x}'_{\mathrm{H},A}), \tag{27}$$

for $\mathbf{x}'_A$ and $\mathbf{x}'_B$ just below $\partial\mathcal{D}_\mathrm{m}$. This equation shows that the reflection response $R^-(\mathbf{x}, \mathbf{x}'_A, \omega)$ for upgoing waves at the bottom of the configuration is unitary when there is a free surface on top of the configuration—see Fig. 1, state A-4.

CONCLUSIONS

We have derived a number of relations between reflection and transmission responses of 3-D inhomogeneous media. The starting point for our derivations is given by the one-way reciprocity theorems of the convolution-type and of the correlation-type (eqs 4 and 5). These reciprocity theorems interrelate flux-normalized downgoing and upgoing waves in two acoustic states A and B. For both reciprocity theorems the medium between $\partial\mathcal{D}_0$ and $\partial\mathcal{D}_\mathrm{m}$ is source-free and the medium parameters are assumed to be identical in both states. For all states specified in Table 1 we have assumed that the half-space below $\partial\mathcal{D}_\mathrm{m}$ is homogeneous, and that above $\partial\mathcal{D}_0$ there is either a free surface or a homogeneous half-space. The one-way convolution-type reciprocity theorem (eq. 4) is exact and applies to lossless as well as to dissipative media. In the derivation of the one-way correlation-type reciprocity theorem (eq. 5) evanescent wave modes have been ignored and the medium between $\partial\mathcal{D}_0$ and $\partial\mathcal{D}_\mathrm{m}$ is assumed to be lossless.

We have considered reflection responses at the upper boundary $\partial\mathcal{D}_0$ and at the lower boundary $\partial\mathcal{D}_\mathrm{m}$ as well as transmission responses between these boundaries. All responses have been considered with and without free-surface multiples; internal multiple scattering was always included. The mutual relations between these responses were obtained by substituting a selection of responses for the two states A and B in either one of the reciprocity theorems (4) or (5). The assumptions discussed above apply of course also to the relations that have been derived from either one of these reciprocity theorems.

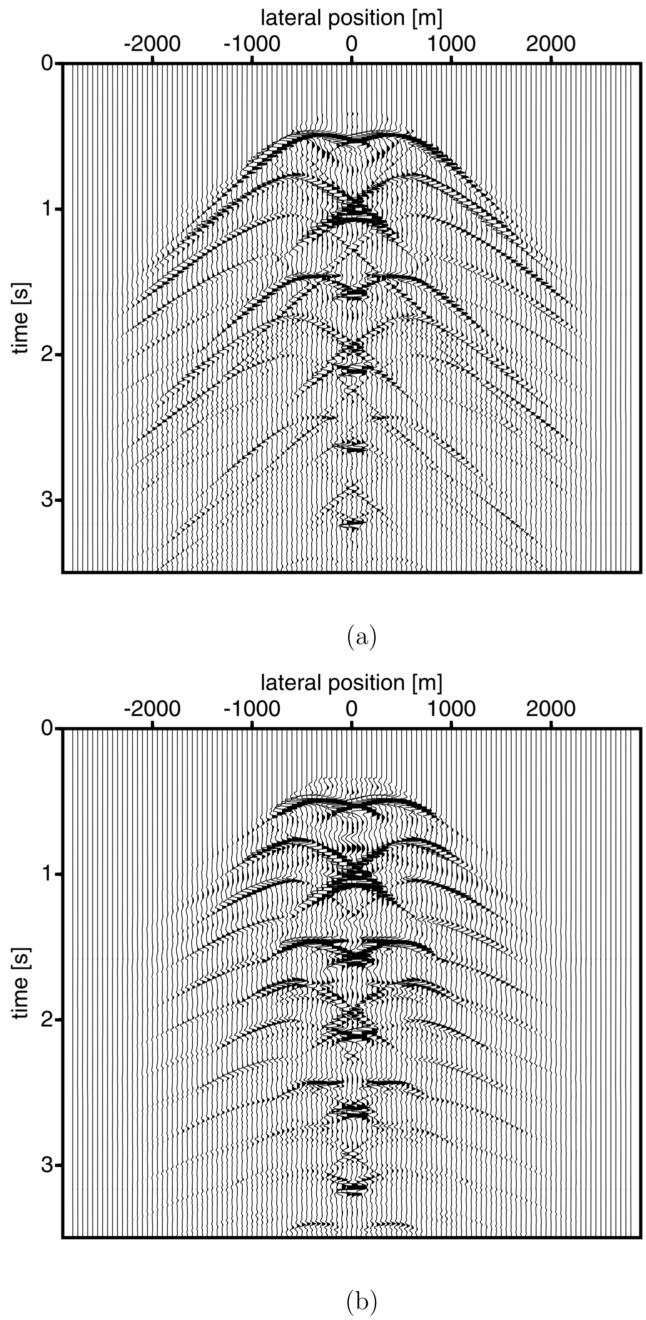


Figure 6. Seismic interferometry: illustration of eq. (24) for the 2-D medium in Fig. 2 and its reflection response in Fig. 3(d). (a) Result of the first integral on the right-hand side of eq. (24) (times −1). (b) Result of the second integral on the right-hand side of eq. (24) (times −1).

First we derived source–receiver reciprocity relations for the various reflection and transmission responses (eqs 11–16). Although this type of relation is well known for full wavefields, we needed to establish them independently for the reflection and transmission responses of downgoing and upgoing wavefields. Since these expressions were derived from the one-way convolution-type reciprocity theorem (eq. 4) they are exact and valid for media with or without anelastic losses. Eqs (11), (12) and (15) were previously derived in the wavenumber domain by Haines (1988).

Next we derived a relation between reflection and transmission responses including free-surface multiples (eq. 19). This is an explicit expression for the reflection response in terms of the cross-correlation of transmission measurements. We noted that this equation confirms Claerbout's conjecture about acoustic daylight imaging and demonstrated this with a numerical example. We derived a second relation between reflection and transmission responses, this time without free-surface multiples (eq. 20). We noted that this equation can be used to derive the transmission coda from the cross-correlation of the reflection measurements. This may be useful in seismic imaging schemes that take internal multiple scattering into account. Eqs (19) and (20) were derived from the one-way correlation-type reciprocity theorem (eq. 5), and hence evanescent wave modes are ignored and they are valid only for lossless media.

We derived two classes of expressions between responses without and with free-surface multiples. The first class of expressions was based on the convolution-type reciprocity theorem (eq. 4) and led to the exact relations (21)–(23). Eq. (21) is similar to those employed by Berkhout (1982), Verschuur *et al.* (1992), Fokkema & van den Berg (1993) and van Borselen *et al.* (1996) for surface-related multiple elimination. The second class of expressions was based on the correlation-type reciprocity theorem (eq. 5) and led to the approximate eq. (24), which confirms one of the expressions for seismic interferometry derived by Schuster (2001).

Finally, we derived three approximate expressions (eqs 25–27) based on the correlation-type reciprocity theorem for the reflection response at the lower boundary $\partial\mathcal{D}_{\mathrm{m}}$. These expressions may be useful for imaging 'from below'.

Note that, because all relations apply to reflection and/or transmission responses for downgoing and/or upgoing wavefields, in practice a decomposition of measurements into downgoing and upgoing wavefields is required as a pre-processing step.

The extensions of all relations to the elastodynamic situation are discussed in Appendix A. These relations can be applied to multicomponent seismic data after decomposition into downgoing and upgoing P and S waves.

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APPENDIX A: EXTENSION TO THE ELASTODYNAMIC SITUATION

To derive relations between reflection and transmission responses for elastodynamic waves in 3-D inhomogeneous elastic media, we have again the choice between reciprocity theorems for two-way and one-way wavefields. In the former case the reciprocity theorems are formulated in terms of particle velocities and stresses in two states (similarly to the propagation invariant introduced by Kennett *et al.* 1990); in the latter case they are formulated in terms of flux-normalized downgoing and upgoing P and S waves in both states. Hence, elastodynamic two-way reciprocity theorems can be applied directly to observable quantities, such as the three components of particle velocity measured by geophones in multi-component seismic data acquisition. However, similarly to in the acoustic case, elastodynamic reflection and transmission responses are transfer functions between downgoing and upgoing P and S waves. For this reason the elastodynamic one-way reciprocity theorems are preferred as the starting point for deriving the relations between the various reflection and transmission responses. To apply the results in practice, the multi-component data should be decomposed into downgoing and upgoing P and S waves as a pre-processing step (Wapenaar *et al.* 1990; Osen *et al.* 1996; Schalkwijk *et al.* 1998).

In the following we discuss the extensions of all results derived for the acoustic situation to the elastodynamic situation.

Elastodynamic one-way reciprocity theorems

For the elastodynamic situation, the one-way reciprocity theorems (4) and (5) are replaced by

$$\int_{\partial\mathcal{D}_0} \left\{(\mathbf{P}_A^+)^{\mathrm{t}}\mathbf{P}_B^- - (\mathbf{P}_A^-)^{\mathrm{t}}\mathbf{P}_B^+\right\} \mathrm{d}^2\mathbf{x} = \int_{\partial\mathcal{D}_\mathrm{m}} \left\{(\mathbf{P}_A^+)^{\mathrm{t}}\mathbf{P}_B^- - (\mathbf{P}_A^-)^{\mathrm{t}}\mathbf{P}_B^+\right\} \mathrm{d}^2\mathbf{x} \tag{A1}$$

and

$$\int_{\partial\mathcal{D}_0} \left\{(\mathbf{P}_A^+)^{\dagger}\mathbf{P}_B^+ - (\mathbf{P}_A^-)^{\dagger}\mathbf{P}_B^-\right\} \mathrm{d}^2\mathbf{x} = \int_{\partial\mathcal{D}_\mathrm{m}} \left\{(\mathbf{P}_A^+)^{\dagger}\mathbf{P}_B^+ - (\mathbf{P}_A^-)^{\dagger}\mathbf{P}_B^-\right\} \mathrm{d}^2\mathbf{x}, \tag{A2}$$

respectively, where ‘t’ denotes transposition and ‘†’ denotes transposition and complex conjugation. The one-way wavefield vectors are defined as follows

$$\mathbf{P}_A^\pm = \begin{pmatrix} \Phi_A^\pm \\ \Psi_A^\pm \\ \Upsilon_A^\pm \end{pmatrix}, \quad \mathbf{P}_B^\pm = \begin{pmatrix} \Phi_B^\pm \\ \Psi_B^\pm \\ \Upsilon_B^\pm \end{pmatrix}, \tag{A3}$$

where $\Phi^\pm$, $\Psi^\pm$ and $\Upsilon^\pm$ represent the flux-normalized downgoing and upgoing P, S_1 and S_2 waves, respectively, for states A and B (Wapenaar & Grimbergen 1996).

Specification of the states for the one-way reciprocity theorems

In Table 1 the scalar acoustic one-way wavefields $P_A^\pm$ need to be replaced by the vector elastodynamic one-way wavefields $\mathbf{P}_A^\pm$. Furthermore, the scalar reflection and transmission responses $R_0^\pm$, $R^\pm$, $T_0^\pm$ and $T^\pm$ have to be replaced by 3×3 matrices $\mathsf{R}_0^\pm$, $\mathsf{R}^\pm$, $\mathsf{T}_0^\pm$ and $\mathsf{T}^\pm$, respectively, which have the following form

$$\mathsf{R}^\pm(\mathbf{x}, \mathbf{x}_A, \omega) = \begin{pmatrix} R_{\phi,\phi}^\pm & R_{\phi,\psi}^\pm & R_{\phi,\upsilon}^\pm \\ R_{\psi,\phi}^\pm & R_{\psi,\psi}^\pm & R_{\psi,\upsilon}^\pm \\ R_{\upsilon,\phi}^\pm & R_{\upsilon,\psi}^\pm & R_{\upsilon,\upsilon}^\pm \end{pmatrix} (\mathbf{x}, \mathbf{x}_A, \omega), \tag{A4}$$

etc., where $R_{\beta,\alpha}^\pm(\mathbf{x}, \mathbf{x}_A, \omega)$ denotes the reflection response in terms of an incident α-wavefield at $\mathbf{x}_A$ and a reflected β-wavefield at $\mathbf{x}$. The scalar free-surface reflection coefficient r^- in Table 1 (with $r^- = -1$) should be replaced by a 3×3 operator matrix $\hat{\mathbf{r}}^-$, where the circumflex denotes a pseudo-differential operator acting on the x_1- and x_2-coordinates. In Appendix B we show that $\hat{\mathbf{r}}^-$ obeys the following properties

$$\{\hat{\mathbf{r}}^-\}^{\mathrm{t}} = \hat{\mathbf{r}}^-, \tag{A5}$$

$$\{\hat{\mathbf{r}}^-\}^{\dagger}\hat{\mathbf{r}}^- = \mathsf{I}, \tag{A6}$$

where I is the 3×3 identity matrix. Finally, in Table 1 we should replace $\delta(\mathbf{x}_\mathrm{H} - \mathbf{x}_{\mathrm{H},A})$ by $\mathsf{I}\delta(\mathbf{x}_\mathrm{H} - \mathbf{x}_{\mathrm{H},A})$, $s_A(\omega)$ by $\mathbf{s}_A(\omega)$ (a 3×1 vector representing source spectra for the various wave types) and 0 by $\mathbf{0}$ (the 3×1 null vector). Similar replacements should be made for state B.

Source–receiver reciprocity

Substituting the expressions for the elastodynamic one-way wavefields of states A-1 and B-1 of the modified Table 1 into the one-way convolution-type reciprocity theorem (eq. A1) we obtain

$$\int_{\partial\mathcal{D}_0} \mathbf{s}_A^{\mathrm{t}}(\omega)\left[\delta(\mathbf{x}_\mathrm{H} - \mathbf{x}_{\mathrm{H},A})\mathsf{R}_0^+(\mathbf{x}_\mathrm{H}, x_{3,0}, \mathbf{x}_B, \omega) - \left\{\mathsf{R}_0^+(\mathbf{x}_\mathrm{H}, x_{3,0}, \mathbf{x}_A, \omega)\right\}^{\mathrm{t}}\delta(\mathbf{x}_\mathrm{H} - \mathbf{x}_{\mathrm{H},B})\right]\mathbf{s}_B(\omega)\, \mathrm{d}^2\mathbf{x} = 0. \tag{A7}$$

Since this expression should hold for any $\mathbf{s}_A(\omega)$ and $\mathbf{s}_B(\omega)$, we obtain

$$\mathsf{R}_0^+(\mathbf{x}_A, \mathbf{x}_B, \omega) = \left\{\mathsf{R}_0^+(\mathbf{x}_B, \mathbf{x}_A, \omega)\right\}^{\mathrm{t}}, \tag{A8}$$

for $\mathbf{x}_A$ and $\mathbf{x}_B$ just above $\partial\mathcal{D}_0$. This is the elastodynamic equivalent of the source–receiver reciprocity relation (11). In a similar way we find the elastodynamic equivalents of the source–receiver reciprocity relations (12)–(16):

$$\mathsf{R}_0^-(\mathbf{x}'_A, \mathbf{x}'_B, \omega) = \{\mathsf{R}_0^-(\mathbf{x}'_B, \mathbf{x}'_A, \omega)\}^{\mathrm{t}}, \tag{A9}$$

$$\mathsf{R}^+(\mathbf{x}_A, \mathbf{x}_B, \omega) = \left\{\mathsf{R}^+(\mathbf{x}_B, \mathbf{x}_A, \omega)\right\}^{\mathrm{t}}, \tag{A10}$$

$$\mathsf{R}^-(\mathbf{x}'_A, \mathbf{x}'_B, \omega) = \left\{\mathsf{R}^-(\mathbf{x}'_B, \mathbf{x}'_A, \omega)\right\}^{\mathrm{t}}, \tag{A11}$$

$$\mathbf{T}_0^+(\mathbf{x}'_A, \mathbf{x}_B, \omega) = \{\mathbf{T}_0^-(\mathbf{x}_B, \mathbf{x}'_A, \omega)\}^{\mathrm{t}}, \tag{A12}$$

$$\mathbf{T}^+(\mathbf{x}'_A, \mathbf{x}_B, \omega) = \{\mathbf{T}^-(\mathbf{x}_B, \mathbf{x}'_A, \omega)\}^{\mathrm{t}}, \tag{A13}$$

for $\mathbf{x}_A$ and $\mathbf{x}_B$ just above $\partial\mathcal{D}_0$ and $\mathbf{x}'_A$ and $\mathbf{x}'_B$ just below $\partial\mathcal{D}_{\mathrm{m}}$. Eqs (A8), (A9) and (A12) were previously derived in the wavenumber domain by Kennett *et al.* (1990). Note that we made use of the first symmetry property of the free-surface reflection operator $\hat{\mathbf{r}}^-$ (eq. A5) in the derivation of eqs (A10), (A11) and (A13).

Using the form introduced in eq. (A4) we obtain for example from eq. (A10) that

$$R^+_{\alpha,\beta}(\mathbf{x}_A, \mathbf{x}_B, \omega) = R^+_{\beta,\alpha}(\mathbf{x}_B, \mathbf{x}_A, \omega). \tag{A14}$$

This equation states that the upgoing α-wavefield at $\mathbf{x}_A$ due to a downgoing β-wavefield at $\mathbf{x}_B$ is identical to the upgoing β-wavefield at $\mathbf{x}_B$ due to a downgoing α-wavefield at $\mathbf{x}_A$.

From transmission to reflection

Using the second symmetry property of the free-surface reflection operator $\hat{\mathbf{r}}^-$ (eq. A6), we find for the elastodynamic version of the relation between reflection and transmission responses with free-surface multiples, analogous to eq. (18),

$$-\hat{\mathbf{r}}^-(\mathbf{x}_A)\mathbf{R}^+(\mathbf{x}_A, \mathbf{x}_B, \omega) - \{\mathbf{R}^+(\mathbf{x}_A, \mathbf{x}_B, \omega)\}^*\{\hat{\mathbf{r}}^-(\mathbf{x}_B)\}^\dagger = \mathbf{I}\delta(\mathbf{x}_{\mathrm{H},B} - \mathbf{x}_{\mathrm{H},A}) - \int_{\partial\mathcal{D}_{\mathrm{m}}} \{\mathbf{T}^-(\mathbf{x}_A, \mathbf{x}, \omega)\}^*\{\mathbf{T}^-(\mathbf{x}_B, \mathbf{x}, \omega)\}^{\mathrm{t}}\, \mathrm{d}^2\mathbf{x}, \tag{A15}$$

for $\mathbf{x}_A$ and $\mathbf{x}_B$ at the free surface, just above $\partial\mathcal{D}_0$. This equation is the basis for resolving the P and S reflection responses from the cross-correlation of the P and S transmission responses and finds its application in acoustic daylight imaging.

From reflection to transmission

Analogous to the acoustic relation (eq. 20) between reflection and transmission responses without free-surface multiples, we obtain for the elastodynamic situation

$$\int_{\partial\mathcal{D}_{\mathrm{m}}} \left\{\mathbf{T}_0^+(\mathbf{x}, \mathbf{x}_A, \omega)\right\}^\dagger \mathbf{T}_0^+(\mathbf{x}, \mathbf{x}_B, \omega)\, \mathrm{d}^2\mathbf{x} = \mathbf{I}\delta(\mathbf{x}_{\mathrm{H},B} - \mathbf{x}_{\mathrm{H},A}) - \int_{\partial\mathcal{D}_0} \left\{\mathbf{R}_0^+(\mathbf{x}, \mathbf{x}_A, \omega)\right\}^\dagger \mathbf{R}_0^+(\mathbf{x}, \mathbf{x}_B, \omega)\, \mathrm{d}^2\mathbf{x}, \tag{A16}$$

for $\mathbf{x}_A$ and $\mathbf{x}_B$ just above $\partial\mathcal{D}_0$. This equation is the basis for resolving the P and S wave transmission codas from the cross-correlation of the P and S reflection responses. These transmission codas may be used to derive inverse propagators that take internal multiple scattering into account in elastodynamic seismic imaging.

Expressions for free-surface multiples

The elastodynamic equivalent of the acoustic relation between reflection responses with and without free-surface multiples (eq. 21) reads

$$\mathbf{R}_0^+(\mathbf{x}_A, \mathbf{x}_B, \omega) - \mathbf{R}^+(\mathbf{x}_A, \mathbf{x}_B, \omega) = -\int_{\partial\mathcal{D}_0} \mathbf{R}_0^+(\mathbf{x}_A, \mathbf{x}, \omega)\hat{\mathbf{r}}^-\mathbf{R}^+(\mathbf{x}, \mathbf{x}_B, \omega)\, \mathrm{d}^2\mathbf{x}, \tag{A17}$$

for $\mathbf{x}_A$ and $\mathbf{x}_B$ just above $\partial\mathcal{D}_0$. A similar relation was obtained previously by Wapenaar *et al.* (1990) and illustrated with a numerical example, demonstrating its application in elastodynamic multiple elimination of decomposed multi-component data in a 2-D inhomogeneous elastic medium.

The elastodynamic equivalents of eqs (22) and (23) read

$$\mathbf{T}_0^+(\mathbf{x}'_A, \mathbf{x}_B, \omega) - \mathbf{T}^+(\mathbf{x}'_A, \mathbf{x}_B, \omega) = -\int_{\partial\mathcal{D}_0} \mathbf{T}_0^+(\mathbf{x}'_A, \mathbf{x}, \omega)\hat{\mathbf{r}}^-\mathbf{R}^+(\mathbf{x}, \mathbf{x}_B, \omega)\, \mathrm{d}^2\mathbf{x} \tag{A18}$$

and

$$\mathbf{T}_0^+(\mathbf{x}'_A, \mathbf{x}_B, \omega) - \mathbf{T}^+(\mathbf{x}'_A, \mathbf{x}_B, \omega) = -\int_{\partial\mathcal{D}_0} \mathbf{T}^+(\mathbf{x}'_A, \mathbf{x}, \omega)\hat{\mathbf{r}}^-\mathbf{R}_0^+(\mathbf{x}, \mathbf{x}_B, \omega)\, \mathrm{d}^2\mathbf{x}, \tag{A19}$$

for $\mathbf{x}_B$ just above $\partial\mathcal{D}_0$ and $\mathbf{x}'_A$ just below $\partial\mathcal{D}_{\mathrm{m}}$.

Seismic interferometry

Analogous to eq. (24) we obtain

$$\begin{aligned}-\hat{\mathbf{r}}^-(\mathbf{x}_A)\mathbf{R}^+(\mathbf{x}_A, \mathbf{x}_B, \omega) = {}& \mathbf{I}\delta(\mathbf{x}_{\mathrm{H},B} - \mathbf{x}_{\mathrm{H},A}) - \int_{\partial\mathcal{D}_0} \left\{\mathbf{R}_0^+(\mathbf{x}_A, \mathbf{x}, \omega)\right\}^*\{\mathbf{R}^+(\mathbf{x}_B, \mathbf{x}, \omega)\}^{\mathrm{t}}\, \mathrm{d}^2\mathbf{x} \\ &- \int_{\partial\mathcal{D}_{\mathrm{m}}} \left\{\mathbf{T}_0^-(\mathbf{x}_A, \mathbf{x}, \omega)\right\}^*\{\mathbf{T}^-(\mathbf{x}_B, \mathbf{x}, \omega)\}^{\mathrm{t}}\, \mathrm{d}^2\mathbf{x},\end{aligned} \tag{A20}$$

for $\mathbf{x}_A$ and $\mathbf{x}_B$ just above $\partial\mathcal{D}_0$. This equation is the basis for elastodynamic seismic interferometry.

Expressions for reflectivity 'from below'

Finally, the elastodynamic equivalents of the expressions for reflectivity 'from below' (eqs 25, 26 and 27) read

$$\int_{\partial\mathcal{D}_\mathrm{m}} \left\{\mathsf{T}_0^+(\mathbf{x},\mathbf{x}_A,\omega)\right\}^\dagger \mathsf{R}_0^-(\mathbf{x},\mathbf{x}'_B,\omega)\,\mathrm{d}^2\mathbf{x} = -\int_{\partial\mathcal{D}_0} \left\{\mathsf{R}_0^+(\mathbf{x},\mathbf{x}_A,\omega)\right\}^\dagger \mathsf{T}_0^+(\mathbf{x}'_B,\mathbf{x},\omega)\,\mathrm{d}^2\mathbf{x}, \tag{A21}$$

for $\mathbf{x}_A$ just above $\partial\mathcal{D}_0$ and $\mathbf{x}'_B$ just below $\partial\mathcal{D}_\mathrm{m}$;

$$\int_{\partial\mathcal{D}_\mathrm{m}} \left\{\mathsf{T}^+(\mathbf{x},\mathbf{x}_A,\omega)\right\}^\dagger \mathsf{R}^-(\mathbf{x},\mathbf{x}'_B,\omega)\,\mathrm{d}^2\mathbf{x} = \hat{\mathsf{r}}^-(\mathbf{x}_A)\mathsf{T}^+(\mathbf{x}'_B,\mathbf{x}_A,\omega), \tag{A22}$$

for $\mathbf{x}_A$ at the free surface just above $\partial\mathcal{D}_0$ and $\mathbf{x}'_B$ just below $\partial\mathcal{D}_\mathrm{m}$; and

$$\int_{\partial\mathcal{D}_\mathrm{m}} \{\mathsf{R}^-(\mathbf{x},\mathbf{x}'_A,\omega)\}^\dagger \mathsf{R}^-(\mathbf{x},\mathbf{x}'_B,\omega)\,\mathrm{d}^2\mathbf{x} = \mathsf{I}\delta(\mathbf{x}'_{\mathrm{H},B}-\mathbf{x}'_{\mathrm{H},A}), \tag{A23}$$

for $\mathbf{x}'_A$ and $\mathbf{x}'_B$ just below $\partial\mathcal{D}_\mathrm{m}$. These expressions are the basis for elastodynamic imaging of the configuration between the boundaries $\partial\mathcal{D}_0$ and $\partial\mathcal{D}_\mathrm{m}$ 'from below', in addition to elastodynamic imaging 'from above'.

APPENDIX B: THE FREE-SURFACE REFLECTION OPERATOR

For the derivation of the properties of the free-surface reflection operator $\hat{\mathsf{r}}^-$ we choose $\partial\mathcal{D}_0$ again at $x_{3,0}+\epsilon$, that is, just below the free surface, but the other integration boundary is chosen above the free surface. Since no wavefields are present in the upper half-space, the integral over this upper boundary is zero. Hence, if we choose the sources for the one-way wavefields below $\partial\mathcal{D}_0$, the elastodynamic one-way reciprocity theorems read

$$\int_{\partial\mathcal{D}_0} \left\{(\mathbf{P}_A^+)^\mathrm{t}\mathbf{P}_B^- - (\mathbf{P}_A^-)^\mathrm{t}\mathbf{P}_B^+\right\}\mathrm{d}^2\mathbf{x} = 0 \tag{B1}$$

and

$$\int_{\partial\mathcal{D}_0} \{(\mathbf{P}_A^+)^\dagger\mathbf{P}_B^+ - (\mathbf{P}_A^-)^\dagger\mathbf{P}_B^-\}\,\mathrm{d}^2\mathbf{x} = 0. \tag{B2}$$

The free-surface reflection operator $\hat{\mathsf{r}}^-$ transforms an upgoing elastodynamic wavefield at $\partial\mathcal{D}_0$ (that is, just below the free surface) into a downgoing wavefield at $\partial\mathcal{D}_0$. Hence, for states A and B we have

$$\mathbf{P}_A^+(\mathbf{x},\omega) = \hat{\mathsf{r}}^-\mathbf{P}_A^-(\mathbf{x},\omega), \tag{B3}$$

$$\mathbf{P}_B^+(\mathbf{x},\omega) = \hat{\mathsf{r}}^-\mathbf{P}_B^-(\mathbf{x},\omega), \tag{B4}$$

for $\mathbf{x}$ at $\partial\mathcal{D}_0$. Substitution into eqs (B1) and (B2) gives

$$\int_{\partial\mathcal{D}_0} \{\mathbf{P}_A^-(\mathbf{x},\omega)\}^\mathrm{t}(\{\hat{\mathsf{r}}^-\}^\mathrm{t} - \hat{\mathsf{r}}^-)\mathbf{P}_B^-(\mathbf{x},\omega)\,\mathrm{d}^2\mathbf{x} = 0 \tag{B5}$$

and

$$\int_{\partial\mathcal{D}_0} \{\mathbf{P}_A^-(\mathbf{x},\omega)\}^\dagger(\{\hat{\mathsf{r}}^-\}^\dagger\hat{\mathsf{r}}^- - \mathsf{I})\mathbf{P}_B^-(\mathbf{x},\omega)\,\mathrm{d}^2\mathbf{x} = 0, \tag{B6}$$

respectively, where $\{\hat{\mathsf{r}}^-\}^\mathrm{t}$ and $\{\hat{\mathsf{r}}^-\}^\dagger$ denote the transposed and adjoint free-surface reflection operators, respectively. Since eqs (B5) and (B6) should hold for any $\mathbf{P}_A^-(\mathbf{x},\omega)$ and $\mathbf{P}_B^-(\mathbf{x},\omega)$, we obtain

$$\{\hat{\mathsf{r}}^-\}^\mathrm{t} = \hat{\mathsf{r}}^-, \tag{B7}$$

$$\{\hat{\mathsf{r}}^-\}^\dagger\hat{\mathsf{r}}^- = \mathsf{I}. \tag{B8}$$