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# The explicit scalar equations of infinitesimal elastic-gravitational motion in the rotating, slightly elliptical fluid outer core of the Earth

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## SUMMARY

Smith (1974) derived a set of scalar equations of infinitesimal elastic-gravitational motion for a rotating, slightly elliptical Earth. In the fluid outer core (FOC), the tangential tractions vanish identically and the differential equations for the transverse displacement fields do not hold any longer, but the derivative of the transverse displacements are needed in other differential equations, it is therefore the purpose of this paper to get the derivative of the transverse displacements as well as other differential equations in an explicit and ready-to-program format. Our derivation is shown to be more direct and explicit than that suggested by Smith (1974).

**Key words:** displacement field, elastic-gravitational excitation, fluid outer core, motion equation.





## 1 INTRODUCTION

In the work of Smith (1974) (hereafter referred to as S74), the generalized surface spherical harmonics (GSSH) representation (see Phinney & Burridge 1973; Huang & Liao 2003, for correctness) and the symmetry properties of a rotating planet (Luh 1973) are used to derive an infinite set of motion equations. These motion equations (eqs 5.22–5.26 and 5.28–5.30 in S74) are scalar, linear, first-order and ordinary differential equations (ODEs) about radius; they govern the infinitesimal free elastic-gravitational oscillations of the Earth. The Earth is assumed to be a rotating, slightly elliptical Earth with an isotropic perfectly elastic constitutive relation and a hydrostatic pre-stress field.

This set of equations is used very often in research communities such as seismology, earth tides, nutation and else. In his work, the final form is eight differential equations for $\partial_r \mathrm{Y}$, where Y is the column-vector of solution combining ten scalars, i.e.

$$\mathrm{Y}(r,\omega) = \left(U_l^m, P_l^m, {\phi_1}_l^m, {g_1}_l^m, V_l^m, W_{l-1}^m, W_{l+1}^m, Q_l^m, R_{l-1}^m, R_{l+1}^m\right)^T, \quad (1)$$

where, $P_l^m$, $Q_l^m$, $R_{l\pm1}^m$ are the three scalars in the GSSH representation of $\hat{n}\cdot\mathbf{T}$ (where $\hat{n}$ is the normal vector and $\mathbf{T}$ is the stress field); $l$, $m$ are the longitudinal degree and azimuthal order in the GSSH expansion, respectively; $U_l^m$, $V_l^m$, $W_{l\pm1}^m$ are the three scalars in the GSSH representation of the displacement field $\mathbf{s}$; ${\phi_1}_l^m$ is for the perturbed gravitational potential; and ${g_1}_l^m$ is another scalar related to the perturbed gravity that is defined in the formula (eq. 5.18) in S74. All these ten scalars are functions of the radius and the frequency (in frequency domain).

For compactness, these ODEs are arranged in the matrix form

$$\partial_r \mathrm{Y}(r,\omega) = A(r,\omega)\mathrm{Y}(r,\omega), \quad (2)$$

where $A(r,\omega)$ is a propagation matrix of dimension of $10\times10$. If the anelasticity is considered in the mantle, i.e. the Lame parameters are complex rather than real, then the matrix $A(r,\omega)$ and the variables $\mathrm{Y}(r,\omega)$ become complex in the mantle.

Because the rigidity $\mu$ vanishes identically in the fluid outer core (FOC), the differential equations for $\partial_r V_l^m$ and $\partial_r W_{l\pm1}^m$ (eqs 5.23 and 5.24 in S74) do not hold any longer. However, all $V_l^m$, $W_{l\pm1}^m$, $\partial_r V_l^m$ and $\partial_r W_{l\pm1}^m$ do not vanish in FOC and they should be kept in the ODEs for $\partial_r U_l^m$, $\partial_r P_l^m$, $\partial_r {\phi_1}_l^m$ and $\partial_r {g_1}_l^m$, so we have to have another pair of ODEs for $\partial_r V_l^m$ and $\partial_r W_{l\pm1}^m$. Although an algorithm to get $\partial_r V_l^m$ and $\partial_r W_{l\pm1}^m$ are described in S74, it is, however, not in detail, the algorithm is complex itself and the explicit form is unavailable. In this paper, we will present a similar but more direct algorithm to obtain them, and the explicit and read-to-program form is given.

For compactness, all the notations in this paper are kept completely the same as in S74 if not notated specially.

## 2 THE ODES IN FOC

As a result of ellipticity, the spherical scalars and toroidal scalars are coupled with each other. One has to truncate them in the form

$$s(\mathbf{r},\omega)=\tau_{l-1}^{m}(\mathbf{r},\omega)+\sigma_{l}^{m}(\mathbf{r},\omega)+\tau_{l+1}^{m}(\mathbf{r},\omega)+\sigma_{l+2}^{m}(\mathbf{r},\omega)+\ldots, \tag{3}$$

where $\sigma$ and $\tau$ are the spheroidal and toroidal scalars of $s$, respectively.

In practice, only the first three terms are kept and all the higher degree terms are ignored, although nobody has tested this truncation is correct or precise enough. Such truncation is also used in this paper and produces ten variables and ten ODEs for these ten variables.

In an ideal fluid like the fluid outer core, the rigidity $\mu$ vanishes identically, the fluid can not support transverse traction, i.e. $Q_l^m = R_{l\pm1}^m = 0$ and the ODEs for $\partial_r(Q_l^m, R_{l\pm1}^m)$ (eqs 5.29 and 5.30 in S74) become meaningless (but we will use them later). Moreover, the ODEs for $\partial_r(V_l^m, W_{l\pm1}^m)$ (eqs 5.23 and 5.24 in S74) do not hold any longer (because $\mu = 0$). Therefore, only the four differential equations for $\partial_r U_l^m$, $\partial_r P_l^m$, $\partial_r \phi_{1l}^{\ m}$ and $\partial_r g_{1l}^{\ m}$ (eqs 5.22, 5.25, 5.26 and 5.28 in S74) are kept.

Following S74, we separate the solution Y into two parts:

$$\tilde{\mathrm{Y}}(r,\omega)=\left(U_l^m, P_l^m, {\phi_1}_l^m, {g_1}_l^m\right)^T \tag{4}$$

and

$$\tilde{\mathrm{X}}(r,\omega)=\left(V_l^m, W_{l-1}^m, W_{1+l}^m\right)^T. \tag{5}$$

### 2.1 Simplified ODEs for $\partial_r\tilde{\mathrm{Y}}$

Using the fact that, in FOC,

$$Q_l^m = R_{l\pm1}^m = 0,\ \mu_0=\mu_2=0,\ \beta_0=\lambda_0,\ \beta_2=\lambda_2, \tag{6}$$

the differential equation for $\partial_r U_l^m$, eq. (5.22) in S74, can be simplified as follows, if truncated as mentioned above,

$$\partial_r U_l^m = f_\lambda P_l^m - \frac{L_0}{r}V_l^m - \frac{2}{r}U_l^m, \tag{7}$$

where

$$f_\lambda \equiv \left(\lambda_0-\lambda_2 J_{0,0,0}^{l,2,l}\right)/(\lambda_0)^2, \tag{8}$$

in which, the compact symbol $J_{0,0,0}^{l,2,l}$ is exactly the same of the so-called $J$-square in S74 by:

$$J_{n,n',n''}^{l,l',l''} \equiv \begin{bmatrix} l & l' & l'' \\ n & n' & n'' \\ m & 0 & m \end{bmatrix} \equiv (2l+1)(-)^{m+n}\begin{pmatrix} l & l' & l'' \\ -n & n' & n'' \end{pmatrix}\begin{pmatrix} l & l' & l'' \\ -m & 0 & m \end{pmatrix} \tag{9}$$

where the last two terms are Wigner 3-j symbols (Edmonds 1960). Then, the term related to the radial component of volume inflation factor, eq. (5.19) in S74, becomes

$$\Delta_l^m = f_\lambda P_l^m. \tag{10}$$

The differential equation for $\partial_r {\phi_1}_l^m$ still takes the form (eq. 5.25) in S74, i.e.

$$\partial_r {\phi_1}_l^m = {g_1}_l^m - 4\pi G\rho_0 U_l^m; \tag{11}$$

while, the differential equation for $\partial_r {g_1}_l^m$, eq. (5.26) in S74, can be simplified a little as

$$\begin{aligned}\partial_r {g_1}_l^m = {} & \frac{l(l+1)}{r^2}{\phi_1}_l^m - \frac{2}{r}{g_1}_l^m - 4\pi G J_{0,0,0}^{l,2,l}\left(\rho_2' U_l^m + f_\lambda \rho_2 P_l^m\right) \\ & + \frac{4\pi G}{r}L_0\left(-\rho_0-\rho_2 J_{+,0,+}^{l,2,l}-\rho_2 J_{0,0,0}^{l,2,l}\right)V_l^m - \frac{4\pi G}{r}L_0\rho_2\left(J_{+,0,+}^{l,2,l-1}W_{l-1}^m + J_{+,0,+}^{l,2,l+1}W_{l+1}^m\right),\end{aligned} \tag{12}$$

where $\rho_2' \equiv \partial_r \rho_2$.

Notice that the first two lines of eq. (5.28) in S74 becomes

$$-\frac{2}{r}P_l^m + \frac{2}{r}\lambda_0\Delta_l^m + \frac{2}{r}J_{0,0,0}^{l,2,l}\lambda_2\Delta_l^m = -\frac{2}{r}P_l^m + \frac{2}{r}\left[1-\left(J_{0,0,0}^{l,2,l}\right)^2\left(\frac{\lambda_2}{\lambda_0}\right)^2\right]P_l^m \approx 0, \tag{13}$$

as a result of the fact that $(\frac{\lambda_2}{\lambda_0})^2$ is a second-order minor with respect to 1 and all the absolute values of $J$-squares are smaller than 1 (in this case, $J_{0,0,0}^{2,2,2} \approx 0.143$ and $J_{0,0,0}^{4,2,4} \approx 0.221$).

Therefore, the differential equation for $\partial_r P_l^m$, eq. (5.28) in S74, is simplified as

$$\partial_r P_l^m = \left(\rho_0 + \rho_2 J_{0,0,0}^{l,2,l}\right)\xi_l^U + \rho_2 J_{0,0,0}^{l,2,l+2}\xi_{l+2}^U, \tag{14}$$

where $\xi$ is a compact combination of the three $\eta$ in S74, i.e.

$$\xi_k^U \equiv \left({\eta_1}_k^m\right)^U + \left({\eta_2}_k^m\right)^U + \left({\eta_3}_k^m\right)^U, \tag{15}$$

where the subscript $k$ may takes $l-1, l, l+1, l+2$ (because $\xi_{l+2}^U$ includes terms of $\sigma_l$).

Analogously, we will combine the other $\eta$ combinations to two $\xi$ as

$$\xi_k^V \equiv \left(\eta_1{}_k^m\right)^V + \left(\eta_2{}_k^m\right)^V + \left(\eta_3{}_k^m\right)^V \tag{16}$$

and

$$\xi_k^W \equiv \left(\eta_1{}_k^m\right)^W + \left(\eta_2{}_k^m\right)^W + \left(\eta_3{}_k^m\right)^W . \tag{17}$$

It is worthy to point out here that the term $\rho_2 P_2 \eta_3$ ($P_2$ is the Legendre function of second order here) is ignored in the eq. (5.27a) and therefore in the eqs (5.28)–(5.30) in S74, but it has been shown that this term should definitely be kept in these equations (see Huang 1999; Rogister 2001). To keep the equations of $\eta_1$, $\eta_2$ and $\eta_3$ in homogenous form, we remove the first coefficient $\rho_0$ from the definition of $\eta_3$ in eqs (5.27d) and (5.37)–(5.39) in S74.

After substituting $\partial_r U_l^m$ and $\Delta_l^m$ into $\eta_3$, all the $\partial_r \tilde{\mathrm{Y}}$ expressions include $\tilde{\mathrm{Y}}$ itself and $\tilde{\mathrm{X}}$, while $\partial_r \tilde{\mathrm{X}}$ only exists in $\partial_r P_l^m$ via $(\eta_3{}_l^m)^U$.

As a result of the regular conditions of the solutions at the center, there are five independent solutions for the ten ODEs in the solid inner core. Usually, one can have five independent initial analytical solutions for a small homogeneous spherical Earth near the center (Pekeris & Jarosch 1958; Takeuchi & Saito 1972), all these five solutions of Y are propagated from this small sphere upward to the Earth surface, by integrating these ten ODEs. When crossing the boundary between inner core and FOC (inner core boundary, ICB), there are only continuity conditions for the quantities in $\tilde{\mathrm{Y}}$ but not for those in $\tilde{\mathrm{X}}$. This means that $\tilde{\mathrm{Y}}$ in the FOC is known after using the boundary conditions and $\tilde{\mathrm{X}}$ is unknown. However, the $\tilde{\mathrm{X}}$ values themselves are required and subsequently needed for the calculation of $\partial_r \tilde{\mathrm{Y}}$. In following sections, we will give the solutions of $\tilde{\mathrm{X}}$, $\partial_r \tilde{\mathrm{X}}$ and finally $\partial_r \tilde{\mathrm{Y}}$ in the fluid core.

## 3 SOLUTION OF $\tilde{\mathrm{X}}$

The solution of $\tilde{\mathrm{X}}$ can be obtained from the degenerated ODEs for $\partial_r Q_l^m$ and $\partial_r R_l^m$ in the eqs (5.29) and (5.30) of S74.

### 3.1 From eq. (5.29) of S74

Let's consider $\partial_r Q_l^m = Q_l^m = 0$ and $\mu_0 = \mu_2 = 0$ in the eq. (5.29) of S74. After applying the truncation rule, we have

$$0 = \frac{2L_0}{r} P_l^m + \rho_0 \xi_l^V + \rho_2 \left( J_{+,0,+}^{l,2,l-1} \xi_{l-1}^W + J_{+,0,+}^{l,2,l} \xi_l^V + J_{+,0,+}^{l,2,l+1} \xi_{l+1}^W + J_{+,0,+}^{l,2,l+2} \xi_{l+2}^V \right), \tag{18}$$

in which we have used

$$\frac{2L_0}{r} \lambda_0 \Delta_l^m + \frac{2L_0}{r} J_{0,0,0}^{l,2,l} \lambda_2 \Delta_l^m \approx \frac{2L_0}{r} P_l^m .$$

Now, we separate all the terms of $V_l^m$, $W_{l-1}^m$ and $W_{l+1}^m$ from these $\xi$. The coefficients before the terms $V_l^m$, $W_{l-1}^m$ and $W_{l+1}^m$ in $\xi_{l-1}^W$ are denoted as $(\xi_{l-1}^W)^{V_l}$, $(\xi_{l-1}^W)^{W_{l-1}}$ and $(\xi_{l-1}^W)^{W_{l+1}}$ respectively; they are:

$$\left(\xi_{l-1}^W\right)^{V_l} = 2\omega\Omega_0 J_{-,0,-}^{l-1,1,l}, \tag{19}$$

$$\left(\xi_{l-1}^W\right)^{W_{l-1}} = -\omega^2 - 2\omega\Omega_0 J_{-,0,-}^{l-1,1,l-1} \tag{20}$$

and

$$\left(\xi_{l-1}^W\right)^{W_{l+1}} = 0, \tag{21}$$

respectively. Analogously, the coefficients before the terms $V_l^m$,$W_{l-1}^m$ and $W_{l+1}^m$ in $\xi_l^V$ are:

$$(\xi_l^V)^{V_l} = -\omega^2 - 2\omega\Omega_0 J_{-,0,-}^{l,1,l} - \frac{2L_0}{r} \frac{\sqrt{3}}{r} \phi_2 J_{0,+,-}^{l,2,l}, \tag{22}$$

$$\left(\xi_l^V\right)^{W_{l-1}} = 2\omega\Omega_0 J_{-,0,-}^{l,1,l-1} + \frac{2L_0}{r} \frac{\sqrt{3}}{r} \phi_2 J_{0,+,-}^{l,2,l-1} \tag{23}$$

and

$$\left(\xi_l^V\right)^{W_{l+1}} = 2\omega\Omega_0 J_{-,0,-}^{l,1,l+1} + \frac{2L_0}{r} \frac{\sqrt{3}}{r} \phi_2 J_{0,+,-}^{l,2,l+1}, \tag{24}$$

respectively; the coefficients before the terms $V_l^m$, $W_{l-1}^m$ and $W_{l+1}^m$ in $\xi_{l+1}^W$ are:

$$\left(\xi_{l+1}^W\right)^{V_l} = 2\omega\Omega_0 J_{-,0,-}^{l+1,1,l}, \tag{25}$$

$$\left(\xi_{l+1}^W\right)^{W_{l-1}} = 0 \tag{26}$$

and

$$\left(\xi_{l+1}^W\right)^{W_{l+1}} = -\omega^2 - 2\omega\Omega_0 J_{-,0,-}^{l+1,1,l+1} \tag{27}$$

respectively; and the coefficients before the terms $V_l^m$, $W_{l-1}^m$ and $W_{l+1}^m$ in $\xi_{l+2}^V$ are:

$$\left(\xi_{l+2}^V\right)^{V_l} = -\frac{2L_0^{(4)}}{r} \frac{\sqrt{3}}{r} \phi_2 J_{0,+,-}^{l+2,2,l}, \tag{28}$$

$$\left(\xi_{l+2}^{V}\right)^{W_{l-1}} = 0 \tag{29}$$

and

$$\left(\xi_{l+2}^{V}\right)^{W_{l+1}} = 2\omega\Omega_0 J_{-,0,-}^{l+2,1,l+1} + \frac{2L_0^{(4)}}{r}\frac{\sqrt{3}}{r}\phi_2 J_{0,+,-}^{l+2,2,l+1} \tag{30}$$

respectively.

Now we re-arrange eq. (18) in a matrix form (here is the first row), which is similar (but a little different) to the eq. (E2) in S74:

$$D(1,1)V_l^m + D(1,2)W_{l-1}^m + D(1,3)W_{l+1}^m = (E\cdot\tilde{\mathrm{Y}})(1) \tag{31}$$

where

$$D(1,1) = \left(\rho_0 + \rho_2 J_{+,0,+}^{l,2,l}\right)\left(\xi_l^V\right)^{V_l} + \rho_2\left[J_{+,0,+}^{l,2,l-1}\left(\xi_{l-1}^W\right)^{V_l} + J_{+,0,+}^{l,2,l+1}\left(\xi_{l+1}^W\right)^{V_l} + J_{+,0,+}^{l,2,l+2}\left(\xi_{l+2}^V\right)^{V_l}\right], \tag{32}$$

$$D(1,2) = \left(\rho_0 + \rho_2 J_{+,0,+}^{l,2,l}\right)\left(\xi_l^V\right)^{W_{l-1}} + \rho_2 J_{+,0,+}^{l,2,l-1}\left(\xi_{l-1}^W\right)^{W_{l-1}}, \tag{33}$$

$$D(1,3) = \left(\rho_0 + \rho_2 J_{+,0,+}^{l,2,l}\right)\left(\xi_l^V\right)^{W_{l+1}} + \rho_2 J_{+,0,+}^{l,2,l+1}\left(\xi_{l+1}^W\right)^{W_{l+1}} + \rho_2 J_{+,0,+}^{l,2,l+2}\left(\xi_{l+2}^V\right)^{W_{l+1}} \tag{34}$$

and

$$(E\cdot\tilde{\mathrm{Y}})(1) = -\frac{2L_0}{r}P_l^m - \left(\rho_0 + \rho_2 J_{+,0,+}^{l,2,l}\right)\zeta_l^V - \rho_2\left(J_{+,0,+}^{l,2,l-1}\zeta_{l-1}^W + J_{+,0,+}^{l,2,l+1}\zeta_{l+1}^W + J_{+,0,+}^{l,2,l+2}\zeta_{l+2}^V\right). \tag{35}$$

While these new $\zeta$ are the residual of $\xi$, all the terms of $V_l^m$ and $W_{l\pm1}^m$ have been removed, i.e.,

$$\zeta_{l-1}^W \equiv \xi_{l-1}^W - \text{ all the } V_l^m, W_{l\pm1}^m \text{ terms in } \xi_{l-1}^W = 4\omega\Omega_0 J_{-,-,0}^{l-1,1,l}U_l^m + 2\frac{\sqrt{3}}{r}\phi_2 J_{+,+,0}^{l-1,2,l} f_\lambda P_l^m, \tag{36}$$

$$\zeta_l^V \equiv \xi_l^V - \text{ all the } V_l^m, W_{l\pm1}^m \text{ terms in } \xi_l^V = -\frac{2L_0}{r}(\phi_1{}_l^m + \tilde{g}U_l^m) - 4\omega\Omega_0 J_{-,-,0}^{l,1,l}U_l^m - \frac{2L_0}{r}\phi_2{}' J_{0,0,0}^{l,2,l}U_l^m + 2\frac{\sqrt{3}}{r}\phi_2 J_{+,+,0}^{l,2,l} f_\lambda P_l^m, \tag{37}$$

$$\zeta_{l+1}^W \equiv \xi_{l+1}^W - \text{ all the } V_l^m, W_{l\pm1}^m \text{ terms in } \xi_{l+1}^W = 4\omega\Omega_0 J_{-,-,0}^{l+1,1,l}U_l^m + 2\frac{\sqrt{3}}{r}\phi_2 J_{+,+,0}^{l+1,2,l} f_\lambda P_l^m \tag{38}$$

and

$$\zeta_{l+2}^V \equiv \xi_{l+2}^V - \text{ all the } V_l^m, W_{l\pm1}^m \text{ terms in } \xi_{l+2}^V = -\frac{2L_0^{(4)}}{r}\phi_2' J_{0,0,0}^{l+2,2,l}U_l^m + 2\frac{\sqrt{3}}{r}\phi_2 J_{+,+,0}^{l+2,2,l} f_\lambda P_l^m. \tag{39}$$

Now, the coefficient matrix $D(1,i)$ $(i = 1, 2, 3)$ depends on the radius only and not $\tilde{\mathrm{X}}$ or $\tilde{\mathrm{Y}}$; the coefficient matrix can then be derived from the Earth model, while the right side of eq. (31) includes only $\tilde{\mathrm{Y}}$ which is also known.

In the following, we will derive the other two similar equations.

### 3.2 From eq. (5.30) of S74

Analogously, let $\partial_r R_l^m = R_l^m = 0$ and $\mu_0 = \mu_2 = 0$, then from eq. (5.30) of S74, where the subscript (longitudinal degree) takes $l-1$ (we will take $l+1$ later), we have

$$0 = \left(\rho_0 + \rho_2 J_{+,0,+}^{l-1,2,l-1}\right)\xi_{l-1}^W + \rho_2\left[J_{+,0,+}^{l-1,2,l}\xi_l^V + J_{+,0,+}^{l-1,2,l+1}\xi_{l+1}^W\right]. \tag{40}$$

After repeating the procedure for eq. (5.29) in the above subsection, we get the second equation and re-arrange it into matrix form to get the second row of this matrix equation, i.e.

$$D(2,1)V_l^m + D(2,2)W_{l-1}^m + D(2,3)W_{l-1}^m = (E\cdot\tilde{\mathrm{Y}})(2), \tag{41}$$

where

$$D(2,1) = \left(\rho_0 + \rho_2 J_{+,0,+}^{l-1,2,l-1}\right)\left(\xi_{l-1}^W\right)^{V_l} + \rho_2 J_{+,0,+}^{l-1,2,l}\left(\xi_l^V\right)^{V_l} + \rho_2 J_{+,0,+}^{l-1,2,l+1}\left(\xi_{l+1}^W\right)^{V_l}, \tag{42}$$

$$D(2,2) = \left(\rho_0 + \rho_2 J_{+,0,+}^{l-1,2,l-1}\right)\left(\xi_{l-1}^W\right)^{W_{l-1}} + \rho_2 J_{+,0,+}^{l-1,2,l}\left(\xi_l^V\right)^{W_{l-1}}, \tag{43}$$

$$D(2,3) = \rho_2 J_{+,0,+}^{l-1,2,l}\left(\xi_l^V\right)^{W_{l+1}} + \rho_2 J_{+,0,+}^{l-1,2,l+1}\left(\xi_{l+1}^W\right)^{W_{l+1}} \tag{44}$$

and

$$(E\cdot\tilde{\mathrm{Y}})(2) = -\left(\rho_0 + \rho_2 J_{+,0,+}^{l-1,2,l-1}\right)\zeta_{l-1}^W - \rho_2 J_{+,0,+}^{l-1,2,l}\zeta_l^V - \rho_2 J_{+,0,+}^{l-1,2,l+1}\zeta_{l+1}^W. \tag{45}$$

All these $\xi$ and $\zeta$ are given in the subsection above.

Analogously, from eq. (5.30) of S74 and taking the subscript as $l+1$, and repeating the procedure described above, we have:

$$0 = \left(\rho_0 + \rho_2 J_{+,0,+}^{l+1,2,l+1}\right)\xi_{l+1}^W + \rho_2\left(J_{+,0,+}^{l+1,2,l-1}\xi_{l-1}^W + J_{+,0,+}^{l+1,2,l}\xi_l^V + J_{+,0,+}^{l+1,2,l+2}\xi_{l+2}^V\right). \tag{46}$$

This equation is also re-arranged into matrix form and we get the third row of the matrix equations, i.e.

$$D(3,1)V_l^m + D(3,2)W_{l-1}^m + D(3,3)W_{l-1}^m = (E\cdot\tilde{\mathrm{Y}})(3), \tag{47}$$

where

$$D(3,1) = \left(\rho_0 + \rho_2 J_{+,0,+}^{l+1,2,l+1}\right)\left(\xi_{l+1}^W\right)^{V_l} + \rho_2\left[J_{+,0,+}^{l+1,2,l-1}\left(\xi_{l-1}^W\right)^{V_l} + J_{+,0,+}^{l+1,2,l}\left(\xi_l^V\right)^{V_l} + J_{+,0,+}^{l+1,2,l+2}\left(\xi_{l+2}^V\right)^{V_l}\right], \quad (48)$$

$$D(3,2) = \rho_2 J_{+,0,+}^{l+1,2,l-1}\left(\xi_{l-1}^W\right)^{W_{l-1}} + \rho_2 J_{+,0,+}^{l+1,2,l}\left(\xi_l^V\right)^{W_{l-1}}, \quad (49)$$

$$D(3,3) = \left(\rho_0 + \rho_2 J_{+,0,+}^{l+1,2,l+1}\right)\left(\xi_{l+1}^W\right)^{W_{l+1}} + \rho_2 J_{+,0,+}^{l+1,2,l}\left(\xi_l^V\right)^{W_{l+1}} + \rho_2 J_{+,0,+}^{l+1,2,l+2}\left(\xi_{l+2}^V\right)^{W_{l+1}} \quad (50)$$

and

$$(E\cdot\tilde{\mathrm{Y}})(3) = -\left(\rho_0 + \rho_2 J_{+,0,+}^{l+1,2,l+1}\right)\zeta_{l+1}^W - \rho_2\left(J_{+,0,+}^{l+1,2,l-1}\zeta_{l-1}^W + J_{+,0,+}^{l+1,2,l}\zeta_l^V + J_{+,0,+}^{l+1,2,l+2}\zeta_{l+2}^V\right). \quad (51)$$

All these $\xi$ and $\zeta$ have been given in the subsection above.

Thus, all the $3\times 3$ components of the matrix **D** and the three components of the column matrix $(E\cdot\tilde{\mathrm{Y}})$ are known, and we can then obtain the solution

$$\tilde{\mathrm{X}} = D^{-1}\cdot(E\cdot\tilde{\mathrm{Y}}). \quad (52)$$

For the purpose of practical calculation of $\tilde{\mathrm{X}}$, it is easy to get $(E\cdot\tilde{\mathrm{Y}})$ directly and we do not need separate $E$ and $\tilde{\mathrm{Y}}$. However, for convenience of writing in the next section, we separate $E$ and $\tilde{\mathrm{Y}}$ by a procedure of separation similar to that of $D$ and $\tilde{\mathrm{X}}$. They are:

$$E(1,1) = -\left(\rho_0 + \rho_2 J_{+,0,+}^{l,2,l}\right)\left(\zeta_l^V\right)^U - \rho_2\left[J_{+,0,+}^{l,2,l-1}\left(\zeta_{l-1}^W\right)^U + J_{+,0,+}^{l,2,l+1}\left(\zeta_{l+1}^W\right)^U + J_{+,0,+}^{l,2,l+2}\left(\zeta_{l+2}^V\right)^U\right], \quad (53)$$

$$E(1,2) = -\frac{2L_0}{r} - \left(\rho_0 + \rho_2 J_{+,0,+}^{l,2,l}\right)\left(\zeta_l^V\right)^P - \rho_2\left[J_{+,0,+}^{l,2,l-1}\left(\zeta_{l-1}^W\right)^P + J_{+,0,+}^{l,2,l+1}\left(\zeta_{l+1}^W\right)^P + J_{+,0,+}^{l,2,l+2}\left(\zeta_{l+2}^V\right)^P\right], \quad (54)$$

$$E(1,3) = \left(\rho_0 + \rho_2 J_{+,0,+}^{l,2,l}\right)\frac{2L_0}{r}, \quad (55)$$

$$E(2,1) = -\left(\rho_0 + \rho_2 J_{+,0,+}^{l-1,2,l-1}\right)\left(\zeta_{l-1}^W\right)^U - \rho_2 J_{+,0,+}^{l-1,2,l}\left(\zeta_l^V\right)^U - \rho_2 J_{+,0,+}^{l-1,2,l+1}\left(\zeta_{l+1}^W\right)^U, \quad (56)$$

$$E(2,2) = -\left(\rho_0 + \rho_2 J_{+,0,+}^{l-1,2,l-1}\right)\left(\zeta_{l-1}^W\right)^P - \rho_2 J_{+,0,+}^{l-1,2,l}\left(\zeta_l^V\right)^P - \rho_2 J_{+,0,+}^{l-1,2,l+1}\left(\zeta_{l+1}^W\right)^P, \quad (57)$$

$$E(2,3) = \frac{2L_0}{r}\rho_2 J_{+,0,+}^{l-1,2,l}, \quad (58)$$

$$E(3,1) = -\left(\rho_0 + \rho_2 J_{+,0,+}^{l+1,2,l+1}\right)\left(\zeta_{l+1}^W\right)^U - \rho_2\left[J_{+,0,+}^{l+1,2,l-1}\left(\zeta_{l-1}^W\right)^U + J_{+,0,+}^{l+1,2,l}\left(\zeta_l^V\right)^U + J_{+,0,+}^{l+1,2,l+2}\left(\zeta_{l+2}^V\right)^U\right], \quad (59)$$

$$E(3,2) = -\left(\rho_0 + \rho_2 J_{+,0,+}^{l+1,2,l+1}\right)\left(\zeta_{l+1}^W\right)^P - \rho_2\left[J_{+,0,+}^{l+1,2,l-1}\left(\zeta_{l-1}^W\right)^P + J_{+,0,+}^{l+1,2,l}\left(\zeta_l^V\right)^P + J_{+,0,+}^{l+1,2,l+2}\left(\zeta_{l+2}^V\right)^P\right], \quad (60)$$

$$E(3,3) = \frac{2L_0}{r}\rho_2 J_{+,0,+}^{l+1,2,l} \quad (61)$$

and

$$E(i,4) = 0 \quad (\text{for } i = 1,2,3). \quad (62)$$

Where, these new $(\zeta_k^{V,W})^{U,P}$ are the coefficient before $U$ or $P$ in $(\zeta_k^V)$ or $(\zeta_k^W)$; they are

$$(\zeta_{l-1}^W)^U = 4\omega\Omega_0 J_{-,-,0}^{l-1,1,l}, \quad (63)$$

$$\left(\zeta_{l-1}^W\right)^P = 2\frac{\sqrt{3}}{r}\phi_2 f_\lambda J_{+,+,0}^{l-1,2,l}, \quad (64)$$

$$\left(\zeta_l^V\right)^U = -\frac{2L_0}{r}\tilde{g} - 4\omega\Omega_0 J_{-,-,0}^{l,1,l} - \frac{2L_0}{r}\phi_2{}' J_{0,0,0}^{l,2,l}, \quad (65)$$

$$\left(\zeta_l^V\right)^P = 2\frac{\sqrt{3}}{r}\phi_2 f_\lambda J_{+,+,0}^{l,2,l}, \quad (66)$$

$$\left(\zeta_{l+1}^W\right)^U = 4\omega\Omega_0 J_{-,-,0}^{l+1,1,l}, \quad (67)$$

$$\left(\zeta_{l+1}^W\right)^P = 2\frac{\sqrt{3}}{r}\phi_2 f_\lambda J_{+,+,0}^{l+1,2,l}, \quad (68)$$

$$\left(\zeta_{l+2}^V\right)^U = -\frac{2L_0^{(4)}}{r}\phi_2{}' J_{0,0,0}^{l+2,2,l} \quad (69)$$

and

$$\left(\zeta_{l+2}^V\right)^P = 2\frac{\sqrt{3}}{r}\phi_2 f_\lambda J_{+,+,0}^{l+2,2,l}. \quad (70)$$

## 4 DIRECT SOLUTION OF $\partial_r \tilde{\mathrm{X}}$

As mentioned above, the ODEs for $\partial_r V_l^m$ and $\partial_r W_l^m$ in eqs (5.23) and (5.24) of S74 no longer hold in the fluid, but the radial gradient of the displacement is required in some research and therefore we need to obtain alternative ODEs. This can be done by taking the derivative of eq. (52):

$$D \cdot \partial_r \tilde{\mathrm{X}} = \partial_r (E \cdot \tilde{\mathrm{Y}}) - (\partial_r D) \cdot \tilde{\mathrm{X}}. \tag{71}$$

In this equation, $D$, $\partial_r D$ and $E$ do not include $\tilde{\mathrm{X}}$ nor $\tilde{\mathrm{Y}}$, but the $\partial_r P_l^m$ in $\partial_r(E \cdot \tilde{\mathrm{Y}})$ includes $\partial_r \tilde{\mathrm{X}}$, therefore, this equation can not be solved directly. One special property of $\partial_r(E \cdot \tilde{\mathrm{Y}})$ is that within it only $\partial_r P_l^m$ includes $\partial_r \tilde{\mathrm{X}}$ [via $({\eta_3}_l^m)^U$] and the other three components [$\partial_r(U_l^m, {\phi_1}_l^m, {g_1}_l^m)$] do not; this property can save our next labour.

Adding the eqs (5.31), (5.34) and (5.37) of S74 together, and applying the truncation rule of this paper, we write out the $\xi_l^U$ and $\xi_{l+2}^U$ in $\partial_r P_l^m$ in eq. (14):

$$\xi_l^U = \zeta_l^U + \left(\frac{\sqrt{3}}{r}\phi_2 J_{0,+,-}^{l,2,l}\right)\partial_r V_l^m + \left(-\frac{\sqrt{3}}{r}\phi_2 J_{0,+,-}^{l,2,l-1}\right)\partial_r W_{l-1}^m + \left(-\frac{\sqrt{3}}{r}\phi_2 J_{0,+,-}^{l,2,l+1}\right)\partial_r W_{l+1}^m, \tag{72}$$

$$\xi_{l+2}^U = \zeta_{l+2}^U + \left(\frac{\sqrt{3}}{r}\phi_2 J_{0,+,-}^{l+2,2,l}\right)\partial_r V_l^m + \left(-\frac{\sqrt{3}}{r}\phi_2 J_{0,+,-}^{l+2,2,l+1}\right)\partial_r W_{l+1}^m, \tag{73}$$

where $\zeta_l^U$ and $\zeta_{l+2}^U$ are the parts of $\xi_l^U$ and $\xi_{l+2}^U$ in which the terms of $\partial_r V_l^m$ and $\partial_r W_{l\pm1}^m$ have been removed, i.e.,

$$\begin{aligned}
\zeta_l^U = {} & {g_1}_l^m + U_l^m \left\{-\left[\omega^2 + \frac{2}{r}(g+\tilde{g})\right] - \left(\frac{2}{r}{\phi_2}' - {\phi_2}''\right) J_{0,0,0}^{l,2,l}\right\} \\
& + V_l^m \left[-\frac{L_0}{r}{\phi_2}' + 2\omega\Omega_0 J_{0,+,-}^{l,1,l} - \frac{L_0}{r}{\phi_2}' J_{0,0,0}^{l,2,l} + \frac{\sqrt{3}}{r} J_{0,+,-}^{l,2,l}\left({\phi_2}' - \frac{1}{r}\phi_2\right)\right] \\
& + W_{l-1}^m \left[-2\omega\Omega_0 J_{0,+,-}^{l,1,l-1} - \frac{\sqrt{3}}{r} J_{0,+,-}^{l,2,l-1}\left({\phi_2}' - \frac{1}{r}\phi_2\right)\right] + W_{l+1}^m \left[-2\omega\Omega_0 J_{0,+,-}^{l,1,l+1} - \frac{\sqrt{3}}{r} J_{0,+,-}^{l,2,l+1}\left({\phi_2}' - \frac{1}{r}\phi_2\right)\right],
\end{aligned} \tag{74}$$

$$\begin{aligned}
\zeta_{l+2}^U = {} & U_l^m \left[-\left(\frac{2}{r}{\phi_2}' - {\phi_2}''\right) J_{0,0,0}^{l+2,2,l}\right] + V_l^m \left[-\frac{L_0^{(4)}}{r}{\phi_2}' J_{0,0,0}^{l+2,2,l} + \frac{\sqrt{3}}{r} J_{0,+,-}^{l+2,2,l}\left({\phi_2}' - \frac{1}{r}\phi_2\right)\right] \\
& + W_{l+1}^m \left[-2\omega\Omega_0 J_{0,+,-}^{l+2,1,l+1} - \frac{\sqrt{3}}{r} J_{0,+,-}^{l+2,2,l+1}\left({\phi_2}' - \frac{1}{r}\phi_2\right)\right].
\end{aligned} \tag{75}$$

We subtract the terms of $\partial_r \tilde{\mathrm{X}}$ from $\partial_r(E \cdot \tilde{\mathrm{Y}})$, move them into the left side of eq. (71) and add their coefficients to $D$ and get a new $3 \times 3$ matrix, $\mathbf{A}$, the new coefficient matrix for $\partial_r \tilde{\mathrm{X}}$ (not the propagation matrix $\mathbf{A}$ in eq. 2). Noting that $\partial_r \tilde{\mathrm{X}}$ exist in $E(i,2) \cdot (\partial_r\ P_l^m)$ only, we have:

$$A(i,1) = D(i,1) + E(i,2)\left[\left(\rho_0 + \rho_2 J_{0,0,0}^{l,2,l}\right)\left(\frac{\sqrt{3}}{r}\phi_2 J_{0,+,-}^{l,2,l}\right) + \rho_2 J_{0,0,0}^{l,2,l+2}\left(\frac{\sqrt{3}}{r}\phi_2 J_{0,+,-}^{l+2,2,l}\right)\right], \tag{76}$$

$$A(i,2) = D(i,2) + E(i,2)\left[\left(\rho_0 + \rho_2 J_{0,0,0}^{l,2,l}\right)\left(-\frac{\sqrt{3}}{r}\phi_2 J_{0,+,-}^{l,2,l-1}\right)\right], \tag{77}$$

$$A(i,3) = D(i,3) + E(i,2)\left[\left(\rho_0 + \rho_2 J_{0,0,0}^{l,2,l}\right)\left(-\frac{\sqrt{3}}{r}\phi_2 J_{0,+,-}^{l,2,l+1}\right) + \rho_2 J_{0,0,0}^{l,2,l+2}\left(-\frac{\sqrt{3}}{r}\phi_2 J_{0,+,-}^{l+2,2,l+1}\right)\right], \tag{78}$$

for $i = 1, 2, 3$.

Thus, eq. (71) becomes

$$A \cdot \partial_r \tilde{\mathrm{X}} = \partial_r (E \cdot \tilde{\mathrm{Y}}) - (\partial_r D) \cdot \tilde{\mathrm{X}}. \tag{79}$$

The above equation is correct if and only if the terms of $\partial_r \tilde{\mathrm{X}}$ are removed from $\partial_r P_l^m$. Alternatively, we write eq. (71) in following exact form

$$A \cdot \partial_r \tilde{\mathrm{X}} = (\partial_r E) \cdot \tilde{\mathrm{Y}} + E \cdot (\partial_r \tilde{\mathbf{y}}) - (\partial_r D) \cdot \tilde{\mathrm{X}}, \tag{80}$$

where, $\partial_r \tilde{\mathbf{y}}$ are exactly the same as $\partial_r \tilde{\mathrm{Y}}$ except that $\partial_r P_l^m$ no longer includes $\partial_r \tilde{\mathrm{X}}$, i.e.

$$\partial_r \tilde{\mathbf{y}} \equiv \left(\partial_r U_l^m, \overline{\partial_r P_l^m}, \partial_r {\phi_1}_l^m, \partial_r {g_1}_l^m\right)^T, \tag{81}$$

where $\overline{\partial_r P_l^m}$ denotes the part of $\partial_r P_l^m$ in which the terms of $\partial_r V_l^m$ and $\partial_r W_{l\pm1}^m$ are removed, i.e.

$$\overline{\partial_r P_l^m} \equiv \left(\rho_0 + \rho_2 J_{0,0,0}^{l,2,l}\right)\zeta_l^U + \rho_2 J_{0,0,0}^{l,2,l+2}\zeta_{l+2}^U, \tag{82}$$

where, $\zeta_l^U$ and $\zeta_{l+2}^U$ are defined in eqs (74) and (75).

In the above matrix equations, $A$, $\partial_r D$ and $\partial_r E$ depend only on the radius ($r$), the density ($\rho_0$, $\rho_2$), the Lame parameters ($\lambda_0$, $\lambda_2$), the frequency ($\omega$), the mean angular velocity ($\Omega_0$), the initial mean gravity ($g$ and $\tilde{g}$, centrifugal term is removed from the latter) and the elliptical term of gravitational potential ($\phi_2$). All these parameters and their radial derivative are known from a given Earth model. $\partial_r D$ and $\partial_r E$ are directly and easily obtained from $D$ and $E$, respectively, and we omit their explicit form here. While, $\tilde{X}$, $\tilde{Y}$ and $\partial_r \tilde{\mathbf{y}}$ (not $\partial_r \tilde{Y}$) have been known. Therefore, we can have $\partial_r \tilde{X}$ now.

Next, one can substitute the $\partial_r \tilde{X}$ in $\partial_r P_l^m$ with the values obtained before and get the exact value of $\partial_r P_l^m$. The solutions for $\partial_r \tilde{Y}$ (not $\partial_r \tilde{\mathbf{y}}$) are then completed [because $\partial_r(U_l^m, {\phi_1}_l^m, {g_1}_l^m)$ have been calculated directly from eqs (7), (11) and (12), respectively].

## 5 SUMMARY AND THE SCHEME OF CALCULATION

In the above sections, we give out the explicit form of the coefficient matrices $D$, $E$ and $A$, from which we get the direct solutions of $\tilde{X}$, $\partial_r \tilde{X}$ and $\partial_r \tilde{Y}$. The scheme of calculation is summarized as follow:

(i) calculate the values of the profiles of $\rho_0$, $\rho_2$, $\lambda_0$, $\lambda_2$, $g$, $\tilde{g}$ and $\phi_2$ (and their radial derivative) from a given Earth model (one may first need to calculate the ellipticity profile from Clairaut equation);

(ii) from (i), all the $(\xi_k^{V,W})^{V,W}$ are calculated from eqs (19)–(30);

(iii) then all nine components of $D(i, j)$ [as well as their radial derivative, $\partial_r D(i, j)$] are obtained from eqs (32)–(34), (42)–(44) and (48)–(50);

(iv) we have the values of $\tilde{Y}$ from the continuation condition at the ICB (in fact, we can only have $\tilde{Y}$ because such a boundary does not provide any information of $\tilde{X}$. See Huang (2001) for example);

(v) with the values of (i) and (iv), one has all the $\zeta_k^{V,W}$ from eqs (36)–(39);

(vi) then the three components of $(E \cdot \tilde{Y})$ are obtained directly from eqs (35), (45) and (51) [or, by another way: by calculating the 12 components of $E$ (as well as $\partial_r E$) from eqs (53)–(62) and multiplying with $\tilde{Y}$)];

(vii) one can thus get the solutions of $\tilde{X}$ from eq. (52);

(viii) we then calculate the new matrix $A$ by eqs (76)–(78);

(ix) we calculate $\partial_r \tilde{\mathbf{y}}$: $\partial_r(U_l^m, {\phi_1}_l^m, {g_1}_l^m)$ are obtained from eqs (7), (11) and (12), respectively, while $\overline{\partial_r P_l^m}$ is obtained from eqs (82) and (74)–(75) (therefore, this $\overline{\partial_r P_l^m}$ calculated here is not the exact value of $\partial_r P_l^m$);

(x) eq. (80) can be solved now and one gets the solutions of $\partial_r \tilde{X}$; Alternatively, one can solve eq. (79) instead, in which the three components of $\partial_r(E \cdot \tilde{Y})$ are calculated by eqs (35), (45) and (51), but remember to replace $\partial_r P_l^m$ with $\overline{\partial_r P_l^m}$ wherever it emerges in all $\partial_r \zeta_k^{V,W}$ (eqs 36–39);

(xi) substituting the $\partial_r \tilde{X}$ into $\xi_l^m$ and $\xi_{l+2}^m$ in eqs (72)–(73), one hence has the exact value of $\partial_r P_l^m$ from eq. (14) with these exact values of $\xi_l^m$ and $\xi_{l+2}^m$.

Thus the exact solutions for $\tilde{X}$, $\partial_r \tilde{X}$ and $\partial_r \tilde{Y}$ are completed. One can use these values and the ODEs for $\partial_r \tilde{X}$ and $\partial_r \tilde{Y}$ to propagate, step by step and upwardly, the solutions from the ICB to the CMB (or downward from CMB to ICB).

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