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Geostatistics for Power Models of Gaussian Fields[1]

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This paper introduces geostatistical approaches (i.e., kriging estimation and simulation) for a group of non-Gaussian random fields that are power algebraic transformations of Gaussian and lognormal random fields. These are power random fields (PRFs) that allow the construction of stochastic polynomial series. They were derived from the exponential random field, which is expressed as Taylor series expansion with PRF terms. The equations developed from computation of moments for conditional random variables allow the correction of Gaussian kriging estimates for the non-Gaussian space. The introduced PRF geostatistics shall provide tools for integration of data that requires simple algebraic transformations, such as regression polynomials that are commonly encountered in the practical applications of estimation. The approach also allows for simulations drawn from skewed distributions.

KEY WORDS: skewed distribution, non-Gaussian kriging, power transform, conditional simulation.

INTRODUCTION

Earth scientists and engineers frequently recur to forecasting an attribute of interest with a polynomial regression relationship. An attribute that is not directly measured is estimated with a polynomial function that contains powers of a set of attributes that may be easier to measure. These regression models do not account for spatial relationships and correlations between spatial locations. A more advanced forecast is to estimate polynomial relationships or other models in the spatial framework using multivariate geostatistics—the data integration approach.

Gaussian geostatistics is able to handle linear combinations such as $W(x) = [\mathbf{Z}(x)]^T\mathbf{A}$ where $W(x)$ is a random field for the forecasted attribute at spatial locations x, $\mathbf{Z}(x) = [Z_1(x)Z_2(x)Z_3(x)\ldots Z_n(x)]$ a second order stationary vector random field, and $\mathbf{A}$ a vector of constant coefficients. Gaussian geostatistics may combine data by kriging the scalar random field $W(x)$, or may cokrige the vector $\mathbf{Z}(x)$ and combine the estimates. Equivalences between both alternatives are in Myers (1983). Combinations of multiple attributes can be estimated with conditional components in a sequential fashion (Vargas-Guzmán, 2003), and also

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may be carried on in the frequency domain (Vargas-Guzmán, Warrick, and Myers, 2002).

The limitation to "linear combinations" in geostatistics presents severe restrictions for many practical applications of data integration in geology, hydrology, soil science, environmental science, remote sensing, and others. An example of a combination using powers is the estimation of intrinsic permeability by empirical regression polynomial functions using other formation attributes such as porosity and textural distribution data. The pedo-transfer functions are regression models to predict hydraulic parameters from vadose zone data (Van Genuchten, Leji, and Yates, 1991). Another example is from petroleum reservoir modeling. Consider Archie's law in a spatial context, one has random fields for the rock resistivity $R_0(x)$ and a formation factor $F(x)$ linearly related by an empirical relation as $R_0(x) = F(x)R_w$, where R_w might be considered a constant resistivity of the saturating brine R_w (Luthi, 2000). It is also known that $F(x)$ is related to the porosity $\phi(x)$ as $\phi(x) = [aF^{-1}(x)]^p$ where a is a constant coefficient and p a constant power. Calling $Z(x) = aF^{-1}(x)$, assume one could estimate $Z(x)$, and using conditional expected values the estimate is $\langle Z(x_0)\rangle$ at a nonsampled location x_0. Note that angular braces will be used to represent conditional expected values. Then, it is easy to show that under dependence $\langle Z^p(x_0)\rangle \neq \langle Z(x_0)\rangle^p$. However, the correct estimate is $\langle Z^p(x_0)\rangle$ and cannot be directly obtained by Gaussian geostatistics unless a correction term is developed. Another example is the power averaging for the change of support of permeability as implicitly proved in Neuman and Orr (1993) and Paleologos, Neuman, and Tartakovsky (1996). They claim their results to be close to empirical power averaging as shown in Desbarats and Dimitrakopoulos (1990). However, these works do not address the case of averaging powers of kriging estimates which are non-Gaussian.

Non-Gaussian geostatistics has attracted a lot of research; Journel (1980), Dowd (1982), Mejia and Rodriguez-Iturbe (1974), Krige (1978), Rendu (1979), and others have contributed to non-Gaussian geostatistics for the lognormal distribution. The lognormal distribution handles the collocated transformation model $\exp(Z(x)) = Y(x)$ or $\ln(Y(x)) = Z(x)$ where ln is the natural logarithm. It is well known that Gaussian estimates need a correction for providing the lognormal estimates. This is $\exp(\hat{Z}(x) + (1/2)\hat{\sigma}_k^2) = \hat{Y}(x)$, where hats are used for the estimates and $\hat{\sigma}_k^2$ is the estimation variance for the Gaussian estimate. Direct kriging of non-Gaussian attributes may need nonlinear estimation, corrections such as the one for lognormal kriging are attractive because they avoid nonlinear geostatistics. An extension of the lognormal correction term for other transformations has been attempted following Cox and Hinkley (1974). It is suggested that it only works for small kriging variances (Chiles and Delfiner, 1999). Other distributions cannot be handled by lognormal kriging (David, 1988). The need for geostatistics for skewed distributions is one factor leading to indicator kriging (Journel, 1983), disjunctive kriging (Matheron, 1976; Rivoirard, 1994), annealing simulation methods

(Deutsch and Cockerham, 1994), and recent multipoint probabilistic geostatistics (Guardiano and Srivastava, 1993). These approaches are not in the scope of this paper. The importance of geostatistics for power averages is stressed in Journel (1999). The computation of exact spatial averaging of powers has forced to direct estimation and simulation of powers. The present research focuses on a different methodology, which is based on Gaussian kriging estimations that are transformed and corrected for non-Gaussian expected values.

In this research, I have introduced power random fields (PRFs) based on the Taylor series expansion of the exponential random field. Using the theory of conditional moments, correction relations are developed for conditional estimation (i.e., kriging) of power transforms. The cases of Gaussian and lognormal distributions for the base attributes are considered. Simulation approaches are explained. The purpose is to develop geostatistical tools for estimation by kriging and simulation of the power transformations that may be used for data integration.

THEORY

The Power Random Field (PRF)

A PRF is an ergodic, non-Gaussian random field $Y(x)$ obtained from a collocated power transformation of a Gaussian random field $Z(x)$ defined on the probability space $(\Omega, ƛ, P)$ where Ω is the sample space, $ƛ$ is the Borel field of subsets of Ω, and P is the Gaussian probability measure on the space defined by $(\Omega, ƛ)$ following Kolmogorov's axioms. Then, $Y(x)$ is not from a spatial transformation, the power transformation of $Z(x)$ is collocated but with a probability space that can be derived from $(\Omega, ƛ, P)$.

Several monomial power transformations can be assembled to model a polynomial transform. A simple case is when all components have the same base $Z(x)$, this is

$$W(x) = 1 + a_1 Z(x) + a_2 Z^2(x) + a_3 Z^3(x) + \cdots + a_n Z^n(x) \tag{1}$$

A classic transformation of a Gaussian random field is the lognormal random field $W(x) = \exp[Z(x)]$. One can use series to decompose this classic case of the lognormal random field as follows:

$$W(x) = \exp[Z(x)] = 1 + Z(x) + \frac{1}{2!}Z^2(x) + \frac{1}{3!}Z^3(x) + \frac{1}{4!}Z^4(x) + \cdots \tag{2}$$

In this case, the monomial components of this series decomposition have the same base $Z(x)$ and may be mutually dependent. In this manuscript, the exponential model is used to check convergence of results. Each monomial in Equation (2) is a second order stationary spatial component of the second order stationary exponential random field. The model above is a collocated transformation where x does not contribute to the analysis. These component PRFs $Y_j(x)$ could be linearized by taking roots p. Note this is a different situation to the one formulated for intrinsic random functions by Matheron (1973) where the terms are from finite differences of $W(x)$.

Another case is a stochastic polynomial. This is

$$W(x) = 1 + Y_1(x) + Y_2(x) + Y_3(x) + \cdots + Y_n(x) \tag{3}$$

Then, in general one could think of polynomial random fields that are made of combinations of PRFs as monomial components as follows:

$$W(x) = 1 + a_1 Z_1(x) + a_2 Z_2^2(x) + a_3 Z_3^3(x) + \cdots + a_n Z_n^n(x) \tag{4}$$

Note that Gaussian fields $Z_i(x)$ in Equation (4) are different to each other and two cases are possible. The first case is when the $Z_i(x)$ fields are independent and a more realistic case is when they are correlated. Correlated cases are considered in a later section.

Perturbation Analysis

The base random fields $Z_i(x)$ are Gaussian, and estimates Z_0 may be done using N data values $Z(x_\alpha)$ that are available at x_α locations. Without loss of generality, the exponential model in Equation (2) may be written for an estimated location in terms of mean and perturbations and taking expected value conditional to the data values yields

$$\langle W(x_0)\rangle = \langle \exp[Z_0 + z(x)]\rangle = 1 + \langle Z_0 + z(x)\rangle + \frac{1}{2!}\langle (Z_0 + z(x))^2\rangle + \frac{1}{3!}\langle (Z_0 + z(x))^3\rangle + \frac{1}{4!}\langle (Z_0 + z(x))^4\rangle + \cdots \tag{5a}$$

the perturbation $z(x)$ has a mean zero and a conditional variance β^2 for each monomial at each location. The effect of perturbation on the conditional mean value of powers provides residual terms $r_p(x)$. The Gaussian conditional mean can be grouped and the error terms that appear are the expected value of power perturbations. In the case of the first and second powers, the residual terms are just function of the estimation variance β^2. For higher terms they are function of powers of the mean and estimation variance. This becomes apparent in next section and it is better to write Equation (5a) as

$$\langle W(x_0)\rangle = \langle \exp[Z_0 + z(x)]\rangle = 1 + Z_0 + \frac{1}{2!}\left(Z_0^2 + r_2(x)\right) + \frac{1}{3!}\left(Z_0^3 + r_3(x)\right) + \frac{1}{4!}\left(Z_0^4 + r_4(x)\right) + \cdots \tag{5b}$$

This provides addible residual terms. These residual terms may allow for reconstruction of non-Gaussian estimates $\hat{W}(x_0) \equiv \langle W(x_0)\rangle$ from Gaussian estimates $\hat{Z}(x_0) \equiv \langle Z_0\rangle$. Recall that the error in the exponential estimate is a residual term $\langle \exp(z(x))\rangle$. Thus, the next step is to evaluate the residual terms for the PRFs that can be exact for the lognormal random field in Equation (5b).

Spatial Estimation Theory for Algebraic Transformations

Conditional Estimates for PRFs of Gaussian Random Fields

Application of conditional expectation theory always requires knowledge of the conditional probability density function (pdf) with known parameters. For the PRFs, one may apply

$$\langle Y_p(x_0)\rangle = a_p \int_{-\infty}^{\infty} [Z(x_0)]^p f(Z(x_0)|[Z(x_\alpha), \alpha = 1 \text{ to } \mathrm{N}])\, dZ \tag{6}$$

where the conditional pdf at one spatial location is

$$f(Z|[Z(x_\alpha), \alpha = 1 \text{ to } \mathrm{N}]) = \frac{1}{\beta\sqrt{2\pi}} e^{-\frac{(Z-m)^2}{2\beta^2}} \tag{7}$$

Note that instead of conditional moments, I refer to conditional expected value of powers of a Gaussian marginal pdf. The estimation standard deviation β is always positive and the power p is considered positive and m is a conditional mean. Under those conditions the integral of Equation (6) is evaluated in terms of hypergeometric functions as follows:

$$\langle Y_p(x_0)\rangle = \frac{1}{\sqrt{2\pi}\beta}\frac{1}{\sqrt{2}(1+p)}\left(\frac{1}{\beta^2}\right)^{-p/2}\left[-2^{p/2}(1+p)\left(\sqrt{2}m\left((-1)^p\left(-\frac{1}{m}\right)^p - (m)^p - \left(-\frac{1}{m}\right)^p m^p\right)\Gamma\left(1+\frac{p}{2}\right)\right)\,{}_1F_1\left[\frac{1}{2}-\frac{p}{2}, \frac{3}{2}, -\frac{m^2}{2\beta^2}\right]\right.$$

Table 1. Conditional Power Estimates M^p From Gaussian Estimates m and Estimation Standard Deviation β

$\langle Z^p(x)\rangle$	
$\langle Z^2(x)\rangle$	$M^2 = m^2 - \beta^2$
$\langle Z^3(x)\rangle$	$M^3 = m(m^2 - 3\beta^2)$
$\langle Z^4(x)\rangle$	$M^4 = m^4 + 6m^2\beta^2 + 3\beta^4$
$\langle Z^5(x)\rangle$	$M^5 = m(m^4 + 10m^2\beta^2 + 15\beta^4)$
$\langle Z^6(x)\rangle$	$M^6 = m^6 + 15m^4\beta^2 + 45m^2\beta^4 + 15\beta^6$
$\langle Z^7(x)\rangle$	$M^7 = m(m^6 + 21m^4\beta^2 + 105m^2\beta^4 + 105\beta^6)$
$\langle Z^8(x)\rangle$	$M^8 = m^8 + 28m^6\beta^2 + 210m^4\beta^4 + 420m^2\beta^6 + 105\beta^8$
$\langle Z^9(x)\rangle$	$M^9 = m(m^8 + 36m^6\beta^2 + 378m^4\beta^4 + 1260m^2\beta^6 + 945\beta^8)$
$\langle Z^{10}(x)\rangle$	$M^{10} = m^{10} + 45m^8\beta^2 + 630m^6\beta^4 + 3150m^4\beta^6 + 4725m^2\beta^8 + 945\beta^{10}$

$$
\begin{aligned}
&-\left((-1)^p\left(-\frac{1}{m}\right)^p(-m)^p + \left(-\frac{1}{m}\right)^p m^p\right)\sqrt{\frac{1}{\beta^2}}\beta^2\Gamma\left(\frac{1+p}{2}\right)\\
&\times {}_1F_1\left[-\frac{p}{2},\frac{1}{2},-\frac{m^2}{2\beta^2}\right] + (\sqrt{2}m(-1)^p(-m)^p - m^p)\left(\frac{1}{\beta^2}\right)^{p/2}\\
&\times {}_PF_Q\left[\left\{\frac{1}{2},1\right\},\left\{1+\frac{p}{2},\frac{3}{2}+\frac{p}{2}\right\},-\frac{m^2}{2\beta^2}\right]\Bigg)\Bigg]
\end{aligned} \tag{8}
$$

where ${}_1F_1$ and ${}_PF_Q$ are the hypergeometric functions and Γ is the gamma function. Table 1 shows specific results obtained for the first 10 integer powers. As will be soon apparent these results are useful in the context of correcting Gaussian kriging estimates. Results of Table 1 have been tested with direct evaluation of first two moments for the non-Gaussian transformed random variables from transformed pdfs.

The equations of Table 1 clearly show that the residual terms are functions of the mean and variance estimated. Figure 1 shows the first 10 powers for the exponential of Equation (4) and estimation standard deviation from 0 to 1, the plot is for a conditional mean $m = 1$. The estimated values are expressed in a logarithmic axis.

If the random fields are independent monomial terms, the resulting combination is a polynomial random field $W(x)$. The truncated sum of monomial power estimates added to a qth term converges to the exponential estimate. Figure 2 shows the fast converge attained when the Gaussian estimated value is zero. In such a case, the monomials of odd powers do not contribute to the exponential estimate because the conditional mean multiplies the expected values (see Table 1). The monomials of even powers are added and displayed in Figure 2. The second power approaches the exponential but the second plus the fourth moment get very close

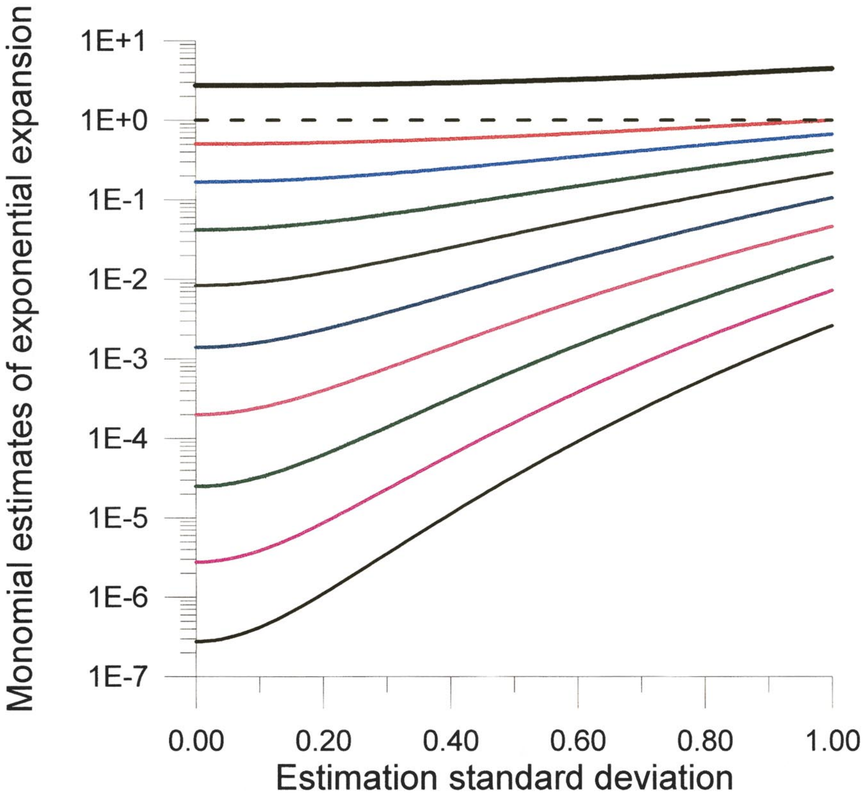

Figure 1. Contribution of each monomial to the exponential expansion for a mean = 1 and kriging or estimation standard deviation. Power terms from top to bottom are {1, 2, 3, 4, 5, 6, 7, 8, 9, and 10th power term}.

and so forth. This convergence is better explained with Figures 3 and 4, where the differences between the true exponential estimate and the truncated polynomial estimate up to a power are plotted.

Figure 3 shows the differences between the true exponential estimates and the curves of added monomials shown in Figure 2. The higher error curve is for power two and the second curve is for power four. It is obvious that the second even power (fourth) substantially contributes to the convergence. However, such a convergence is a function of the mean, as was proven in the equations in Table 1. The convergence is slower for larger values of conditional mean estimates. This is shown in Figure 4 corresponding to a Gaussian conditional mean = 3. Larger values of the estimated attribute will force the practitioner to use higher order powers. Figure 4 allows the use of a probability scale because the plot has no negative values.

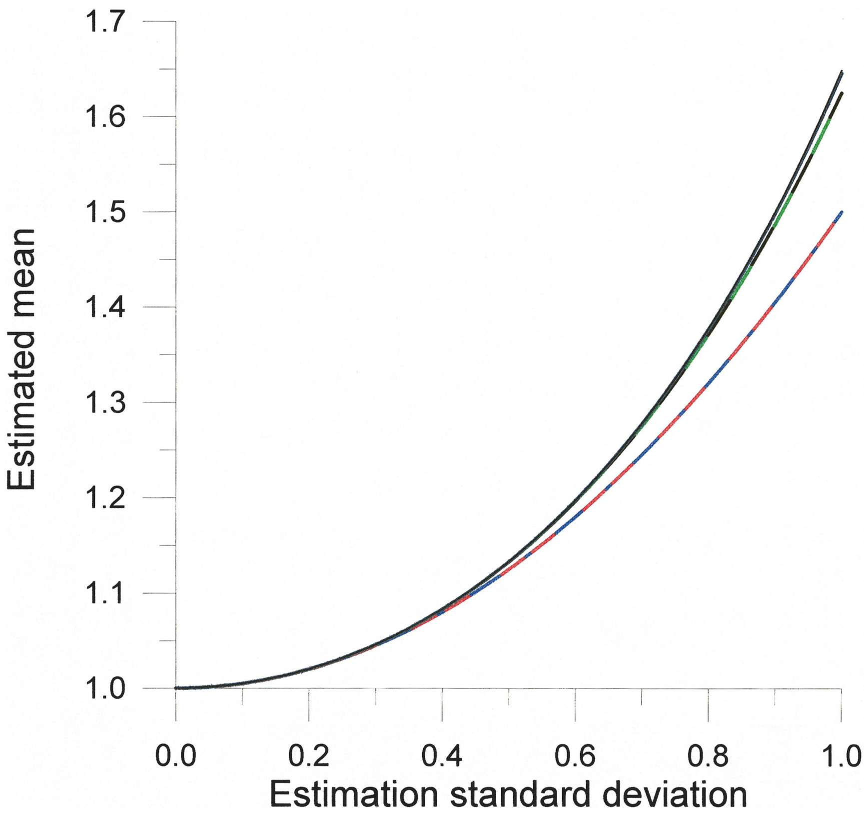

Figure 2. Fast convergence to the exponential conditional mean estimate with the first five moments for mean = 0 and variable kriging or estimation standard deviation. From bottom to top curves cumulate even powers.

The convergence observed in Figures 3 and 4 provides additional evidence that corrections following Table 1 should be applied to the kriging estimates of a PRF. Thus, PRFs could be components of an exponential random field if coefficients are properly chosen.

Conditional Estimates for PRFs of Lognormal Random Fields

Recall the example of the estimation of porosity powers explained in the Introduction section, and one may assume the porosity sample values are lognormal distributed. Afterwards, the marginal pdf for the attribute could be normalized using

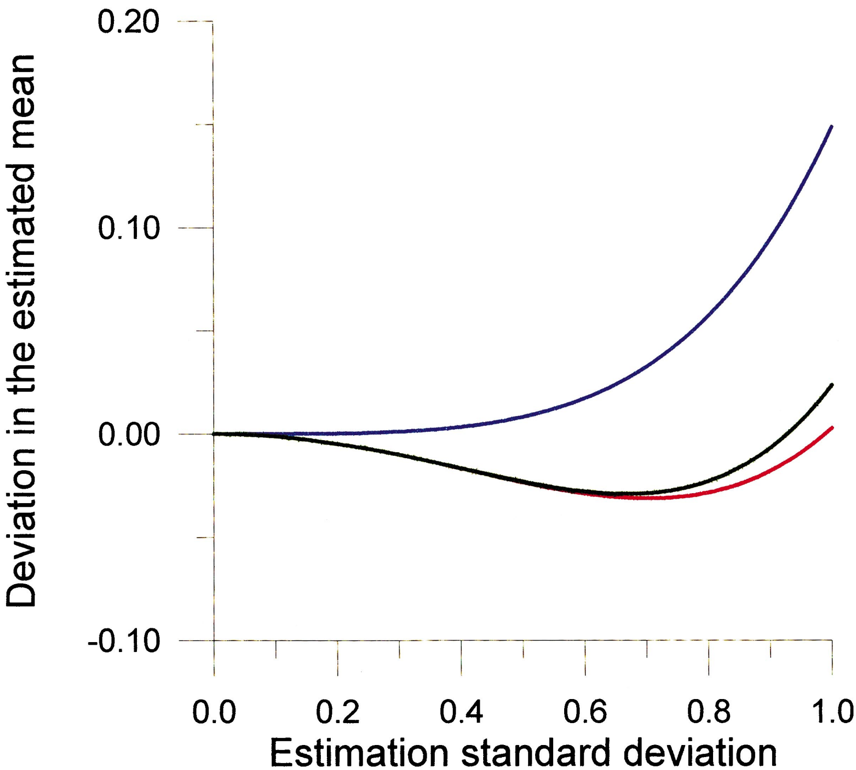

Figure 3. Deviation of the estimate from the true exponential for $m = 0$ (each two moments) and kriging or estimation standard deviation. From top to bottom added powers up to {2, 4, and 6}.

the log transform. Then standard lognormal kriging may be applied (Dowd, 1982; Journel, 1980; Rendu, 1979). It is well known that the correct lognormal estimates $\hat{Y}(x)$ are obtained as

$$\hat{Y}(x) = \exp\left(\hat{Z}(x) + \frac{\beta^2}{2}\right) \tag{9}$$

The square of the lognormal estimate needs a new correction. In general, any power of a lognormal estimate may be corrected. Inclusion of the power in the exponential transform is

$$Y(x) = (\exp Z(x))^q \tag{10}$$

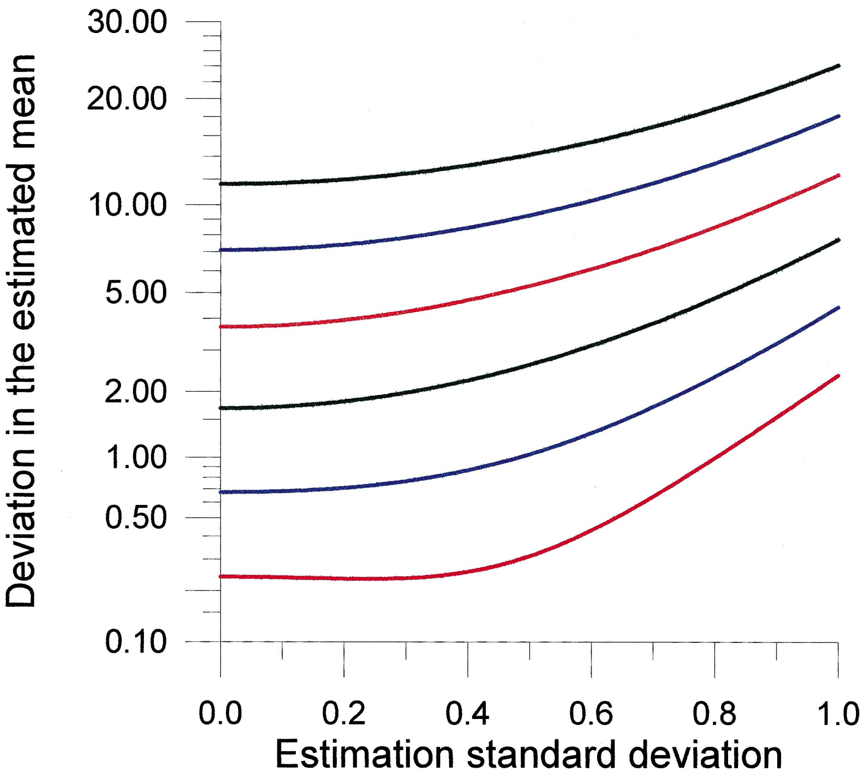

Figure 4. Deviation of the estimate from the true exponential for $m = 3$ and conditional or estimation standard deviation. From top to bottom powers are added up to {2, 3, 4, 5, 6, and 7}.

This allows the computation of the expected value of a power of an exponential as follows:

$$\langle Y_p(x_0)\rangle = \int_{-\infty}^{\infty} \left[e^{Z(x_0)}\right]^q f(Z(x_0)|[Z(x_\alpha), \alpha = 1 \text{ to } \mathrm{N}])\, dZ \tag{11}$$

This yields

$$\langle Y_p(x_0)\rangle = \exp\left(a_q \frac{q}{2}(2m_0 + q\beta^2)\right) \tag{12}$$

Note that Equation (9) is a particular case of Equation (12) for $q = 1$.

One can attempt more complicated estimates of monomials such as the product of the power by the power of the exponential. This is

$$\langle Y_p(x_0)\rangle = \int_{-\infty}^{\infty} Z^p(x_0)\left[e^{a_q Z(x_0)}\right]^q f(Z(x_0)|[Z(x_\alpha), \alpha = 1 \text{ to } \mathrm{N}])\, dZ \qquad (13)$$

The result is a cumbersome expression in terms of hypergeometric functions that we do not reproduce here. However, the numerical integration of Equation (13) allows solving it easily for specific cases. For other complicated transforms, numerical integration may correct the Gaussian estimates to non-Gaussian conditional expected values.

PRF Kriging

The kriging estimation is straightforward using the well-known best linear unbiased estimator (BLUE). This is

$$\hat{Z}(x_0) = \boldsymbol{\lambda}^T \mathbf{Z}(x_\alpha) \qquad (14)$$

where $\mathbf{Z}(x_\alpha)$ is a vector made of N data values at locations $x_\alpha|\alpha = 1 \ldots N$. The vector of weights $\boldsymbol{\lambda}$ is obtained by solving the classic system of simple kriging equations. The power estimates are obtained by considering $m_0 = \hat{Z}(x_0)$ and the estimation variance is $\sigma_k^2 = \boldsymbol{\lambda}^T C_{\alpha 0}$, where $C_{\alpha 0}$ is the covariance between data and the estimated location. Plugging the result into the equation for the desired power in Table 1, the correct estimate $\hat{Y}(x)$ is obtained. To construct a polynomial estimate such as in Figure 2, the polynomial coefficients must be included. In the case of the exponential expansion the coefficients contain the factorial terms. Direct estimation of the powers could be attempted developing a nonlinear geostatistics but that is beyond the scope of collocated transformations as in this paper. The proposed approach is to use the BLUE, make the transformations with the powers, and obtain corrected results following Table 1. Note this methodology is also applicable to the exponential transforms mentioned in the previous section.

Equations in Table 1 could still be used for ordinary kriging estimates by considering that the estimation variance is $\sigma^2 - \sigma_{\mathrm{OK}}^2 + 2\mu$, where σ^2 is the Gaussian random field variance, σ_{OK}^2 is the ordinary kriging variance, and μ is the Lagrange multiplier. This is equivalent to considerations for ordinary lognormal kriging (e.g., Journel, 1980). An alternative that avoids the computations in the Gaussian space is to substitute kriging by an iterative solution for lognormal kriging suggested as the basis for the inverse problem in hydrology (Vargas-Guzmán and Yeh, 2002).

An interesting property is that the kriging variance of the power estimates can be obtained by the following simple difference:

$$\sigma_p(x_0) = \langle Z^{2p}(x_0)\rangle - (\langle Z^p(x_0)\rangle^2) \tag{15}$$

where the expected values are computed following the corresponding transforms. As is obvious, independent monomials allow the estimation variance to be added. If the monomials are not independent the problem becomes a cokriging one.

Transform of PRF Covariances

The Gaussian random field has a covariance structure $C_Z(h)$ that may be transformed to obtain the covariance for a transformed field $Y(x)$. The general approach is to compute the product moment using the bivariate Gaussian for a given correlation $\rho = \rho(h) = \frac{C_Z(h)}{\sigma^2}$ as

$$E\left(Z_1^P Z_2^P\right) = \int_{-\infty}^{\infty} Z_1^P Z_2^P \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2\sqrt{1-\rho^2}}\frac{(Z_1-m_1)^2}{\sigma_1^2}+2r\frac{(Z_1-m_1)(Z_2-m_2)}{\sigma_1\sigma_2}+\frac{(Z_2-m_2)^2}{\sigma_2^2}} dZ_1 dZ_2 \tag{16}$$

where the covariance is

$$C_Y(h) = E\left(Z_1^P Z_2^P\right) - E\left(Z_1^P\right)E\left(Z_2^P\right) \tag{17}$$

Analytical integration may lead to very long expressions, except in the case of the lognormal where it is relatively simple to demonstrate that

$$C_Y(h) = e^{2m+\sigma^2(1+\rho(h))} - e^{m+\frac{\sigma^2}{2}} \tag{18}$$

Computations are carried on numerically for covariance powers and modeling may be applied afterwards. Notice that the previous kriging did not use these covariances.

Since tranformation models are available, a more practical alternative to compute covariances is by sensitivity analysis using sensitivity Jacobian matrices $\mathbf{J}$, when a model relating the Gaussian attribute to the non-Gaussian as $Y(x) = g(Z)$ is known. If the covariance matrix $\mathbf{c}_{ZZ}$ for the Gaussian attribute is known, then the covariance for the transformed random field is $\mathbf{c}_{YY} = \mathbf{J}\mathbf{c}_{ZZ}\mathbf{J}^T$ and the cross-covariance is $\mathbf{c}_{YZ} = \mathbf{J}\mathbf{c}_{ZZ}$. In the case of direct transformations, as in this paper, the Jacobian matrices become diagonal. Since both attributes Y and Z need to be known for computing Jacobians, derivatives of the forward model are required.

See also Vargas-Guzmán and Yeh (2002) for an inverse iterative stochastic Gauss–Newton approach to estimate transformations with models that are not explicitly invertible such as partial differential equations.

Conditional Geostatistical Simulations of PRF

Simulation of PRFs requires the knowledge of the corresponding conditional cumulative distribution function (ccdf) at each location. One alternative is to use the sequential Gaussian simulation (SGS) method (e.g., Isaaks, 1990) or a fast simulation of a random field by residuals (Vargas-Guzmán and Dimitrakopoulos, 2002). Following the classic SGS and using kriging for powers explained above, each randomly visited location is estimated by BLUE and a conditional mean estimate $\hat{Y}(x_0)$ obtained with the corresponding transformation (Table 1). Then, a residual term is simulated from the standard normal scaled by the kriging standard deviation and transformed to $y(x_0)$. The transformed residual is added to the estimated mean $Y(x_0) = \hat{Y}(x_0) + y(x_0)$. This result is back transformed to normal space by the root transformation and included for kriging of next estimated point.

Another alternative is to draw the residual from the integral of the skewed distribution for the transformed random variable $Y(x_0) = g(Z(x_0))$. Considering all r real roots of the transformation $Y(x)$, this is

$$Y = g(Z_1) = \cdots = g(Z_r) \tag{19}$$

and using the derivatives (e.g., Papoulis, 1984). The well-known way to obtain the model pdfs is

$$f(Y) = \frac{f(Z_1)}{g'(Z_1)} + \cdots + \frac{f(Z_r)}{g'(Z_r)} \tag{20}$$

For example the pdf for the square transform is

$$f(Y) = \frac{1}{\sigma\sqrt{2\pi Y}} e^{-\frac{Y}{2\sigma^2}} \tag{21}$$

and for the cubic transform is

$$f(Y) = \frac{1}{3(Y^2)^{1/3}\sigma\sqrt{2\pi}} e^{-\frac{(Y^2)^{1/3}}{2\sigma^2}} \tag{22}$$

where the conditional variance parameter σ^2 is from the Gaussian estimate.

The Application to Vector Random Fields

If the components of a polynomial such as Equation (4) are correlated, then the estimation and simulation of each PRF cannot be made for each monomial separately. In this more general case, the terms are cross-correlated. Some polynomials may have constant powers but the base terms may correspond to different attributes. This is for example

$$W(x) = a_1 Z_1^2(x) + a_2 Z_2^2(x) + a_3 Z_3^2(x) + \cdots + a_n Z_n^2(x) \tag{23}$$

The base random fields form a vector random field as $\mathbf{Z}(x) = [Z_1(x) Z_2(x) Z_3(x) \cdots Z_n(x)]$ and a joint cokriging may need to be solved. However, this brings complications to the corrections and analysis developed in the previous sections. The conditional pdfs at one point become multivariate and integrations may become cumbersome. However, there is a simple way of getting out of such complications by applying conditional component random fields. This is a factorization in linearly independent random fields as introduced from an analysis in the frequency domain by Vargas-Guzmán (2003). Briefly, I recall those results as follows.

The conditional components are second order stationary random fields. They are derived from the Fourier transform of the covariance matrix of the vector random field introducing *conditional spectral functions*. The conditional spectral functions provide covariance functions for second order stationary conditional components that are mutually orthogonal. This is a sequential conditioning of random fields and it is not based on data conditioning. The order of conditioning between the fields has been proved to be irrelevant to the final numerical result, however in simulations the order of conditioning may be swap as desired. Notice that this is different of the developments for sequential cokriging (Vargas-Guzmán and Yeh, 1999), where the random variables are conditional to data and therefore conditional covariances are not stationary. For the purposes of this paper, the conditional components may be computed and estimated or simulated separately following the approach for PRF explained in previous sections. The final results are assembled as explained in Vargas-Guzmán (2003) and not repeated here.

DISCUSSION

This paper has introduced a novel concept of a power random field (PRF), which may be combined to form polynomials. A particular PRFs family is obtained from the Taylor series expansion of the exponential random field, and estimates converge to the lognormal random field conditional means. Convergence of estimates has been found to be dependent on the Gaussian kriging values or conditional means. Converge for large values of estimates need higher order power monomials to be added to the polynomial as opposed to when conditional Gaussian estimates are smaller values. In the particular case of a Gaussian estimate equal to zero, only even powers contribute to the convergence and just three even powers are needed to approach the lognormal. However, convergence is slower for estimates larger than zero. Convergence to exponential is not an issue for individual monomials because estimates are statistically exact.

The role of the estimation variance (i.e., kriging variance) in the estimates has been investigated, and results also show that convergence will be faster for a lower estimation variance.

The case where the terms are independent allow for separate geostatistics of each power. Kriging of a PRF has been found from transformation equations developed for the Gaussian estimates. The procedure follows the same paradigm as the lognormal kriging, where the Gaussian estimate is modified to provide the non-Gaussian estimate. This avoids dealing with covariances for powers or higher order moments and other well-known complications for kriging non-Gaussian random fields. It has been shown here that the use of power transformations of Gaussian random fields does not present much theoretical difficulties, and kriging estimates are easily corrected. However, it is important to underline that the approach presented here is not for the general non-Gaussian and nonlinear random fields which is a more general problem.

This theory has avoided the use of a multivariate skewed probability density function (pdf) and instead has confined the analysis for marginal conditional distribution functions in spatial locations. This provides results compatible with the theory of moments for a single marginal random variable. This is a powerful simplification that can be used analytically as in this paper or numerically for other complicated skewed distributions. The use of the marginal conditional distributions is validated by the fact that the algebraic transformations are collocated. These transformations do not involve convolutions.

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