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On the Equivalence of the Cokriging and Kriging Systems[1]

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Simple cokriging of components of a p*-dimensional second-order stationary random process is considered. Necessary and sufficient conditions under which simple cokriging is equivalent to simple kriging are given. Essentially this condition requires that it should be possible to express the cross-covariance at any lag* $\boldsymbol{h}$ *using the cross-covariance at* $|\boldsymbol{h}| = 0$ *and the auto-covariance at lag* $\boldsymbol{h}$*. The mosaic model, multicolocated kriging and the linear model of coregionalization are examined in this context. A data analytic method to examine whether simple kriging of components of a multivariate random process is equivalent to its cokriging is given*

KEY WORDS: cokriging, kriging, intrinsic coregionalization, spatial orthogonality, mosaic model.

INTRODUCTION

Several applications of geostatistics lead to models involving multivariate random processes. Fitting of valid auto- and cross-covariances and solving of large cokriging systems are often required in such applications. Similar problems may also arise in the univariate case when several functions of a random process are of interest. Matheron (1979) introduced the notion of autokrigeability. A component of a multivariate random process is said to be autokrigeable if its cokriging coincides with its simple kriging. When all components of the random process have been observed at all data locations (i.e. the isotopic case), Matheron (1979) obtained conditions under which a linear combination of these components is autokrigeable. These results are also given in Wackernaegel (1995). Autokrigeability is also discussed in Chiles and Delfiner (1999) and Rivoirard (2002).

In the absence of autokrigeability, to make the task of modelling covariances (or equivalently the variograms) simpler, and that of solving cokriging

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systems less arduous, it is desirable to obtain simplifications wherever possible. The common theme of simplifications in dealing with such problems has been to examine whether a linear transformation would lead to spatial orthogonality. Suro-Perez and Journel (1991) suggested the use of principal components derived from variance covariance matrices at lag $\mathbf{h} = 0$ or at small values of $\mathbf{h}$. This method is justified if these principal components are spatially orthogonal. Goovaerts (1993) gives conditions under which principal components are spatially orthogonal. Another linear transformation was suggested by Xie and Myers (1995) which examines whether the family of variance-covariance matrices at all $\mathbf{h}$ are simultaneously diagonalizable. In the univariate case, Subramanyam and Pandalai (2001) give conditions under which linear transforms of functions of a univariate random process are spatially orthogonal. Another interesting procedure is the min/max autocorrelation factors (MAF) introduced by Switzer and Green (1984), a procedure that is akin to extracting canonical variables which has been used by Desbarats and Dimitrakopoulos (2000) along with the linear model of coregionalization to achieve simplifications in simulation.

In this paper simple cokriging of components of a p-dimensional second-order stationary random process is considered. A necessary and sufficient condition for the simple cokriging of the components of a multivariate random process to be equivalent to their simple kriging in the isotopic case is given. Although this condition is available in Matheron (1979), Wackernaegel (1995), Chiles and Delfiner (1999), and Rivoirard (2002), the derivation of equivalence conditions presented here leads to a complete analysis of autokrigeable random variables. It is shown here that cokriging of a component of a vector-valued *RF* is equivalent to its simple kriging in the isotopic case if and only if components of the *RF* can be divided into groups such that components belonging to one group have proportional auto- and cross-covariances while being spatially orthogonal to components in any other group. Proof for this is not explicitly available in published literature.

The mosaic model, multicolocated kriging, and the linear model of coregionalization are discussed in this context.

SIMPLE COKRIGING

Let $(\mathbf{Y}(x),\ x \in D\}$ be a second-order stationary random function taking values in $\mathbf{R}^p$ i.e. $\mathbf{Y}(x) = (Y_1(x), Y_2(x), \ldots Y_p(x))^\top$. Assume $E[\mathbf{Y}(x)] = \boldsymbol{\mu}$ and $\mathrm{Cov}\,(Y_i(x), Y_j(x+h)) = C_{ij}(h)\ \ \forall\ x \in D$. It is assumed that $\boldsymbol{\mu}$ is known and hence is taken to be $\mathbf{0}$. It may be noted here that $\mathrm{Cov}\,(Y_j(x), Y_i(x+h) = \mathrm{Cov}\,(Y_i(x+h), Y_j(x)) = \mathrm{Cov}\,(Y_i(h), Y_j(0)) = \mathrm{Cov}\,(Y_i(0), Y_j(-h))$, i.e.,

$$C_{ji}(h) = C_{ij}(-h) \tag{1}$$

In particular, if $C_{ji}(h)$ is an even function,

$$C_{ji}(h) = C_{ij}(h) \quad \forall\, h.$$

The process $\mathbf{Y}(x)$ has been observed at locations $x_1, x_2 \ldots, x_N$. It is of interest to predict the value of $\mathbf{Y}(x_0)$ which involves prediction of each $Y_i(x_0)$, $i = 1, 2, \ldots p$. To begin with, consider prediction of one of the variables, say the first. The cokriging equations for predicting $Y_1(x_0)$ can be set up (after Matheron, 1965) as follows.

Let $\boldsymbol{\lambda}_1^c, \boldsymbol{\lambda}_2^c, \cdots \boldsymbol{\lambda}_p^c$ be N-dimensional vectors such that $\boldsymbol{\lambda}_i^c$ is the weight associated with $Y_i(x_\alpha)$, $\alpha = 1, 2 \ldots N$ for cokriging $Y_1(x_0)$. Further, let $\mathbf{A}_{ij} = [a_{ij}(\alpha, \beta)]$, where $a_{ij}(\alpha, \beta) = \mathrm{Cov}\,(Y_i(x_\alpha), Y_j(x_\beta)) = C_{ij}(x_\beta - x_\alpha)$. Denote $\mathrm{Cov}\,(Y_1(x_0), Y_j(x_\beta))$ by $d_j^1(\beta) = C_{1j}(x_\beta - x_0)$ and let

$$\mathbf{d}_j^1 = \begin{bmatrix} d_j^1(1) \\ d_j(2) \\ \vdots \\ d_j^1(N) \end{bmatrix}$$

The simple cokriging system is then

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \cdots & \mathbf{A}_{1p} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \cdots & \mathbf{A}_{2p} \\ \vdots & \vdots & \ldots & \\ \mathbf{A}_{p1} & \mathbf{A}_{p2} & \cdots & \mathbf{A}_{pp} \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda}_1^c \\ \boldsymbol{\lambda}_2^c \\ \vdots \\ \boldsymbol{\lambda}_p^c \end{bmatrix} = \begin{bmatrix} \mathbf{d}_1^1 \\ \mathbf{d}_2^1 \\ \vdots \\ \mathbf{d}_p^1 \end{bmatrix} \tag{2}$$

The system of equations for simple kriging of $Y_1(x_0)$ is given by

$$\mathbf{A}_{11}\boldsymbol{\lambda}^k = \mathbf{d}_1^1 \tag{3}$$

where

$$\boldsymbol{\lambda}^k = \begin{pmatrix} \lambda_1^k \\ \lambda_2^k \\ \vdots \\ \lambda_N^k \end{pmatrix}$$

is the vector of weights corresponding to the simple kriging of $Y_1(x_0)$ with $Y_1(x_\alpha)$, $\alpha = 1, 2, \ldots N$. System (3) has the solution $\boldsymbol{\lambda}^k = \mathbf{A}_{11}^{-1}\mathbf{d}_1^1$
Simple cokriging of $Y_1(x_0)$ is equivalent to simple kriging means that

$$\begin{aligned}
\boldsymbol{\lambda}_{(1)}^c &= \boldsymbol{\lambda}_{(1)}^k \\
\boldsymbol{\lambda}_{(2)}^c &= \mathbf{0} \\
\boldsymbol{\lambda}_{(3)}^c &= \mathbf{0} \\
&\vdots \\
\boldsymbol{\lambda}_{(N)}^c &= \mathbf{0}
\end{aligned}$$

must be a solution to system (2). This gives,

$$\begin{aligned}
\mathbf{A}_{11}\boldsymbol{\lambda}_1^c &= \mathbf{d}_1^1 \\
\mathbf{A}_{21}\boldsymbol{\lambda}_1^c &= \mathbf{d}_2^1 \\
&\vdots \\
\mathbf{A}_{p1}\boldsymbol{\lambda}_1^c &= \mathbf{d}_p^1
\end{aligned}$$

satisfies the first set of equations $\mathbf{A}_{11}\boldsymbol{\lambda}_1^c = \mathbf{d}_1^1$. In order that it satisfies the $\boldsymbol{\lambda}_1^c = \boldsymbol{\lambda}_1^k$ and remaining set of equations, there must exist matrices $\mathbf{B}_2, \mathbf{B}_3 \ldots \mathbf{B}_p$ such that

$$\begin{aligned}
\mathbf{A}_{21} &= \mathbf{B}_2\mathbf{A}_{11} \text{ and } \mathbf{d}_2^1 = \mathbf{B}_2\mathbf{d}_1^1 \\
\mathbf{A}_{31} &= \mathbf{B}_3\mathbf{A}_{11} \text{ and } \mathbf{d}_3^1 = \mathbf{B}_3\mathbf{d}_1^1 \\
&\vdots \qquad\qquad\qquad \vdots \\
\mathbf{A}_{p1} &= \mathbf{B}_p\mathbf{A}_{11} \text{ and } \mathbf{d}_p^1 = \mathbf{B}_p\mathbf{d}_1^1.
\end{aligned}$$

Since it is assumed that $\{\mathbf{Y}(x), x \in D\}$ is a second order stationary process, it can be deduced that $\mathbf{B}_2, \mathbf{B}_3, \ldots, \mathbf{B}_p$ are $N \times N$ diagonal matrices. To see this, consider the equation

$$\mathbf{d}_2^1 = \mathbf{B}_2\mathbf{d}_1^1$$

which gives

$$\begin{aligned}
\mathrm{Cov}\,(Y_1(x_0), Y_2(x_\alpha)) = b_{\alpha 1}^{(2)}\mathrm{Cov}\,(Y_1(x_0), Y_1(x_1)) + b_{\alpha 2}^{(2)}\mathrm{Cov}\,(Y_1(x_0)Y_1(x_2)) \\
+ \ldots + b_{\alpha\alpha}^{(2)}\mathrm{Cov}\,(Y_1(x_0)Y_1(x_\alpha)) + \ldots \\
+ b_{\alpha N}^{(2)}\mathrm{Cov}\,(Y_1(x_0)Y_1(x_N))
\end{aligned}$$

i.e.

$$C_{12}(x_\alpha - x_0) = \sum_{i=1}^{N} b_{\alpha i}^{(2)} C_{11}(x_i - x_0).$$

For second order stationarity of $\{\mathbf{Y}(x)\}$ the RHS must be a function only of $(x_\alpha - x_0)$ and so $b_{\alpha i}^{(2)} = 0$ for $i \neq \alpha$, giving

$$C_{12}(x_\alpha - x_0) = b_{\alpha\alpha}^{(2)} C_{11}(x_\alpha - x_0) \quad \forall\, \alpha = 1, 2, \ldots, N \tag{4}$$

i.e. $\mathbf{B}_2$ is a diagonal matrix. Similarly $\mathbf{B}_3, \mathbf{B}_4, \ldots, \mathbf{B}_p$ are all diagonal.

Taking $x_0 = x_\alpha$ in equation (4) we obtain

$$\operatorname{Cov}(Y_1(x_\alpha), Y_2(x_\alpha)) = b_{\alpha\alpha}^{(2)} \operatorname{Cov}(Y_1(x_\alpha), Y_1(x_\alpha)) \;\; \forall\, \alpha = 1, 2, \ldots N$$

$$b_{\alpha\alpha}^{(2)} = \frac{\operatorname{Cov}(Y_1(x_\alpha) Y_2(x_\alpha))}{\operatorname{Cov}(Y_1(x_\alpha), Y_1(x_\alpha))} = \frac{C_{12}(0)}{C_{11}(0)} \quad \forall\, \alpha = 1, 2, \ldots, N. \tag{5}$$

Since the RHS of equation (5) is independent of α, one has $b_{\alpha\alpha}^{(2)} = b_{\beta\beta}^{(2)} = b^{(2)}$, say.

Thus $\mathbf{B}_2$ is a scalar matrix and $C_{12}(h) = b^{(2)} C_{11}(h)$. Repeating this argument for $\mathbf{B}_3, \mathbf{B}_4, \ldots, \mathbf{B}_p$, it can be shown that these matrices are all scalar, and

$$C_{1j}(h) = b^{(j)} C_{11}(h) \;\; \forall\, j = 2, 3, \ldots, p. \tag{6}$$

In addition, one has $C_{1j}(-h) = b^{(j)} C_{11}(-h) = b^{(j)} C_{11}(h)$ as the autocovariance is an even function. This gives (using 6 and 1),

$$C_{1j}(-h) = C_{1j}(h) = C_{j1}(h), \tag{7}$$

showing that the cross-covariances are also even functions.

To summarize, the simple cokriging of $Y_1(x_0)$ is equivalent to simple kriging in the second order stationary case if and only if

$$C_{1j}(h) = b_1^{(j)} C_{11}^{(h)} \;\; \forall\, j = 2, 3, \ldots, p,$$

where $b_1^{(j)} = b^{(j)}$ in (6).

Generalizing this to the simple cokriging of $Y_i(x_0)$, it can be seen that under second order stationarity of $\{\mathbf{Y}(x)\}$ the simple cokriging system is equivalent to the simple kriging system if and only if

$$C_{ij}(h) = b_i^{(j)} C_{ii}(h) \;\; \forall\, i \neq j, \quad i, j = 1, 2, \ldots p \tag{8}$$

Condition (8) holds in the following two cases:

Case 1 (Spatial Orthogonality of $\{\mathbf{Y}_{(x)})\}$. When $C_{ij}(h) = 0 \;\; \forall\, h$ and $i \neq j$, condition (8) holds with $b_i^{(j)} = 0 \;\; \forall\, i \neq j$. It may be noted here that there is no condition on $C_{ii}(h)$.

Case 2 (Proportional Covariance Structure). From (8) it can be seen that

$$\begin{aligned} C_{ij}(h) &= b_i^{(j)} C_{ii}(h), \text{ and} \\ C_{ji}(h) &= b_j^{(i)} C_{jj}(h) \qquad \forall\, i, j. \end{aligned}$$

But since $C_{ij}(h) = C_{ji}(h)$, one has

$$b_i^{(j)} C_{ii}(h) = b_j^{(i)} C_{jj}(h), \text{ giving (if } b_j^{(i)} \neq 0),$$

$$C_{jj}(h) = \frac{b_i^{(j)}}{b_j^{(i)}} C_{ii}(h) \;\; \forall\, i, j. \tag{9}$$

Equation (9) implies that all covariances are proportional to a common covariance structure, say $C_{11}(h)$. Thus,

$$C_{jj}(h) = \beta^{(j)} C_{11}(h).$$

Often one may be interested in cokriging a linear combination $Y_{1^*}(x_0) = \sum_{k=1}^{p} a_k Y_k(x_0)$ using $Y_{1^*}(x), Y_i(x) \;\; i = 2, 3, \ldots, p$.

From (6) Y_{1^*} is autokrigeable if and only if

$$\sum_{k=1}^{p} a_k C_{b^* j}(h) = b^{(j)} \sum_{k=1}^{p} \sum_{k'=1}^{p} a_k a_{k'} C_{kk'}(h) \quad \forall\; j = 2, 3, \ldots, p.$$

This is the condition obtained by Matheron (1979).

EQUIVALENCE CONDITIONS

In this section it is shown that simple cokriging is equivalent to simple kriging only under *Case 1* or *Case 2* or a mixture of the two. Denote by $\mathcal{S}$ the covariance matrix of the vector

$$\begin{aligned} &[(Y_1(x_1)\; Y_2(x_1) \ldots Y_p(x_1)),\; (Y_1(x_2)\; Y_2(x_2) \ldots Y_p(x_2)), \\ &\quad \cdots\; (Y_1(x_N)\; Y_2(x_n) \ldots Y_p(x_N))]^\top \end{aligned}$$

$$\mathcal{S} = \begin{bmatrix} \mathbf{S}(0) & \mathbf{S}(x_2 - x_1) & \mathbf{S}(x_3 - x_1) & \ldots & \mathbf{S}(x_N - x_1) \\ \mathbf{S}(x_1 - x_2) & \mathbf{S}(0) & \mathbf{S}(x_3 - x_2) & \ldots & \mathbf{S}(x_N - x_2) \\ \mathbf{S}(x_1 - x_3) & \mathbf{S}(x_2 - x_3) & \mathbf{S}(0) & \ldots & \mathbf{S}(x_N - x_3) \\ \vdots & & & & \\ \mathbf{S}(x_1 - x_N) & \mathbf{S}(x_2 - x_N) & \ldots & \ldots & \mathbf{S}(0) \end{bmatrix}$$

where the ijth element of $\mathbf{S}(x_\beta - x_\alpha)$ is $\mathrm{Cov}\,(Y_i(x_\alpha), Y_j(x_\beta)) = \mathrm{Cov}\,(Y_i(0), Y_j(x_\beta - x_\alpha)) = C_{ij}(x_\beta - x_\alpha)$.

If condition (8) holds,

$$\begin{aligned} C_{ij}(x_\beta - x_\alpha) &= b_i^{(j)} C_{ii}(x_\beta - x_\alpha) \\ &= b_i^{(j)} C_{ii}(x_\alpha - x_\beta) \\ &= C_{ji}(x_\alpha - x_\beta) \\ &= C_{ji}(x_\beta - x_\alpha) \end{aligned}$$

Hence each matrix $\mathbf{S}(x_\beta - x_\alpha)$ is symmetric and so is $\mathcal{S}$.

Consider one of the off-diagonal blocks $\mathbf{S}(x_\beta - x_\alpha)$ of $\mathcal{S}$. Taking $x_\alpha = x_\beta + h$, one has

$$\mathbf{S}(h) = [C_{ij}(h)] = \left[b_i^j C_{ii}(h)\right] = C_h \mathbf{W}.$$

where

$$\begin{aligned} \boldsymbol{C}_h &= \mathrm{Diag}\,[C_{ii}(h)], \quad \text{and} \\ \mathbf{W} &= [b_i^j]. \end{aligned}$$

Since $\mathbf{W}$ is independent of h, it is identified by taking $h = 0$, as

$$\mathbf{W} = \boldsymbol{C}_0^{-1}\mathbf{S}(0)$$

Thus, $\mathbf{S}(h) = \boldsymbol{C}_h \boldsymbol{C}_0^{-1}\mathbf{S}(0)$. Since $\mathbf{S}(h)$ and $\mathbf{S}(0)$ are symmetric, and $\boldsymbol{C}_h$ and $\boldsymbol{C}_0$ are diagonal, it can be seen that $\boldsymbol{C}_h \boldsymbol{C}_0^{-1}\mathbf{S}(0) = \mathbf{S}(0)\boldsymbol{C}_0^{-1}\boldsymbol{C}_h = \mathbf{S}(0)\boldsymbol{C}_h\boldsymbol{C}_0^{-1}$. Denoting $\mathbf{C}_h\mathbf{C}_0^{-1}$ by $\rho(h)$, the equation above can be rewritten as

$$\rho(h)\mathbf{S}(0) = \mathbf{S}(0)\rho(h) = \mathbf{S}(h) \tag{10}$$

If $\mathbf{S}(h)$ is a commuting family, then there exists $\mathbf{A}$ such that $\mathbf{Y} = \mathbf{AZ}$ where Y's are autokrigeable (see for example, Subramanyam and Pandalai, 2001). Condition

(10) implies that $\mathbf{S}(h)$ is a commuting family and allows, in addition, to take $\mathbf{A} = \mathbf{I}$.

Relation (10) can be interpreted as follows: Cokriging is equivalent to kriging if it is possible to construct the cross covariance structure $\mathbf{S}(h)$ using the cross-covariance structure at $h = 0$ and the autocovariance structure.

In *Case 1* this is easy because $\mathbf{S}(h)$ and $\mathbf{S}(0)$ are themselves diagonal, i.e. all cross-covariances are zero. One can formally check that (10) holds by seeing that $\mathbf{S}(h) = \mathbf{C}_h$ and $\mathbf{S}(0) = \mathbf{C}_0$ giving

$$\mathbf{S}^{-1}(0)\mathbf{S}(h) = \mathbf{C}_0^{-1}\mathbf{C}_h = \rho(h).$$

If *Case 2* holds, $C_{ij}(h) = \beta_{ij}C_{11}(h)$, say, and in particular $C_{ii}(h) = \beta_{ii}C_{11}(h)$. Hence

$$\boldsymbol{\rho}(h) = \text{ Diag }\left[\frac{\beta_{ii}C_{11}(h)}{\beta_{ii}C_{11}(0)}\right] = \text{ Diag }\left[\frac{C_{11}(h)}{C_{11}(0)}\right] = \frac{C_{11}(h)}{C_{11}(0)}[\mathbf{I}] = \rho_h[\mathbf{I}], \text{ say.}$$

This gives,

$$\boldsymbol{\rho}(h)\mathbf{S}(0) = \rho_h[\mathbf{I}]]\mathbf{S}(0) = \rho_h\mathbf{S}(0).$$

Under *Case 2*, since $C_{ij}(h) = \beta_{ij}C_{11}(h)$, clearly $\beta_{ij} = \dfrac{C_{ij}(0)}{C_{11}(0)}$. This gives

$$\mathbf{S}(h) = [C_{ij}(h)] = \left[\frac{C_{ij}(0)}{C_{11}(0)}C_{11}(h)\right] = [\rho_h C_{ij}(0)] = \rho_h\mathbf{S}(0)$$

and thus verifies that relation (10) holds under *Case 2*.

To find all possible cases under which (10) holds, it may be noted that $\boldsymbol{\rho}(h)$ and $\mathbf{S}(0)$ must commute for all h. If for some h all diagonal elements of $\boldsymbol{\rho}(h)$ are distinct, then $\mathbf{S}(0)$ has to be diagonal for it to commute with $\boldsymbol{\rho}(h)$ which in turn implies that $\mathbf{S}(h)$ is diagonal. This can be seen as follows.

Since $\boldsymbol{\rho}(h)$ and $\mathbf{S}(0)$ commute for all h, one has

$$\rho_i(h)C_{ij}(0) = \rho_j(h)C_{ij}(0).$$

Thus, if for some h

$$\rho_i(h) \neq \rho_j(h), \quad C_{ij}(0) = 0 \tag{11}$$

i.e., Y_i and Y_j are orthogonal at $h = 0$. If there is no such h, i.e., if one has $\rho_i(h) = \rho_j(h)\ \forall\ h$, one has

$$\frac{C_{ii}(h)}{C_{ii}(0)} = \frac{C_{jj}(h)}{C_{jj}(0)} \quad \forall\ h$$

i.e., the autocovariance functions are proportional. The processes $\{Y_i(x), x \in D, i = 1, 2, \ldots p\}$ can then be divided into r groups, the jth group containing k_j elements in such a way that processes in the same group have autocovariance functions proportional to a common covariance structure. Since by (11), processes in different groups are locally orthogonal, the matrix $\mathbf{S}(0)$ is block diagonal with the jth block being a $k_j \times k_j$ matrix. Further, since $\mathbf{S}(h) = \boldsymbol{\rho}(h)\mathbf{S}(0)$ and $\boldsymbol{\rho}(h)$ is diagonal, $\mathbf{S}(h)$ is also block diagonal. Thus simple cokriging being equivalent to simple kriging implies $\mathbf{S}(h)$ is block diagonal and $\rho_i(h) = \rho_j(h)$ if i and j are in the same block.

On the other hand, if $\mathbf{S}(h)$ is block diagonal because groups are orthogonal, to cokrige a member of one group, only variables from the same group need be used. But variables belonging to the same group have proportional covariances and thus fall under *Case 2*. Hence simple cokriging of the variable is equivalent to its simple kriging. In conclusion, simple cokriging is equivalent to simple kriging if and only if $\mathbf{S}(h)$ is block diagonal and *RF*s belonging to the same block have proportional covariance.

COKRIGING IN THE MOSAIC MODEL

Cokriging in the mosaic model is discussed in Matheron (1982a). Owing to the importance of this model in geostatistics, a detailed discussion is given below. Consider the process $\{Y(x), x \in D\}$. In the mosaic model of Matheron (1982a) it is assumed that D is divided into disjoint compartments called tiles and that $Y(x)$ takes the same value for all x within a tile while $Y(x)$, $Y(x+h)$ are independent if $x, x+h$ are in different tiles. Let the probability that x and $x+h$ belong to the same tile be $\rho(h)$. Then the joint distribution function of $Y(x)$, $Y(x+h)$ is given by

$$\begin{aligned} T_{ij}(h) = P(Y(x) \geq i, Y(x+h) \geq j) &= \rho(h)P(Y(x) \geq i \vee j) \\ &\quad + (1-\rho(h))P(Y(x) \geq i)P(Y(x) \geq j) \end{aligned} \tag{12}$$

where $(i \vee j)$ denotes maximum of i and j.

Let f and g be any two functions and consider $C^{fg}(h) = \text{Cov}\,[f(Y(x)), g(Y(x+h))]$. It can be shown that under the mosaic model,

$$C^{fg}(h) = \rho(h) C^{fg}(0). \tag{13}$$

Also,

$$C(h) = \text{Cov}\,(Y(x), Y(x+h)) = \rho(h) C(0), \tag{14}$$

where $C(0) = \mathit{Var}\,(Y(x))$.

Then, from (13) and (14)

$$\text{Cov}\,[f(Y(x)), g(Y(x+h))] = \text{Cov}\,(Y(x), Y(x+h)) \frac{C^{fg}(0)}{C(0)}$$

i.e.

$$C^{fg}(h) = C(h) \frac{C^{fg}(0)}{C(0)} = C(h)\alpha^{fg}, \text{ say.}$$

Thus all cross-covariances are proportional to the covariance structure of $\{Y(x)\}$. Hence simple cokriging of $f_1(Y(x)),\ f_2(Y(x)), \ldots,\ f_m(Y(x))$, for any functions $f_1, f_2, \ldots, f_m$ reduces to simple kriging under *Case 2* discussed in section 2.

It is interesting to examine the converse i.e., if for every function $f_1, f_2, \ldots, f_m$ of $Y(x)$ simple cokriging reduces to simple kriging, can $Y(x)$ be described by the mosaic model? It is shown below that if for every function $f_1, f_2, \ldots, f_m$ of $Y(x)$ simple cokriging is equivalent to simple kriging, the bivariate distribution of $\{Y(x)\}$ is the same as (12).

To see this, using (8) one may write

$$C^{f_1, f_2}(h) = \alpha^{12} C^{f_1 f_1}(h), \text{ say} \tag{15}$$

Taking f_2 to be the identity function one has

$$C^{f_1 f_2}(h) = \beta C^{f_1 f_1}(h)$$

$$\text{and} \quad C^{f_2 f_1}(h) = \gamma C^{f_2 f_2}(h) = \gamma C(h)$$

Since $C^{f_1 f_2}(h) = C^{f_2 f_1}(h)$ one has

$$C^{f_1 f_1}(h) = \frac{\gamma}{\beta} C(h) = \beta^1 C(h). \tag{16}$$

This gives

$$C^{f_1 f_2}(h) = \alpha^{12} \beta^1 C(h). \tag{17}$$

Taking limit $h \to 0$ in (15) and (16), one has

$$\alpha^{12} = \frac{C^{f_1 f_2}(0)}{C^{f_1 f_1}(0)} \quad \text{and} \quad \beta^1 = \frac{C^{f_1 f_1}(0)}{C(0)}.$$

Thus

$$C^{f_1 f_2}(h) = C^{f_1 f_2}(0) \frac{C(h)}{C(0)} = C^{f_1, f_2}(0) \cdot \rho_{(h)}, \text{ say.} \tag{18}$$

Clearly, $-1 \leq \rho_{(h)} \leq 1$ and $\rho_{(h)} \geq 0$ if $C(h) \geq 0$.
Taking $f_1 = I_{Y(x) \geq i}$ and $f_2 = I_{Y(x) \geq j}$,

$$\begin{aligned} C^{f_1 f_2}(h) &= \operatorname{Cov}(I_{Y(x) \geq i}, I_{Y(x+h) \geq j}) \\ &= P(Y(x) \geq i, Y(x+h) \geq j) - P(Y(x) \geq i) P(Y(x+h) \geq j) \\ &= T_{ij}^h - T_i T_j, \text{ say.} \end{aligned} \tag{19}$$

Further,

$$\begin{aligned} C^{f_1 f_2}(0) &= P(Y(x) \geq i, Y(x) \geq j) - P(Y(x) \geq i) P(Y(x) \geq j) \\ &= P(Y(x) \geq i \vee j) - P(Y(x) \geq i) P(Y(x) \geq j) \\ &= T(i \vee j) - T_i T_j. \end{aligned}$$

Hence (19) becomes

$$C^{f_1 f_2}(h) = T_{ij}^h - T_i T_j = \{T(i \vee j) - T_i T_j\} \frac{C(h)}{C(0)} = (T(i \vee j) - T_i T_j) \rho_{(h)}, \text{ say.} \tag{20}$$

Rearranging terms in (20) one has

$$T_{ij}^{h} = T(i \vee j)\rho_{(h)} + (1 - \rho_{(h)})T_{(i)}T_{(j)},$$

which is the same as (12).

Therefore, if simple cokriging and simple kriging are equivalent for every $f_1, f_2, \ldots, f_m$, the process $\{Y(x)\}$ must have the bivariate distribution given by (12). This of course does not imply the existence of tiles.

MULTICOLOCATED COKRIGING

In this section, the application of condition (10) to transforms of the original data is examined. In particular, the example of multicolocated cokriging is examined.

Let $\mathbf{Y}_(x) = (Y_1(x), Y_2(x))^\top$ be a stationary process with mean $\mathbf{0}$ observable for $x \in D$. Denote by $C_{ij}(h)$, $i = 1, 2,\ j = 1, 2$ the covariance between $Y_i(x)$ and $Y_j(x+h)$. Let $\mathbf{U}(x) = (U_1(x), U_2(x))^T$, where $U_1(x)$ and $U_2(x)$ are linear combinations of $Y_1(x)$ and $Y_2(x+h)$, chosen such that Cov $(U_1(x), U_2(x)) = 0$. Let Cov $(\mathbf{U}(x), \mathbf{U}(x+h))$ be denoted by $\mathbf{S}_u(h)$. By assumption, $\mathbf{S}_u(0)$ is a diagonal matrix. Using (10) for the $\mathbf{U}$ process, the condition for simple cokriging to be equivalent to kriging is given by

$$\mathbf{S}_u(h) = \rho_u(h)\mathbf{S}_u(0) \tag{21}$$

since $\rho_u(h)$ is diagonal, $\mathbf{S}_u(h)$ is also diagonal i.e., if the off-diagonal elements, Cov $(U_1(x+h), U_2(x))$ and Cov $(U_1(x), U_2(x+h))$ of $\mathbf{S}_u(h)$ are zero, the condition for simple cokriging to be equivalent to simple kriging is satisfied.

An important case is when $U_1 = Y_1 - rY_2$ and $U_2 = Y_2$ where $r = \frac{C_{12}(0)}{C_{22}(0)}$ which always satisfies Cov $(U_1(x), U_2(x)) = 0$ since U_1 is the residue of the regression of Y_1 on Y_2. For this choice of U_1 and U_2, the diagonal elements of $\mathbf{S}_u(h)$ are $C_{21}(h) - rC_{22}(h)$ and $C_{12}(h) - rC_{22}(h)$ and are *both* equal to 0 iff

$$C_{12}(h) = rC_{22}(h) \tag{22}$$

Condition (22) is the same as Model 2a of Rivoirard (2001), the reverse Markov model of Chiles and Delfiner (1999) and the MM2 model of Journel (1999) that is used for multicollocated cokriging. As pointed out by Rivorard (2001), under condition (22) when $Y_1(x_0)$ is to be estimated and Y_2 is observed at more locations than Y_1, including at x_0, one needs to use only $Y_2(x_0)$ and data from location where both Y_1 and Y_2 are observed resulting in the estimation procedure known as such multicollocated cokriging.

THE LINEAR MODEL OF COREGIONALIZATION

A multivariate p-dimensional stationary random process $\mathbf{Z}$ may often be considered as a linear combination of n unobserved p-dimensional stationary random processes $\mathbf{Y}^u(x)$ called factor processes (Goovaerts, 1993). In this section, conditions on the factor processes $\mathbf{Y}^u(x)$ under which simple cokriging of $Z_i(x_0), i = 1, 2, \ldots, p$ reduces to kriging are examined. The linear model of coregionalization (Matheron, 1982b; Wackernagel, 1988) can be written as

$$\mathbf{Z}(x) = \mathbf{M}_1^T \mathbf{Y}^1(x) + \mathbf{M}_2^T \mathbf{Y}^2(x) + \cdots + \mathbf{M}_n^T \mathbf{Y}^n(x),$$

where

$$\mathbf{Y}^u(x) = \left(Y_1^u(x), Y_2^u(x) \ldots Y_p^u(x)\right)^T, \quad u = 1, 2, \ldots, n,$$

with Cov $\left(Y_i^{u'}(x), Y_j^u(x)\right) = 0 \;\; u \neq u'$ or $i \neq j$,

$$\text{Cov}\left(Y_i^u(x), Y_i^u(x+h)\right) = c^u(h), \quad u = 1, 2, \ldots, n, \quad i = 1, 2, \ldots, p$$

and $\mathbf{M}_u, \; u = 1, 2, \ldots, n$ are $p \times p$ matrices. Also, $c^u(0)$ can be taken to be 1 for all u.

If $\mathbf{C}_\mathbf{Z}(h) = \text{Cov}\,(\mathbf{Z}(x), \mathbf{Z}(x+h))$, it can be seen from the above that

$$\mathbf{S}(h) = \mathbf{C}_\mathbf{Z}(h) = \sum_{u=1}^{n} c^u(h) \mathbf{M}_u^T \mathbf{M}_u = \sum_{u=1}^{n} c^u(h) \mathbf{B}(u). \tag{23}$$

Since $c^u(0) = 1$, $\mathbf{C}_\mathbf{Z}(0) = \sum_{u=1}^n \mathbf{B}_u$. It may be noted here that the quantity $c^u(h)\mathbf{B}_u$ can be interpreted as the contribution of the uth factor process to the cross-covariance structure of $\mathbf{Z}$.

From (10), for simple cokriging to be equivalent to simple kriging, one has the condition,

$$\mathbf{C}_\mathbf{Z}(h) = \boldsymbol{\rho}(h) \mathbf{C}_\mathbf{Z}(0) \tag{24}$$

This gives

$$\sum_{u=1}^{n} c^u(h) \mathbf{B}_u = \rho(h) \mathbf{C}_\mathbf{Z}(0) = \rho(h) \sum_{u=1}^{n} \mathbf{B}_u \tag{25}$$

Here $\boldsymbol{\rho}(h)$ is diagonal, the ith element being

$$\frac{C_{ii}(h)}{C_{ii}(0)} = \frac{\sum_{u=1}^{i} c^u(h) b_{ii}(u)}{\sum_{u=1}^{h} c^u(0) b_{ii}(u)} = \frac{\sum_{u=1}^{n} c^u(h) b_{ii}(u)}{\sum_{u=1}^{n} b_{ii}(u)}. \tag{26}$$

From (25), it can be seen that if no conditions are imposed on $\mathbf{B}_u$'s condition (10) holds if and only if

$$\frac{C_{ii}(h)}{C_{ii}(0)} = \rho_h, \text{ say } \quad \forall\, i. \tag{27}$$

This implies that

$$\rho_h = \frac{\sum_{u=1}^{n} c^u(h) b_{ii}(u)}{\sum_{u=1}^{n} b_{ii}(u)} \quad \text{for } i = 1, 2, \ldots, p.$$

If $c^1(h), c^2(h), \ldots, c^n(h)$ are linearly independent,

$$\left.\begin{aligned} \rho_h &= \sum_{u=1}^{n} c^u(h) \beta_u \\ \text{where } \beta_u &= \frac{b_{ii}(u)}{\sum_{u=1}^{n} b_{ii}(u)} \quad \forall\, i \\ \text{and } \sum_{u=1}^{n} \beta_u &= 1. \end{aligned}\right\} \tag{28}$$

In other words, the process $\boldsymbol{Z}$ has to be intrinsic with

$$\mathbf{C}_{\mathbf{Z}}(h) = \rho_h \mathbf{C}_{\mathbf{Z}}(0)$$

i.e. all auto and cross-covariances have to be proportional to ρ_h.

In the case when $\sum_{u=1}^{n} \mathbf{B}_u$ is diagonal, $\mathbf{C}_{\mathbf{z}}(h) = \rho(h) \sum_{u=1}^{n} \mathbf{B}_u$. Condition (10) then holds for any diagonal matrix $\rho(h)$. Here $\mathbf{C}_{\mathbf{z}}(h)$ is diagonal, implying that the process $\mathbf{Z}$ is itself spatially orthogonal. It is interesting to note here that $\sum_{u=1}^{n} \mathbf{B}_u$ is diagonal if and only if each $\mathbf{B}_u$ is diagonal. To see this, one may equate the i, jth element of the matrices in the equation above, to obtain

$$\sum_{u=1}^{n} c^u(h) b_{ij}(u) = \left(\frac{\sum_{u=1}^{n} c^u(h) b_{ii}(u)}{\sum_{u=1}^{n} b_{ii}(u)} \right) \sum_{u=1}^{n} b_{ij}(u).$$

When $i \neq j$, one has $\sum_{u=1}^{n} b_{ij}(u) = 0$, which implies that each $b_{ij}(u) = 0$ as $c^1(h)$, $c^2(h), \ldots, c^n(h)$ are linearly independent.

Further, if $\mathbf{M}_u = \mathbf{M} \;\; \forall\, u$, it can be seen that

$$\mathbf{B}_u = \mathbf{B}$$

and

$$\mathbf{C}_{\mathbf{Z}}(h) = \mathbf{B} \sum_{u=1}^{n} c^u(h) = \mathbf{B}\rho_h, \text{ say,}$$

where

$$\rho_h = \sum_{u=1}^{n} c^u(h).$$

Here (10) holds without linear independence of $c^1(h), c^2(h), \cdots, c^n(h)$.

The observations made above are useful in exploratory data analysis. One may first model the auto-covariance of $\mathbf{Z}$ to identify spatial structures i.e. to suitably identify $\hat{c}^1(h), \hat{c}^2(h), \ldots, \hat{c}^n(h)$ along with $\hat{b}_{ii}(1), \hat{b}_{ii}(2), \ldots, \hat{b}_{ii}(n)$ for $i = 1, 2, \ldots, p$. If it is found that $\frac{\hat{b}_{ii}(u)}{\Sigma_{u=1}^{n} \hat{b}_{ii}(u)} = \frac{\hat{b}_{jj}(u)}{\Sigma_{u=1}^{n} \hat{b}_{jj}(u)} = \hat{\beta}_u$ for all i, j, one obtains $\rho_h = \sum_{u=1}^{n} \hat{c}^u(h)\hat{\beta}_u$. The variance-covariance matrix $\hat{\mathbf{C}}_{\mathbf{Z}}(0)$ can be computed from the data and one may check whether $\hat{\mathbf{C}}_{\mathbf{Z}}(h) \simeq \hat{\rho}_h \hat{\mathbf{C}}_{\mathbf{Z}}(0)$. By this process the spatial structures are identified and it can be checked whether components of $\mathbf{Z}$ may be kriged individually.

It may be noted that in the method suggested above, the modeled auto-covariances are used to check whether $\widehat{\rho}(h)\widehat{\mathbf{C}}_{\mathbf{Z}}(0)$ is an adequate model for $\mathbf{C}_{\mathbf{Z}}(h)$ by comparing $\widehat{\mathbf{C}}_{\mathbf{Z}}(h)$ with $\widehat{\rho}(h)\widehat{\mathbf{C}}_{\mathbf{Z}}(0)$. This method provides an alternative to comparing $\frac{\widehat{C}_{ij}(h)}{\widehat{C}_{ii}(h)}$ directly.

A variation of the above may occur if it is found that some of the off-diagonal elements of $\hat{\mathbf{C}}_{\mathbf{Z}}(0)$ are negligible and $\hat{\mathbf{C}}_{\mathbf{Z}}(0)$ is close to a block-diagonal matrix. The procedure outlined above can be applied separately to variables belonging to each block to check whether they may be kriged individually.

CONCLUSIONS

In the case of a p-dimensional second-order stationary random process $\{\mathbf{Y}(x)\}$, simple cokriging of its components is equivalent to their simple kriging if and only if condition (10) holds. In other words, it should be possible to construct the cross-covariance structure $\mathbf{C}_{\mathbf{Y}}(h)$ using the cross-covariance structure $\mathbf{C}_{\mathbf{Y}}(0)$ at $h = 0$ and the auto-covariance structures (specified by the elements of a diagonal matrix $\rho(h)$). It is shown that

$$\mathbf{C}_{\mathbf{Y}}(h) = \rho(h)\mathbf{C}_{\mathbf{Y}}(0) = \mathbf{C}_{\mathbf{Y}}(0)\rho(h) \quad \forall \;\; h$$

is a necessary and sufficient condition for simple cokriging to be equivalent to simple kriging.

Since the diagonal matrix $\rho(h)$ and $\mathbf{C}_{\mathbf{Y}}(0)$ must commute for all h, it is clear that simple cokriging is equivalent to simple kriging if and only if $\mathbf{C}_{\mathbf{Y}}(h)$ is block diagonal and *RF*s belonging to the same block have proportional covariance. Two extreme cases are i) when there is only one block and all auto- and cross-covariances are proportional to a common covariance structure, and ii) when the number of blocks equals p, i.e. when $\mathbf{C}_{\mathbf{Y}}(h)$ is diagonal.

The mosaic model, multicollocated kriging, and the linear model of coregionalization are discussed in the context of the above. It is shown that if for the random process $\{\mathbf{Y}(x)\}$ simple cokriging of every function is equivalent to its kriging, then the bivariate distributions of $\{\mathbf{Y}(x)\}$ are identical to that of the mosaic model. This however does not necessarily imply the existence of tiles. In the case of multicolocated kriging, condition (10) holds after suitable transformation of data.

For the general case, a method for checking whether simple cokriging of components of $\{\mathbf{Y}(x)\}$ are equivalent to their simple kriging is provided by an analysis of the linear model of coregionalization. Estimation of the autocovariance functions of $\{\mathbf{Y}(x)\}$ and the diagonal elements of matrices as $\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_m$ corresponding to m nested covariance structures allows determination of whether $\rho(h)$ and $\mathbf{C}_{\mathbf{Y}}(0) = \sum_{u=1}^{m} \mathbf{B}_u$ commute for all h.

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