Classical logic and the problem of uncertainty

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Abstract: The uncertainty of knowledge, in contrast to that of data, can be assessed by its probability in the logical sense. The logical concept of probability has been developed since the 1930s but, to date, no complete and accepted framework has been found. This paper approaches this problem from the point of view of logical entailment and natural sequential calculus of Classical logic. It is shown herein that probability can be comprehended in terms of a set of formal theories built in similar language. This measure is compliant with general understanding of probability, can be both conditional and unconditional, accounts for learning new evidence and complements Bayes's rule. The approach suggested is practically infeasible at present and requires further theoretical research in the domain of geoscience. Nevertheless, even within the framework of existing methods of expert judgement processing, there is a way of implementing logic that will improve the quality of judgements. Also, to reach the state of formalization necessary to use logical probability, techniques of knowledge engineering are required; this paper explains how logical probabilistic methods relate to such techniques, and shows that the perfect formalization of a domain of knowledge requires these methods. Hence, the lines for future research should be: (1) the development of a strategy of co-application of existing expert judgement-processing techniques, knowledge engineering and classical logic; and (2) further research into logic enabling the development of formal languages and theories in geoscience.

Among the sources of uncertainty reported in literature, several relate to scientific reasoning and language. Probabilistic methods used to handle them require conditions that are difficult to meet, for example, an absolutely objective and bias-free supervisor of expert judgements, a statistically representative number of experts, or a kit of test datasets and questions that are guaranteed to be previously unknown to the tested experts and to be absolutely perfect themselves (Aspinall & Woo 1994; Aspinall & Cooke 1998).

Another option, in the author's view, is the study of reasoning itself, which is known to be governed by formal rules that are similar for humans and computers. This paper explores the applicability of Classical logic in assessing and reducing various kinds of uncertainty. To approach this goal, it is necessary to:

(1) give an overview of known sources and measures of uncertainty;
(2) investigate how logic can cope with uncertainty;
(3) discuss the role and place of logic among other uncertainty-reducing methods.

Each of these tasks is addressed in this paper.

Sources and measures of uncertainty

Science has developed a wide vocabulary of hesitation (e.g. uncertainty, probability, possibility, inaccuracy, imprecision, fuzziness, error, disagreement) and there is little consensus in the understanding of these terms (e.g. Woo 1999; Virrantaus 2003; Baddeley *et al.* 2004). Nonetheless, to the author's knowledge, most researchers are inclined to use 'uncertainty' as a more-or-less umbrella term for situations in which confidence in a scientific result is lacking. In this paper, it will be used as the general term for all kinds of measurable (at least principally or hypothetically) lack of confidence.

The principle of the complex formal approach to treating information (Pshenichny 2003) seems

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to serve as an appropriate basis for correct and concise classification of sources and measures of uncertainty. According to this approach, any information about the object of study can be regarded as data or knowledge. *Data* are anything expressed as a singular statement (in which the predicate is related to a singular subject), for example, 'The sample 72039 contains 65 wt% silica'. It is commonly accepted in science that data are the result of observation or measurement. *Knowledge* is anything expressed as a general statement (in which the predicate is related to a general subject), for example, 'The rock [meaning '*all* or *some* of the studied samples'] contains (contain) 65 wt% silica'. As shown in Pshenichny (2003), this distinction is actually context-dependent and what is considered knowledge at a local scale (e.g. the scale of an outcrop) can become data at greater scales (e.g. the scale of a region), and vice versa. Nevertheless, in every particular case, knowledge and data can (and should) be clearly separated. Data and knowlege are then treated by different formal approaches.

Sources of uncertainty that refer to data and means of their processing include: sampling, observational and measurement errors, errors of mathematical evaluation of data and propagation of errors (Bardossy & Fodor 2001); estimation errors, scale issues, ignorance and human error (Bowden 2004); temporal, structural, metric-related and translational uncertainties (Rowe 1988); and possibly some others. A spectrum of probabilistic, possibilistic (fuzzy) and joint probabilistic-possibilistic methods (Bardossy & Fodor 2001), including computer applications (e.g. UncertaintyAnalyzer, see http://www.isgmax.com) can be applied to analyse such sources. Here the probability is meant in a statistical or frequentist sense (Baddeley *et al*. 2004; and Woo 1999, respectively).

Consideration of data-related uncertainty and its measures in more detail is beyond the scope of the present paper. The sources of knowledge-related uncertainty are reasoning and language. Woo (1999) recognized two different kinds of such uncertainty: *conceptual* (*epistemic*) and *aleatory*. Epistemic uncertainty is rooted in the knowledge itself, and aleatory uncertainty is rooted in the belief in knowledge. This can be illustrated by the following examples 2004:

> How can we express the likelihood of a new volcano forming in non-volcanic regions or of a new active fault forming in regions far from current active faults?
>
> How can we determine whether or not a volunteer site located in a region outside the 'obvious' exclusion zones for the site selection criteria should be included or excluded from the next investigation stage?

In the former case, for instance, it is implicitly supposed that there is a model or models (called 'probability models' in the literature) saying whether a volcano could form in a given non-volcanic region. Every model has its own intrinsic approximations and associated uncertainty: the epistemic uncertainty. In addition, every volcanologist may trust each model more or less: this is the aleatory uncertainty of a model attributed by individual scientists.

The degree of trust is determined by the researcher's *bias* (e.g. Halpern 1996; Baddeley *et al*. 2004; and references therein). It is accepted that personal or group bias erodes the quality of expert judgements and should be reduced where possible, e.g. by elicitation procedures (see reviews in Baddeley *et al*. 2004; Curtis & Wood 2004). However, bias is a virtue of geology and other largely hermeneutic (interpretative) sciences (e.g. history, psychology, medicine, economics).

> In effect, hermeneutics rejects the claim that facts can ever be completely independent of theory ... we always come to our object of study with a set of prejudgements: an idea of what the problem is, what type of information we are looking for, and what will count as an answer (Frodeman 1995).

Among these prejudgements, Frodeman lists as 'crucial ... and often discounted ... the social and political structures of science'. He acknowledges that exactly these structures and personal preferences, and not a pursuit of absolute truth, route the science.

Once bias is not only unavoidable but even determines the direction of research, then the very idea of decreasing bias becomes questionable, even though there is the obvious involvement of quite diverse circumstances (e.g. from rock composition to standard of living) into any single geoscientific rationale. At best we can substitute personal bias with the most appropriate collective one. Apparently, the most appropriate bias is not the one that is most logically correct or statistically proven, but the one shared by the largest and/or strongest group, for example, the bias that the entire world should adopt democratic values or that humankind should survive; the latter has given ground to the whole study of risk assessment. In general, science seems to have only two options: (1) to ignore bias and fight for objective knowledge, as prescribed by analytical philosophy, or (2) to follow the Continental philosophy apprehension and accept bias, reveal, assess and manage it, but not try to decrease it. This point will be dealt with below.

Discussing present measures of uncertainty, Woo (1999) considers three notions of probability: (1) *frequentist*, which relates to data in the classification above; also termed 'objective' (Baddeley *et al.* 2004) or statistical (Carnap 1950); (2) *subjective* (Baddeley *et al.* 2004); and (3) *logical*. Bardossy & Fodor (2001) add fuzzy measures.

Measures of both epistemic and aleatory uncertainty in most cases is subjective or epistemic (Woo 1999; Fitelson 2003), i.e. probability formulated in probabilistic judgements (Baddeley *et al.* 2004). Fitelson (2003) states:

> On epistemic interpretations of probability, $Pr_M(H)$ is (roughly) the degree of belief an epistemically rational agent assigns to H, according to a probability model M of the agent's epistemic state. A rational agent's background knowledge K is assumed ... to be 'included' in any epistemic probability model M, and therefore K is assumed to have an unconditional probability of 1 in M. $Pr_M(H\,|\,E)$ is the degree of belief an epistemically rational agent assigns to H upon learning that E is true (or upon the supposition that E is true ...), according to a probability model M of the agent's epistemic state ... So, roughly speaking, (the probabilistic structure of) a rational agent's epistemic state evolves (in time t) through a series of probability models $\{M_t\}$, where evidence learnt at time t has probability 1 in all subsequent models $\{M_{t'}\}$, $t'>t$.

A probabilistic framework for this evolving state of knowledge in the geosciences is described in Wood & Curtis (2004).

However, neither the rationality of the agent (i.e. the expert) nor the structure of K, M and E are explained by Fitelson (2003), and involvement of time (t) brings a healthy spirit of physiology into mathematical consideration, because evolution of epistemic state through time would depend, *inter alia*, on the expert's metabolism which determines the rate of digestion of new evidence.

The main criteria of quality of probabilistic statements (and, hence, the probability of subjective probability values) are qualification and objectivity of experts, as, for example, in the approach of Cooke (1990). In this approach, applied by Aspinall & Woo (1994) to assess the expert's qualification and objectivity, each expert is asked some test questions. It is supposed that correct answers for these questions are somehow known, and any other answer can be calibrated against these. Discussing the possible ways to obtain such 'bullet-proofed' answers, Aspinall & Cooke (1998) desire 'an experienced technical facilitator' of expert judgement elicitation procedure. The question remains unanswered as to who evaluates his or her answers.

One of the weakest points of subjective probability, to the author's mind, is that it is rather artificially and voluntarily normalized to unity. Probability, be it frequentist or any other form, requires the condition of additivity (e.g. Bardossy & Fodor 2001) that implies mutually exclusive cases or objects (populations). This readily entails normalization as the ratio n/N, where n is the number of cases or objects for which something is true, and N is the total number of said entities, for example, number of tails v. total number of tossings of a coin. However, if one expert says that he feels. A has the probability of 0.7 while another expert gives 0.6, or if an expert is asked to give a probability distribution based on his intuition, it is unlikely that this can be considered 'true' normalization.

Nevertheless, this fits fairly well with the concept of a membership function of fuzzy logic. This has made Bardossy & Fodor (2001) express a preference for the use of possibilistic rather than probabilistic approaches to processing expert judgements. One problem, however, is that these judgements are commonly used, firstly, in Bayesian approaches for prior probability values where no distinction is made between them and frequentist values, and secondly, in probability trees where they are processed according to the theorems of probability theory. Hence, there is a need for probabilistic, not just fuzzy, estimation of knowledge-related uncertainty. As was pointed by Aspinall & Woo (1994):

> It has been shown theoretically ... that there is no essential distinction between probability assignments based on numerical frequencies and those based on (general – *CP*) judgments: both can be incorporated into a computation of hazard (or any other item of interest – *CP*) if the correct procedures are adopted.

Further pursuit of these procedures should be considered an important task for future research.

A possible alternative might be one which has been rejected with enthusiasm by the proponents of probabilistic and then fuzzy thinking in geoscience: the deterministic model. Once knowledge is not data, probability assessments based on general judgements must be of a different nature than those based on frequencies. On first sight, a 'deterministic probability model' is nonsense. Nevertheless, Woo shows that even the most strict and objective knowledge has, or may have, its own genuine uncertainty. The simplest example is division by zero in arithmetic. Woo also quotes: the Fourier expansion in a Hamiltonian system describing the planet orbits, which, in some cases, also reduces to a necessary division by zero; Heisenberg's principle of uncertainty in quantum mechanics; deterministic chaos in meteorology; and other examples where the uncertainty 'emerges from the midst of determinism' (Woo 1999, p. 72). Perhaps, one of the sources of uncertainty in geoscience cited by Bowden (2004), that of modelling constraints, also falls in this field.

The measure of this 'deterministic sort' of epistemic uncertainty, by Woo, is the third and last kind of probability that he lists: logical probability. Likewise, Virrantaus (2003, Fig. 1) mentions among the sources of uncertainty the 'validity of the model', which includes, *inter alia*, 'the logic of algorithm'. However, in contrast to frequentist and subjective notions of probability considered in much detail, neither of these authors explain *how* logical probability can work. Woo only mentions the principle of indifference (or equiprobability) in absence of additional information.

The logical concept of probability has been developed since the 1930s by Johnson (see references in Fitelson 2003, Carnap 1950 and later works, and others). To evaluate this concept, Classical logic must be introduced. This is discussed in the next section.

To summarize this section, using the complex formal approach of Pshenichny (2003), uncertainty can be classified as that of knowledge and that of data. The uncertainty of data can be considered: to satisfy additivity and be measured by probability in the frequentist sense; to be 'fuzzy' and be measured by a variety of fuzzy parameters; or both. Reasoning and language contribute to the uncertainty of knowledge: the epistemic uncertainty and its refinement, the aleatory uncertainty. As for the measure of epistemic uncertainty, if one approaches knowledge intuitively, the measure may only be fuzzy; if one treats knowledge rationally the measure, if it exists, should be probabilistic in a logical, but not in a subjective, sense.

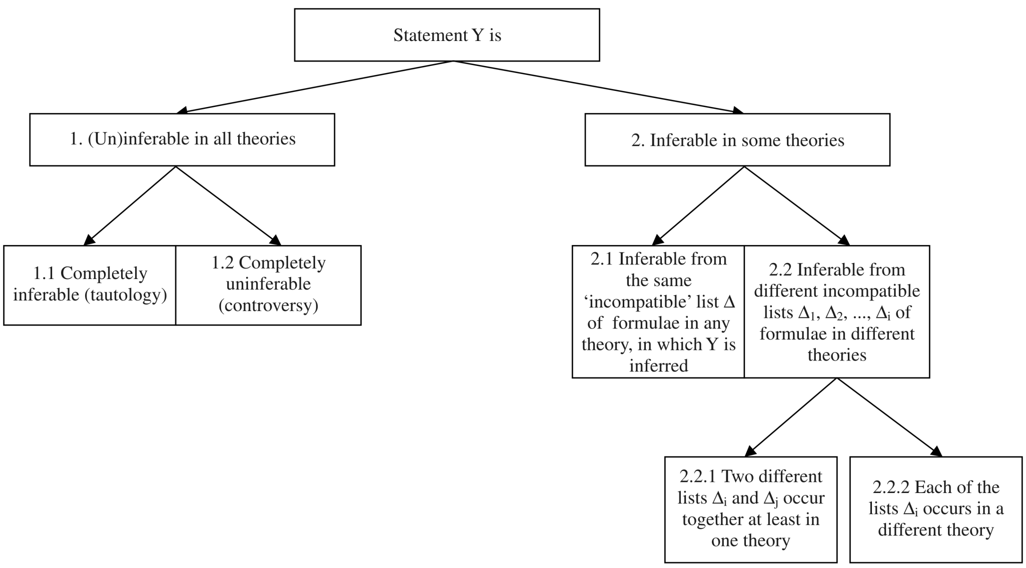

Fig. 1. Inferability of a statement Y in a set of theories built in a similar language (see comments in the text).

The study of logic and its relation to uncertainty

Classical logic, first described by Aristotle, was elaborated in its modern form in the second half of the nineteenth to the first half of the twentieth century by Frege (1896), Whitehead & Russell (1910), Kleene (1952), Hilbert & Bernyce (1956) and some others. A brief account will be given here, based dominantly on Kleene (1952). More details can be found in Pshenichny *et al.* (2003).

Essentially, the logic consists of two parts: propositional logic and predicate logic. They differ in the mode of record of simple statements (or narrative sentences) of natural language (English, Russian, Chinese etc.). By 'simple statements' we mean those grammatically consisting of one subject and one predicate, for example, 'Some magmas ascend to the Earth surface' or 'All volcanoes erupt'.

Propositional logic takes them as indivisible elements (*propositional variables*, or *propositions*), that necessarily have one, and only one, of two *logical* (or *truth*) *values*: *TRUE* and *FALSE*. From propositions *compound statements* are formed by the *logical connectives* ('not', 'and', 'or', 'either, or', 'if, then', 'is equivalent to' and possibly some others – see denotations and definitions in Table 1). For example:

Some magmas ascend to the Earth surface – p

All volcanoes erupt – q

Some magmas do not ascend to the Earth surface – $\neg p$

Some magmas ascend to the Earth surface and all volcanoes erupt – $p \;\&\; q$

If some magmas ascend to the Earth surface, then all volcanoes erupt – $P \supset q$.

Predicate logic, on the contrary, treats such statements as constructions consisting of an *individual variable* ('magma' – x; 'volcano' – y), the *predicate* ('ascend to the Earth surface' – P; 'erupt' – Q) and *quantifier* (*existential* one meaning 'some' – $\exists$, and *universal* one meaning 'all' – $\forall$). Then the same simple statements will be recorded as follows:

Some magmas ascend to the Earth surface – $\exists x P(x)$

All volcanoes erupt – $\forall y Q(y)$.

Literally, these statements mean: 'For some magmas it is true that they ascend to the Earth surface' and 'For all volcanoes it is true that they erupt', respectively. Obviously, predicate logic offers a better opportunity to study the anatomy of a statement. However, any concept can be expressed as an array of statements lined up by *logical inference* (see below), and basic rules of inference are the same for both types of logic.

Let us define both types of record of simple statements – $p, q, \ldots$, and $\exists x P(x), \forall y Q(y), \ldots$ – as *logical formulae* (henceforth generally denoted: A, B, C, ...) and postulate that, if A and B are logical formulae, then $\neg$A, A&B, A $\vee$ B, A$\supset$B, A$\equiv$B (see Table 1) are formulae too.

The truth value of a formula is that of its main connective. There are formulae that can have only one truth value (i.e. are always true or always false) with any values of variables; for example, the formula $(p \vee \neg p)$ is always true and $(p \& \neg p)$ is always false (see Table 2).

If a formula is always true, it is called a *logical law*, or *tautology*. If it is always false, it is a *controversy*. (This is not to be confused with controversy as the relationship between any two statements, one of which is the negation of the other; the conjunction of such statements gives the controversy in the sense meant here. To avoid ambiguity, this relation is also called *contradiction*.) All the remaining formulae may take both truth values and are called *neutral*, or *satisfiable*.

This gives us, according to Kleene (1952), a strict and definite answer to the question 'What means "to follow" in relation to thinking?' If A, B, C, ..., X, Y are formulae (i.e. statements or thoughts), then A, B, C, ..., X is a *finite list* of formulae. *Y follows from A, B, C, ..., X, if and only if* the following condition is met: *provided A, B, C, ..., X are true, Y is true*. The same is

Table 1. *General truth table for basic logical connectives*

A	B	¬A	A&B	A ∨ B	A ⊃ B	A ≡ B
True	True	False	True	True	True	False
False	True	True	False	True	True	True
True	False		False	True	False	True
False	False		False	False	True	False

Table 2. *Truth table for most simple tautology* $(p \vee \neg p)$ *and controversy* $(p \& \neg p)$

p	$\neg p$	$p \vee \neg p$	$p \& \neg p$
True	False	True	False
False	True	True	False

meant by the expression 'Y is inferred from A, B, C, ..., X' and denoted by the arrow: A, B, C, ..., Y → X.

A number of types of inference have been suggested in the logical literature. Here we will briefly describe the *natural sequential calculus* elaborated by Gentzen (1934).

(1) *Sequence* is expression A_1, A_2, ..., $A_m \rightarrow B$, where A_1, A_2, ..., A_m, B are formulae. A_1, A_2, ..., A_m are front members of the sequence, and B is a back member. There may be no front members at all, but the back one must always be present: → B.

(2) *Inference in natural-sequential calculus* consists of a number of sequences. Each of these is either a *main sequence* or is derived from a previous one by a structural transformation or a rule of inference (see below). The last sequence of inference has no front members and its back member is a finite formula.

(3) There are two types of *main sequences*, called *logical* and *mathematical*. A *logical main sequence* is a sequence of general form C → C, where C is a formula (this sequence arises if the inference is based on an assumption expressed by C). A *mathematical main sequence* is a sequence of general form → D, where D is an axiom of mathematics.

(4) Allowed structural transformations (a horizontal line means that, if the above sequence is present, the below one is correct) are:

(4.1) Transposition of two front members:

$$\frac{C, D, \Gamma \rightarrow \Delta}{D, C, \Gamma \rightarrow \Delta}$$

(4.2) Withdrawal of a front member, which is the same as another front member:

$$\frac{C, C, \Gamma \rightarrow \Delta}{C, \Gamma \rightarrow \Delta}$$

(4.3) Addition of any propositional formula to front members:

$$\frac{\Gamma \rightarrow \Delta}{C, \Gamma \rightarrow \Delta}$$

(5) Rules of inference. Let A, B and C denote any propositional formulae and Γ, Δ and Θ – any (possibly empty) lists of formulae divided by commas. The formulae of these lists are front members of some sequences. The following rules of inference of natural-sequential calculus are applicable to propositional logic.

- Introduction of conjunction (henceforth Rule IC):

$$\frac{\begin{array}{l}\Gamma \rightarrow A \\ \quad \Delta \rightarrow B\end{array}}{\Gamma, \Delta \rightarrow (A \& B)}$$

- Elimination of conjunction (henceforth Rule EC):

$$\frac{\Gamma \rightarrow A \& B}{\Gamma \rightarrow A}$$

$$\frac{\Gamma \rightarrow A \& B}{\Gamma \rightarrow B}$$

- Introduction of disjunction (henceforth Rule ID):

$$\frac{\Gamma \rightarrow A}{\Gamma \rightarrow A \vee B}$$

$$\frac{\Gamma \rightarrow B}{\Gamma \rightarrow A \vee B}$$

- Elimination of disjunction (henceforth Rule ED):

$$\frac{\begin{array}{l}\Gamma \rightarrow A \vee B \\ A, \Delta \rightarrow C \\ \quad B, \Theta \rightarrow C\end{array}}{\Gamma, \Delta, \Theta \rightarrow C}$$

- Introduction of implication (henceforth Rule II):

$$\frac{A, \Gamma \rightarrow B}{\Gamma \rightarrow A \supset B}$$

- Elimination of implication (henceforth Rule EI):

$$\frac{\begin{array}{l}\Gamma \rightarrow A \\ \Delta \rightarrow A \supset B\end{array}}{\Gamma, \Delta \rightarrow B}$$

- Introduction of negation (henceforth Rule IN):

$$\frac{\begin{array}{c}A, \Gamma \rightarrow B \\ A, \Delta \rightarrow \neg B\end{array}}{\Gamma, \Delta \rightarrow \neg A}$$

- Elimination of double negation (henceforth Rule EN):

$$\frac{\Gamma \rightarrow \neg\neg A}{\Gamma \rightarrow A}$$

- Introduction of universal quantifier $\forall$:

$$\frac{F \rightarrow G(x)}{F \rightarrow \forall x G(x)}.$$

- Introduction of existential quantifier $\exists$:

$$\frac{G(x) \rightarrow F}{\exists x G(x) \rightarrow F}$$

Here x is an individual variable; F is any correctly built formula independent of x, G(x) is predicate.

In the above expressions, symbols put to the left of the arrow can be regarded as the 'memory' of the inference, and those to the right are its 'working part'.

Hence, the conclusion is not a matter of scientist's intuition, but a certain operation verifiable by an objective tool of logical inference. This can be illustrated by the following example relevant to a situation of volcanic crisis like that in Montserrat.

A lava dome is growing; the direction of growth may change with time. Hot avalanches originate on the growing lava dome. Old domes occur nearby. X is a settlement located in the vicinity of the lava domes, very accessible to avalanches. Query, will the avalanches reach X or not? Volcanologists say, if the new dome 'chooses' a direction toward, one of the old domes and reaches it, hot avalanches may stop, but only if the growing dome does not overwhelm the older one. Otherwise avalanches may resume and reach X. Logic allows us to verify whether this rationale is correct. If we define:

p – fresh portion of magma extrudes

q – block of old lava dome occurs on the fresh portion's way

r – hot avalanches reach X

s – fresh portion of magma overwhelmes the block of old dome,

then the whole rationale takes the form

$$(p \supset r) \vee ((p\&q) \supset (s \supset r)).$$

There are at least two ways to verify this. The first is using a truth table (Table 3).As is seen from the table, the statement is satisfiable and true except in two cases, when p, q, and s are true and r is false, and when p and s are true and q and r are false.

Another method of verification is to use logical calculus. This is necessary when we are

Table 3. *Calculation of truth values of the satisfiable formula* $(p \supset r) \vee ((p\&q) \supset (s \supset r))$ *and tautology* $r \supset ((p \supset r) \vee ((p\&q) \supset (s \supset r)))$ *(see comments in the text)*

p	q	r	s	$p \supset r$	p&q	$s \supset r$	$(p\&q) \supset (s \supset r)$	$(p \supset r) \vee ((p\&q) \supset (s \supset r))$	$r \supset ((p \supset r) \vee ((p\&q) \supset (s \supset r)))$
T	T	T	T	T	T	T	T	T	T
F	T	T	T	T	F	T	T	T	T
T	F	T	T	T	F	T	T	T	T
F	F	T	T	T	F	T	T	T	T
T	T	F	T	F	T	F	F	F	T
F	T	F	T	T	F	F	F	T	T
T	F	F	T	F	F	F	F	F	T
F	F	F	T	T	F	F	F	T	T
T	T	T	F	T	T	T	T	T	T
F	T	T	F	T	F	T	T	T	T
T	F	T	F	T	F	T	T	T	T
F	F	T	F	T	F	T	T	T	T
T	T	F	F	F	T	T	T	T	T
F	T	F	F	F	F	T	T	T	T
T	F	F	F	F	F	T	T	T	T
F	F	F	F	F	F	T	T	T	T

interested in inferability of a formula from a given set of premises. Premises can be any correctly built formulae. For instance, let us assume that:

(1) $p \to p$
(2) $q \to q$
(3) $r \to r$
(4) $s \to s$.

Then the inference of the considered statement (continued from line 5, as lines 1–4 are assumptions) is:

(5) $p, r \to r$ Addition of formula assumed in line 1 to front members for line 3
(6) $r, p \to r$ Transposition of two front members for line 5
(7) $r \to p \supset r$ Rule II for line 6
(8) $s, r \to r$ Addition of formula assumed in line 4 to front members for line 3
(9) $r, s \to r$ Transposition of two front members for line 8
(10) $r \to s \supset r$ Rule II for line 9
(11) $p, q \to p\&q$ Rule IC for lines 1, 2
(12) $p\&q,\ r \to s \supset r$ Addition of formula inferred in line 11 to front members for line 10
(13) $r \to (p\&q) \supset (s \supset r)$ Rule II for line 12
(14) $r, r \to (p \supset r) \vee ((p\&q) \supset (s \supset r))$ Rule ID for lines 7, 13
(15) $r \to (p \supset r) \vee ((p\&q) \supset (s \supset r))$ Withdrawal of a front member for line 14
(16) $\to r \supset ((p \supset r) \vee ((p\&q) \supset (s \supset r)))$ Rule II for line 15. Statement proved.

The statement $r \supset ((p \supset r) \vee ((p\&q) \supset (s \supset r)))$ is a tautology, i.e. true with any values of the variables (see Table 3). Similarly, one can add the next assumption (p, q or s) left of the arrow and move it to the right adding the implication, for example:

(17) $\to r \supset ((p \supset r) \vee ((p\&q) \supset (s \supset r)))$
(18) $s \to r \supset ((p \supset r) \vee ((p\&q) \supset (s \supset r)))$
(19) $\to s \supset (r \supset ((p \supset r) \vee ((p\&q) \supset (s \supset r))))$.

and so forth, up to the sequence $\to p \supset (q \supset (s \supset (r \supset ((p \supset r) \vee ((p\&q) \supset (s \supset r))))))$ including all the premises (assumptions). Starting with $r \supset ((p \supset r) \vee ((p\&q) \supset (s \supset r)))$, only tautologies will be yielded at every new step of this inference. This inference indicates that the considered rationale is correct with the given premises.

However, as one might notice, nothing changes in the above inference if one were to substitute location X, say, with Edinburgh. This is because propositional logic ensures only formal correctness of thinking regardless of the contents of statements (their meaning in the real world). To deal with the contents, it should be 'enhanced' with the armoury of predicate logic. All rules of propositional logic are maintained in predicate logic but, in addition, contents-determined predicates are defined, special axioms can be formulated in terms of these predicates and individual variables and quantifiers can be involved (see last two inference rules above).

All this opens an opportunity to formulate an *ad hoc strict language* for a domain of knowledge and to construct in this language a *strict theory*, in which every statement is either an axiom or a theorem inferred from axioms by the rules of inference, either general (see above) or specific. To become a theory, a set of statements should comply with the following conditions: (1) the language for the theory is defined, (2) a correctly built formula is defined for this theory, (3) axioms are identified among the formulae of the theory, (4) rules of inference are defined in the theory. The theory must be self-consistent (which implies the impossibility to infer A and not-A from any set of axioms and/or theorems in the theory), complete (meaning that, for any statement A of the theory, either A or not-A is a theorem), deducible (i.e. that there is an algorithm to identify whether A is a tautology, controversy or is satisfiable in the theory for any statement A of the theory), and independent of axioms (Takeuti 1975).

Hilbert & Bernyce (1956) presented a strict theory of arithmetics. Tarski (1959) formalized Euclidean geometry. Another important example is Kolmogorov's axiomatization of probability theory (e.g. Woo 1999). An example of strict theory, prone to formalization by predicate logic, is Mendeleev's periodic system of elements and its implications. Also, attempts have been made to apply this method in physics (e.g. Dirac 1964). However, these examples refer to exact and experimental sciences, which, by Frodeman (1995), can satisfy the condition of bias-free observation and objective knowledge posed by analytic philosophy.

However, there has been a temptation in logic to conquer vaster terrains of mental activity than the artificial world of abstraction or experiment. Bacon (1620) in *The Novum Organon* first claimed that logic must be a tool for obtaining new knowledge and suggested a scheme for inductive entailment. Later this idea was developed in the eighteenth to nineteenth centuries by Hume, Mill and many others (historical background outlined in Fitelson 2003) and evolved into the project of inductive logic, aiming to build an inference from particular to general, relative to which the deductive inference would be an extreme case (Carnap 1950; Fitelson 2003). So far, such a formalism has not been constructed, and inductive logic as-is can generalize data but not introduce logical connectives as do logical calculi.

Another idea was to account for natural variability of the world by prescribing more than two truth values to statements. Thus three-, four-, n-, infinite-valued and, finally, fuzzy logics appeared from the work of Vasiliev, Heiting, Post, Levin, Zadeh and others (see, e.g., system and bibliography at http://www.earlham.edu/~peters/courses/logsys/nonstbib.htm). Despite their wide application in technique, medicine, geoscience and other fields, none of them has suggested its own inference allowing the introduction of connectives. Hence, these, so-called non-Classical logical systems are not logic *sensu stricto*. The same can be said, in the author's opinion, about the modal logic (see bibliography mentioned above). In the author's understanding, inductive, multi-valued and even modal logics all serve to process data and proceed, in an essentially inductive manner, from data to knowledge but not from knowledge to strict theory (see Pshenichny 2003). Knowledge remains as crisp and black-white, two-fold and true false as it was in Aristotelean times, and data are as incomplete and prone to non-unique interpretation as they used to be.

The only actual attack on this concept, to the author's knowledge, was Brouwer's approach in mathematics, professing that thinking is essentially intuitive and can be best understood not through logical formalisms but through particular cases, one of which is logic itself. Consideration of this idea, so sensational in mathematics in the 1950s, and its relationship to (discordance with?) the approach of Pshenichny (2003) followed here, is outside of the scope of the present paper.

Another challenge to deal with the 'real' world was a logical approach to probability developed by Carnap (1950 and later works) and yet earlier by Johnson (see references in Fitelson 2003). Carnap formulated unconditional and conditional probabilities in the language of classical (i.e. the first-order predicate) logic to make them fit Bayes's formula. However, problems arose with formulation of conditional probability to account for acquisition of new evidence that made Carnap elaborate more and more complicated theories, reviewed by Fitelson (2003). In general, this problem was not solved. The methodological reason for the lack of success perhaps could be under-estimation of the difference between data and knowledge and inapplicability, or quite limited applicability, of formalisms developed to process ideas (i.e. predicate logic) to handle data – an incorrectness symmetrical to the application of probabilistic methods to study reasoning (see above).

The approach of Gentzen (1934) was free from this consideration. Developing his logical calculi, he simply tried to simulate thinking in the natural science by virtue of one of them: the natural sequential calculus (see above). As has been demonstrated in this paper, (1) this calculus does work for its purpose, but (2) on its own it has little sense outside of a strict theory (e.g. in the real world).

The author deems that it is exactly the logical concept of strict theory that may give us the clue to understanding logical probability. Herewith, we should keep in mind the distinction between knowledge and data and resist the temptation to speak about the *actual* process of learning that refers to data acquisition and processing, not to logic *sensu stricto*.

Learning in a logical sense can be paralleled with the concept of inference (see above). If A, B, C, ..., X is a finite list of formulae (statements), then Y may be *learnt* from this list if, and only if, Y follows from it. The latter is unequivocally identified by a logical calculus. In the simplest case, axioms follow from themselves, $A \rightarrow A$.

If a number of theories are built in one language, then, as shown in Figure 1, a similar statement Y may be inferable in: all of them (if Y is tautology – case 1.1); in some of them (if Y is satisfiable – case 2); or in none of them (if Y is a controversy – case 1.2). The second case is of interest to us. In this case Y follows from some lists of formulae. Let us consider only the lists where: (1) no operation of addition of formulae to front members has been made, (2) no similar formulae are present (that is, there is nothing to reject), and (3) transposition of formulae is ignored, which means that lists A, B and B, A are considered to be the same.

These three conditions allow to us to consider (and define) such lists as *incompatible* with each other (even if there happens to be only one such list over the whole set of theories; Fig. 1, case 2.1). Incompatible lists can be quite different. For instance, if $Y \equiv (Y_1 \& Y_2)$, it may be deduced in one theory from a formula $(Y_1 \& Y_2) \vee Z$ by the rule of elimination of disjunction, and in another theory, say, from independent premises $Y_1 \rightarrow Y_1$ and $Y_2 \rightarrow Y_2$ by introduction of conjunction, and in yet another theory be taken for the axiom itself, $Y_1 \& Y_2 \rightarrow Y_1 \& Y_2$.

Given a language (roughly, the set of predicates and individual variables), a number (N) of mutually exclusive strict theories can be constructed. For instance, in addition to Euclidean geometry, there is Lobachevsky's geometry, as well as a few other geometries. All of them operate with similar objects (points and lines, from which angles, figures and other entities are derived) and predicates (betweenness, equidistance, parallel lines, etc.; see, e.g., Nam 1995) but are incompatible with each other. Yet simpler examples of an infinite set of theories from a similar language are Smullyan's puzzles about knights, knaves and werewolves (Smullyan 1978).

Let us denote one incompatible list A, B, C, …, X, from which Y is inferred, by Δ_1, another list by Δ_2, yet another one by Δ_3 and so forth, up to Δ_k, k being a natural number. Each of Δ_i occurs in n_i theories in the given language. Then three cases are possible, 2.1, 2.2.1 or 2.2.2 (see Fig. 1). Case 2.2.1 is rather complicated for further consideration; to begin with, let us assume for simplicity that if Δ_i and Δ_j occur in one theory, then either of them is arbitrarily excluded, so the theory is considered solely Δ_i-containing or Δ_j-containing to make the alternatives incompatible. For cases 2.1 and 2.2.2, if the total number of theories in which Y is inferred from any incompatible list of formulae Δ_i is n, $n = n_1 + n_2 + n_3 + \cdots + n_k$, the *total logical probability* of *Y over a set of theories in a given language* can be defined exactly in the same way as a frequentist probability, i.e.

$$\begin{aligned} P_{LT}(Y) &= n/N = P_L(\Delta_1 \to Y) \\ &\quad + P_L(\Delta_2 \to Y) + \cdots + P_L(\Delta_k \to Y) \\ &= n_1/N + n_2/N + \cdots + n_k/N \\ &= (n_1 + n_2 + \cdots + n_k/N. \end{aligned}$$

The additivity condition is satisfied by incompatibility of theories and lists of formulae. A more accurate extension of this consideration to case 2.2.1 (Fig. 1) will be reported elsewhere.

The same can be said not only about Y but about every formula in every list Δ_i as well. Each of them would have a probability over a set of theories in a given language. The probability of a list, apparently, should be considered as the sum of probabilities of all its formulae B_1, B_2, …, B_h, i.e.

$$\begin{aligned} P_L(\Delta_i) &= P_{LT}(B_1) + P_{LT}(B_2) + \cdots + P_{LT}(B_h) \\ &= n_{LT\text{-}B1}/N + n_{LT\text{-}B2}/N + \cdots + n_{LT\text{-}Bh}/N \\ &= (n_{LT\text{-}B1} + n_{LT\text{-}B2} + \cdots + n_{LT\text{-}Bh})/N, \end{aligned}$$

the probabilities $P_L(B_1)$, $P_L(B_2)$, …, $P_L(B_h)$ defined in the way shown in previous formula. *A priori logical conditional probability* of Y given any of Δ_i, $P_{LC}(Y \mid \Delta_i)$, is nothing else but the probability of logical inference $\Delta_i \to Y$, which is obviously equal to 1. Hence, *a posteriori conditional logical probability* of any (new) Δ_i given Y, $P_{LC}(\Delta_i \mid Y)$, in accordance with Bayes's theorem, would take the following form:

$$\begin{aligned} P_{CL}(\Delta_i \mid Y) &= P_L(\Delta_i)P_{CL}(Y \mid \Delta_i)/P_{LT}(Y) \\ &= P_L(\Delta_i)/P_{LT}(Y) \\ &= (P_{LT}(B_1) + P_{LT}(B_2) \\ &\quad + \cdots + P_{LT}(B_h))N/ \\ &\quad (n_1 + n_2 + \cdots + n_k) \\ &= (n_{LT\text{-}B1} + n_{LT\text{-}B2} + \ldots + n_{LT\text{-}Bh})/ \\ &\quad (n_1 + n_2 + \ldots + n_k), \end{aligned}$$

with the relative likelihood ratio expressed by the simple term $1/P_{LT}(Y)$. Thus, the posterior logical probability of some knowledge (expressed as a list of formulae) depends directly on the number of theories in which the formulae from this list are inferred (not necessarily together), depends inversely on the numbers of theories in which the consequence from this knowledge (Y) is inferred from various lists, and does not depend at all on the total number (N) of theories.

The N strict theories can readily be taken, in the author's opinion, for N probability models. In the general case, N is infinite.

Construction of strict theories and a logical account of probability based on these is how both the intellectual ambition of logicians and the practical need of geoscientists in probabilistic estimation of knowledge-related uncertainty (see above) can be satisfied. Moreover, this may substantially alter (and optimize, in the author's opinion) 'science's own understanding of the nature of science', as put by Frodeman (1995). Also, a minor yet important benefit is that elaboration of a logical account of probability frees us from a delicate mission of measuring experts' metabolism to estimate the speed of passage of experts through an array of probability models.

To summarize the account of logic, its main virtue is the relation of logical inference, or deducibility. Logic *sensu stricto* consists of two major parts: propositional logic and predicate logic. Propositional logic is a simple tool useful to introduce the logical calculi that actualize the relation of inference in certain formal procedures. However, alone, it is insufficient to process knowledge. Predicate logic incorporates all laws of propositional logic but offers specific means for construction of strict theories, which have been successfully tested in exact and experimental sciences. Consideration of inference of a statement in a set of strict theories composed in similar language opens an opportunity to define and calculate logical probability, which meets general requirements to probability and, in particular, fits well the Bayesian approach.

Discussion: logic and existing approaches

Consideration of logic as a tool to obtain probabilistic estimates of knowledge-related uncertainty raises three principal questions:

(1) How good is logic in comparison with existing methods?
(2) Is it competitive or complimentary with them?
(3) What can be done to make it work best?

Addressing point (1) first, logic may provide a means to assess probability of knowledge just as well as the existing methods of expert judgment processing which lead to subjective probability. However, contrary to subjective probabilities, the logical probability is normalized to 1, complies with the condition of additivity, and is based on absolutely strict foundations and is objective. This definitely favours application of logical rather than subjective probability in the Bayesian approach and elsewhere. Nevertheless, it imposes very strict conditions on knowledge:

(1) it requires a formal language sufficiently well describing the field of interest;
(2) the theories should be formulated in this language, each satisfying the rigours of self-consistency, completeness, deducibility and independence of axioms;
(3) even among these formally correct theories it cannot be excluded that some will appear senseless and would be rejected by scientists, and this might revive the undesirable 'aleatory' component in the logical approach to probability. Though, in contrast with the existing methodology, even if present, here it will be formulated in crisp yes/no mode (to include or not to include a theory in the set N), instead of assigning unverifiable values.

However, perhaps the most palpable consideration is that the geoscientific community is psychologically not ready to accept a formal treatment of knowledge, possibly because of impending consequence of changing, by Frodeman (1995), the 'self-understanding of science'. Only in some exceptional cases like EYDENET, an expert system supporting decisions on landslide hazard warning in northern Italy (Lazzari *et al.* 1997; Woo 1999), a kind of propositional logic is used.

Besides, even if we advance far in logical formalization of some domains of geo-knowledge, there will always be problems unreached by logic but requiring urgent decision making. Moreover, even concerning the well-formalized fields, to relate new knowledge resulting from generalization of new data or from intuition of a new expert, the conventional methods of expert judgement processing will be required. Thus, logic is a theoretically better, but practically less feasible option than the existing methods of evaluating epistemic uncertainty, and in any case it cannot replace these methods.

Addressing point (2), the above notwithstanding, there is still room for logic in the framework of existing procedures. Logical deducibility is the criterion for correctness of formulation of judgements. However, along with deducibility, there are a number of other logical relations between statements and concepts (e.g. controversy, compatibility, subordination, incompatibility), elucidation of which can be useful at least in the reconciliation of views, in processing the results of collective brainstorming (e.g. Morgan & Henrion 1990), in producing collective scientific opinion by a decision support system (Woo 1999), in various elicitation protocols (Baddeley *et al.* 2004; Curtis & Wood 2004) and in the compilation of test datasets and questions to qualify experts in Cooke's method (Aspinall & Woo 1994).

To identify these relations between the statements or concepts formulated in natural language, Classical logic described above is necessary but not sufficient. Psychological, linguistic and other aspects should be accounted for in order to extract crisp sense from the loose and 'woolly' record in natural language, be it in text or speech.

It should be recalled herewith that logic in its modern form was developed only a few decades ago, while for centuries, since Aristotle, it existed in semi-intuitive, verbal form, perhaps with the single exception of syllogistic deduction. Until now the so-called 'traditional', or Aristotelean logic (e.g. http://plato.stanford.edu/entries/aristotle-logic/) occupied the first half of textbooks on logic for first-year students, serving as the introduction to the strict study of reasoning. It offers rules, quite clear by intuition, to define and classify concepts, to identify the relations between statements and to conclude from premises. However, these rules give unambiguous results only in a limited context (i.e. the number of involved entities – objects and/or properties – is finite). Hence, it should be accompanied by means of distinguishing data from knowledge and relevant knowledge from noise, recognition of linguistic ambiguities, extraction of knowledge from texts and speech and its compact presentation in various form (textual, graphical etc.) forms. All this is the virtue of the new field rapidly developing in recent years, *knowledge engineering* (thousands of publications every year and hundreds of thousands of references on the Internet (e.g. the website of Knowledge Engineering Review, http://titles.cambridge.org/journals/journal_catalogue.asp?mnemonic = ker; or a comprehensive, though rather old, review of Lukose 1996), and information technologies that utilize it (e.g. Loudon 2000; Smyth 2003; http://www.jiscmail.ac.uk/files/GEO-REASONING/dist.ppt). It incorporates a variety of tools united by goal rather than by methodology, ranging from psychological approaches to interviewing experts to statistical methods of extracting relevant words from texts or speech. The possibilistic or fuzzy logic approach advocated by Bardossy & Fodor (2001) is likely to be related to knowledge engineering when used to evaluate opinions. Virrantaus (2003) pointed out the necessity to involve knowledge engineering to decrease the uncertainty ('imprecision') of knowledge. The work on building ontologies for various fields of geoscience reported by Smyth (2003) is one of the first examples of this.

Likewise, the methods of expert judgement processing developed *ad hoc* to estimate subjective probability and support decision making can also be considered a part of knowledge engineering. In terms of the approach of Pshenichny (2003), they all fall into the field of methods enabling the passage from loose knowledge to strict theory and are naturally connected with logic. Therefore, a strategy is needed for efficient co-application of existing approaches of expert judgement processing, knowledge engineering and Classical logic to obtain the best estimation of epistemic uncertainty by possibilistic measures or by subjective probability where necessary, and to proceed to estimation of logical probability with time.

Addressing point (3), the passage from a loose knowledge to a structured finite domain, then to formal language and on to strict theories, is the optimal way to implement logic to estimate uncertainty. This may count as the answer to Question 3 at the beginning of this section. However, the outlined succession generates three successive sub-questions:

(1) How do you confine a context?
(2) How do you make a language from it?
(3) How do you create N theories in this language?

A serious problem in limiting geoscientific contexts is the involvement of diverse sources of knowledge, some of which are, in addition, largely subjective. Observation in many cases can hardly be distinguished from interpretation. Modelling, especially at a regional and planetary scale, often relies on a good portion of imagination. Putting together, say, a field description of a sandstone, equations from mechanics and dialectical laws widen the context rapidly. To limit it, it is pertinent, in the author's view, to avoid 'global' (philosophical, natural-scientific, physical, chemical and even regional geological) terms whenever possible. Rather, one should focus on a set of features furnished by the object in question and akin to objects, recognized at a given scale.

However, even if feasible, this is only part of a solution because interpretation of an outcrop, specimen or thin section unavoidably bears personal or common bias (see above). For instance, when describing a thin section, the optical constants of minerals are determined by what Frodeman calls, after Heidegger, the 'fore-havings' and 'fore-structures' – properties of the microscope and of the eye of the petrographer. While the parameters of the microscope are at least objective, and it can definitely be said that they have nothing to do with the mineral in question, what the eye sees is not only subjective, but we cannot even firmly decide whether it is relevant to the context or not. In a psycho-physiological sense (the ability of the eye to see and discern that depends on the eye proper and on the personality of the researcher), it should not be relevant to the nature of the object studied, though certainly is influential on the result of the study! Just as irrelevant is the capacity of the microscope, while in an epistemiological sense the eye is in fact a combination of 'eye and mind', by the locution of Merleau-Ponty quoted by Johnson (see reference in Frodeman 1995), and therefore is loaded with prior knowledge about what minerals are, and is obviously relevant. Hence, focusing on the set of features furnished by the object in question, one should abstract not only from too 'general' terms but also from the 'side' (technical, physiological) terms if possible, *or*, if these terms are crucial, involve them somehow in the context. The latter is again illustrated by Frodeman (1995, p. 962):

> If the energy crisis is defined as a problem of supply ('we need more oil'), we will find a different set of facts and a different range of possible solutions than if it is defined as a problem of demand ('we need to conserve').

This recalls the point made in the 'Sources and measures of uncertainty' section, that science has two alternatives: either to ignore bias or to accept, reveal, assess and manage, but not to decrease it. As is clear now, the latter case implies that bias is explicitly formulated and included in the context of consideration and, further, into the strict language and, ultimately, in the theories describing, for example, the (im)possibility of a new volcano forming in non-volcanic regions. This reflects the 'division of labour' acknowledged by Frodeman between Continental and Analytical philosophies in hermeneutic (interpretive) sciences like geology: like the humanities, geology is not bias-free, but like mathematics, it *should* be strict.

In the limited context where the number of involved entities (objects and/or properties) is finite, a strict language can be constructed. The main problem here, in the author's view, is minimization and strict formulation of terms, making them universal for expression of various standpoints.

For instance, a peculiar context of volcanology is the formation of rocks of transitional lava–pyroclastic outlook. In a few decades, a number of concepts about their formation were suggested. These include:

(1) pyroclastic flows (Smith 1960; Ross & Smith 1961; and others)
(2) lava flows (Abich 1882)
(3) two immiscible lavas in one flow (Steiner 1960)
(4) 'boiling' lava turning into foam (Boyd 1954; and others)
(5) hydrothermally altered lava (Naboko 1971 – see reference in Sheimovich 1979)
(6) products of subaqueous lava flows (Ivanov 1966 – see reference in Sheimovich 1979)
(7) product of redeposition of ash in lakes (Karolusova-Kočiščákova 1958 – see reference in Sheimovich 1979).

Despite the striking diversity of views, they form a united and long-lived context. In some periods, for example, in the 1980s to early 1990s, a particular concept (e.g. formation of all rocks of transitional outlook from pyroclastic flows) was popular (e.g. Fisher & Schmincke 1984). However, later some objects (Rooiberg Felsite in South Africa for instance) were reported that can hardly be explained by this concept (Twist & Elston 1989) as well as the cases of formation of patches of pyroclastic-looking material in the rocks confidently interpreted as lavas (Allen 1989; Fink 1989) and intrusive bodies filled with the material of the same outlook (Stasiuk *et al.* 1996). Still, these authors, contrary to early adepts of the 'effusive' concept, reconciled their points with the concept of pyroclastic flow. Another concept, that of rheomorphism, proclaimed by Smith (1960) and Ross & Smith (1961), was developed that actually showed a possibility of 'effusive' environment in a thick body of pyroclastic flow material (Milner *et al.* 1992; Streck & Grunder 1995) after, and (according to the most recent works) before and during the deposition (Branney & Kokelaar 2002). We cannot exclude the possibility of seemingly weird ideas of hydrothermal alteration of lava, subaqueous lava flows or redeposition of ash in lakes, reformulated in new terms, being revived and successfully used for interpretation of the same rocks in the future. Moreover, perhaps the maturity of a hermeneutic science might be understood as the moment when new concepts are no longer elaborated, and new data continue to support or slightly modify some of the previously suggested concepts, none of which can ever be totally rejected or totally adopted, so that further development of science is an endless 'championship' of concepts, in which no one wins the cup or leaves the league.

However, once the 'league' has been formed, the language should be able to describe the concepts from, and relationships between each 'player' – and to do that in as strict and concise form as possible in order to be translatable into predicate logic language. In this language, the standpoints are expected to become strict theories, but some of them, if they appear compliant, would merge to produce one theory, and new theories may emerge by automatic operation with variables and predicates. These 'artificial' theories should be examined by scientists and either adopted or rejected (herewith the 'aleatory component' may emerge again, as discussed above). Nevertheless, as the logical studies show, correct formulation of a large number of theories in a given language is a task requiring time and patience, with a high risk of human error, even for a simple case of Smullyan's puzzles (Moukhachov & Netchitailov 2001). At the same time, the degree of formality principally allows its complete automatization, and this work is actually ongoing (Moukhachov pers. comm.). Needless to say, for geoscience, such an automated 'theory generator' is perhaps the only way to make the assessment of logical probability practically achievable.

Summing up the discussion, logic is a theoretically better, but practically less feasible option than the existing methods of evaluating epistemic uncertainty, but in any case it can neither be a total substitute for these methods nor totally abolish the aleatory uncertainty. Therefore, a strategy is needed for efficient co-application of existing approaches of expert judgement processing, knowledge engineering and classical logic to assess epistemic uncertainty by possibilistic measures or by subjective probability where necessary, and to proceed to logical probability with time. This strategy should aim to confine the context of geoscientific study but this context may include, where relevant, the forestructures and bias. To advance the logical estimation of probability, an automated generator of logical tasks (theories) is a prerequisite.

Conclusions

(1) Based on the approach of Pshenichny (2003), uncertainty can be classified as being either that of knowledge or that of data. The uncertainty of data might be considered: (a) in terms of additivity and measured by probability in the frequentist sense, (b) in fuzzy terms and measured by a variety of fuzzy parameters, or (c) jointly. Reasoning and language contribute to the uncertainty of knowledge, or epistemic uncertainty, and its refinement, aleatory uncertainty. As for the measure of epistemic uncertainty, if the approach to knowledge is intuitive, it may only be fuzzy; if the approach is to treat knowledge objectively, in the tradition of analytical philosophy, this measure, if it exists, should be probabilistic in a logical but not in a subjective sense.

(2) The main virtue of logic is the relation of logical inference, or deducibility. Logic *sensu stricto* consists of two major parts: propositional logic and predicate logic. Propositional logic is a simple tool, useful to introduce the logical calculi that actualize the relationship of inference in certain formal procedures. However, it is insufficient alone to process knowledge. Predicate logic incorporates all of the laws of propositional logic but offers specific means for construction of strict theories, which have been tested successfully in exact and experimental sciences. Consideration of inference of a similar statement in a set of strict theories composed in similar language opens the opportunity to define and calculate logical probability, which, being a measure of epistemic uncertainty, meets the general requirements of probability and, in particular, fits well with the Bayesian approach.

(3) Logic is a theoretically better but practically less feasible option than the existing methods of evaluating epistemic uncertainty, but in any case it can neither be a total substitute for these methods, nor totally abolish the aleatory uncertainty.

(4) Urgent tasks for future research include, firstly, a strategy for co-application of existing approaches of expert judgement processing, knowledge engineering and classical logic, able to incorporate prejudgements and bias if necessary, and secondly, an automated generator of logical tasks (strict theories) in a given language.

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