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Inversion for elastic parameters in weakly anisotropic media

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SUMMARY

Linearized inversion equations of quasi *P* waves (*qP*) and quasi *S* waves (*qS*) in arbitrary weakly anisotropic media are presented. The equations can be used for local determination of weak anisotropy (WA) parameters in inhomogeneous media. The equations for *qS* waves hold not only in regular directions but also in singularities. In the case of *qS*-wave coupling, the first arrival of *qS* waves can be used alone for inversion. We obtain three equations for each wave expressed in terms of WA parameters. One of the equations relates the WA parameters to slowness and the other two relate the WA parameters to polarization vectors. By eliminating one or two components of the slowness vector from the three equations, we get a new equation system which enables us to perform inversion when two (or one) components of the slowness vector are known. Since a coordinate system is selected to be identical to the polarization vectors, the inversion formulae for *qS* waves become linear ones. A synthetic multi-azimuthal multiple-source walkaway vertical seismic profile experiment for an orthorhombic inhomogeneous medium with tilted symmetry axes is carried out to illustrate the applicability and efficiency of the method.

Key words: inversion, polarization vector, *qP* wave, *qS* waves, slowness vector, weak anisotropy.

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1 INTRODUCTION

Elastic waves propagating in an anisotropic medium satisfy the Christoffel equation. The phase velocities and polarization vectors corresponding to the eigenvalues and eigenvectors of the Christoffel matrix depend on the propagation direction. In general, the polarization vectors are neither parallel nor perpendicular to the wave normal. For a few simple symmetries, such as a transversely isotropic medium, we have analytic expressions for the eigenvalues and eigenvectors and thus for phase velocities and polarization vectors. Based on the observation that most anisotropic media are only weakly anisotropic, Backus (1965) first derived an approximate equation for the phase velocity of a *qP* wave in arbitrary weakly anisotropic media. Thomsen (1986) presented angle-dependent formulae for wave velocities appropriate for weak anisotropy by using anisotropy parameters. Sayers (1994), Mensch & Rasolofosaon (1997) and Pšenčík & Gajewski (1998) obtained approximate equations for the phase velocity of a *qP* wave in an arbitrary weakly anisotropic medium. The advantage of these approximations for a *qP* wave is that they are linear in terms of elastic parameters. But for *qS* waves, the approximate equations have a non-linear form (Jech & Pšenčík 1989) and it is difficult to apply them to inverse problems. In the cases of *qS*-wave singularities and coupling, the standard zeroth-order ray theory fails. Modifications of the ray theory such as the coupling theory (Chapman & Shearer 1989; Coates & Chapman 1990) or the quasi-isotropic approximation of the ray theory (Pšenčík 1998; Zillmer *et al.* 1998) have been used to reproduce *qS* waves correctly. Vavryčuk (1999) showed that the failure of the zeroth-order ray theory does not necessarily implicate a failure of the high-order ray theory or the ray theory at all. He showed that *qS*-wave coupling, ignored by the zeroth-order ray approximation, can be reproduced well by the high-order ray approximations.

To derive the approximate equations for *qP* and *qS* waves, one often uses the perturbation method. Perturbation of the Christoffel matrix can be caused either by perturbation of medium parameters or by perturbation of the wave normal. Jech & Pšenčík (1989) mainly concentrated on the perturbation caused by perturbation of medium parameters. In this paper we use perturbation theory to derive approximate equations for slowness and polarization vectors of *qP* and *qS* waves. Not only is the perturbation of medium parameters considered, but also the perturbation of the wave normal. The medium is assumed to be weakly anisotropic, but of arbitrary symmetry. We obtain three basic inversion equations which linearly depend on the 21 weak anisotropy parameters for the *qP* wave and the two *qS* waves, respectively. From the three equations, one or two components of the slowness vector can be eliminated. These equations can be used to retrieve weak anisotropy parameters locally at the target point surrounded by an inhomogeneous anisotropic medium. The equations for *qS* waves hold not only in regular directions but also in singularities. In the case of *qS*-wave coupling, the first arrival can be incorporated into the inversion. The use of *qS* waves together with a *qP* wave provides much more information than the use of a qP wave alone. It enables us to retrieve all 21 weak anisotropy parameters for a general anisotropic medium. We carry out a synthetic multi-azimuthal multiple-source walkaway vertical seismic profile (VSP) experiment in which a vertical inhomogeneous orthorhombic (ORT) medium with tilted symmetry axes is used. The medium contains singularities for the synthetic experiment, which generates qS-wave coupling. The results from the synthetic experiment demonstrate that the method is applicable and is capable of dealing with such complicated problems. We use the component notation of vectors and matrices. Einstein summation convention is applied to repeated indices.

2 CHRISTOFFEL EQUATION AND PERTURBATION METHOD

The basic equation in plane wave theory or ray theory in isotropic and anisotropic media is the Christoffel equation (Červený 2001)

$$(\Gamma_{jk} - G_m \delta_{jk})\mathbf{g}_j^{(m)} = 0, \tag{1}$$

where $\Gamma_{jk} = \boldsymbol{a}_{ijkl}\mathbf{p}_i\mathbf{p}_l$ is the Christoffel matrix, $\boldsymbol{a}_{ijkl}$ is density-normalized elastic tensor, $\mathbf{p}_i = \mathbf{n}_i/v_m$ is a slowness vector, $\mathbf{n}_i$ is the wave normal and v_m is the phase velocity. The symbol G_m ($m = 1, 2, 3$) represents three eigenvalues of the Christoffel matrix. Each m corresponds to one of the three waves propagating in anisotropic media, $m = 1, 2$ for qS_1 and qS_2 waves and $m = 3$ for the qP wave. $\mathbf{g}_i^{(m)}$ ($m = 1, 2, 3$) represents polarization vectors of the corresponding waves (no summation over m).

In the following, we use an alternative form of the Christoffel eq. (1):

$$(\bar{\Gamma}_{jk} - \bar{G}_m \delta_{jk})\mathbf{g}_j^{(m)} = 0, \tag{2}$$

where $\bar{\Gamma}_{jk} = \boldsymbol{a}_{ijkl}\mathbf{n}_i\mathbf{n}_l$, $\bar{G}_m = v_m^2$ is the square of phase velocity.

Because many real materials are weakly anisotropic (Thomsen 1986), we are therefore motivated to use perturbation theory to simplify the equations governing wave propagation in anisotropic media. Now we consider a weakly anisotropic medium and take an isotropic medium as a reference one. The elastic parameters of a weakly anisotropic medium can be expressed as the sum of elastic parameters of a reference isotropic medium $\boldsymbol{a}_{ijkl}^0$ and their perturbations $\Delta\boldsymbol{a}_{ijkl}$:

$$\boldsymbol{a}_{ijkl} = \boldsymbol{a}_{ijkl}^0 + \Delta\boldsymbol{a}_{ijkl}. \tag{3}$$

In the reference isotropic medium, we introduce three mutually perpendicular unit vectors $\mathbf{e}_i^{(1)}$, $\mathbf{e}_i^{(2)}$ and $\mathbf{e}_i^{(3)}$ so that the vector $\mathbf{e}_i^{(3)}$ is identical with the wave normal $\mathbf{n}_i$. The vector $\mathbf{n}_i$ is also parallel to the polarization vector of the P wave since the wave normal and the polarization vector of the P wave are identical in an isotropic medium. The vectors $\mathbf{e}_i^{(1)}$ and $\mathbf{e}_i^{(2)}$ can be chosen arbitrarily in the plane perpendicular to $\mathbf{n}_i$. A practical choice of vectors $\mathbf{e}_i^{(1)}$ and $\mathbf{e}_i^{(2)}$ expressed in terms of components of the vector $\mathbf{e}_i^{(3)}$ is as follows (Pšenčík & Gajewski 1998)

$$\mathbf{e}^{(1)} = D^{-1}\left(n_1 n_3, n_2 n_3, {n_3}^2 - 1\right), \quad \mathbf{e}^{(2)} = D^{-1}(-n_2, n_1, 0), \quad \mathbf{e}^{(3)} = \mathbf{n} = (n_1, n_2, n_3), \tag{4}$$

where

$$D = \left({n_1}^2 + {n_2}^2\right)^{1/2}, \qquad {n_1}^2 + {n_2}^2 + {n_3}^2 = 1. \tag{5}$$

If the wave normal is specified as

$$\mathbf{n} = (\cos\varphi \sin\vartheta, \sin\varphi \sin\vartheta, \cos\vartheta), \tag{6}$$

where φ denotes an azimuthal angle and ϑ a polar angle ($0 \le \varphi \le 2\pi$, $0 \le \vartheta \le \pi$), then $D = \sin\vartheta$ and vectors $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$ can be rewritten in the form

$$\mathbf{e}^{(1)} = (\cos\varphi \cos\vartheta, \sin\varphi \cos\vartheta, -\sin\vartheta), \qquad \mathbf{e}^{(2)} = (-\sin\varphi, \cos\varphi, 0). \tag{7}$$

For simplicity, we use eqs (4) in terms of components of the wave normal in deriving the following formulae. But in the computer programs, specifications (6) and (7) are used. The latter choice avoids artificial problems when D equals zero or approximately equals zero in eq. (4).

The slowness vector $\mathbf{p}_i$ in a weakly anisotropic medium can be expressed as

$$\mathbf{p}_i = \mathbf{p}_i^0 + \Delta\mathbf{p}_i \tag{8}$$

or

$$\mathbf{p}_i = \mathbf{p}_i^0 + \Delta\xi i_i + \Delta\zeta j_i + \Delta\eta k_i = (\xi + \Delta\xi)i_i + (\zeta + \Delta\zeta)j_i + (\eta + \Delta\eta)k_i, \tag{9}$$

where $\mathbf{p}_i^0$ is a slowness vector in the reference isotropic medium and $\Delta\mathbf{p}_i$ is its perturbation. i_i, j_i and k_i represent the ith components of basic vectors along axes x, y and z respectively. ξ, ζ and η denote projections of the slowness vector $\mathbf{p}_i^0$ onto i_i, j_i and k_i, respectively. $\Delta\xi$, $\Delta\zeta$ and $\Delta\eta$ denote perturbations of the slowness vector $\mathbf{p}_i^0$.

The slowness vector $\mathbf{p}_i^0$ in the reference isotropic medium has the form

$$\mathbf{p}_i^0 = v_0^{-1}\mathbf{n}_i \tag{10}$$

and its components (see eq. 9) are

$$\xi = \frac{n_1}{v_0}, \quad \zeta = \frac{n_2}{v_0}, \quad \eta = \frac{n_3}{v_0}, \tag{11}$$

where $v_0 = \alpha$ is the P-wave velocity and $v_0 = \beta$ is the S-wave velocity.

If the second-order terms of $\Delta\xi$, $\Delta\zeta$ and $\Delta\eta$ are neglected, we get from eq. (9) the first-order approximation of the square of slowness of qP and qS waves

$$v^{-2} = \mathbf{p}_i\mathbf{p}_i = v_0^{-2}\left(1 + 2v_0^2\xi\Delta\xi + 2v_0^2\zeta\Delta\zeta + 2v_0^2\eta\Delta\eta\right). \tag{12}$$

The polarization vector $\mathbf{g}_i^{(m)}$ in a weakly anisotropic medium can be expressed as

$$\mathbf{g}_i^{(m)} = \mathbf{g}_i^{(m)0} + \Delta\mathbf{g}_i^{(m)}, \tag{13}$$

where $\mathbf{g}_i^{(m)0}$ is a polarization vector in the reference isotropic medium and $\Delta\mathbf{g}_i^{(m)}$ is its perturbation.

$\bar{\Gamma}_{jk}$ can be expressed as

$$\bar{\Gamma}_{jk} = \bar{\Gamma}_{jk}^0 + \Delta\bar{\Gamma}_{jk}, \tag{14}$$

where $\bar{\Gamma}_{jk}^0 = \boldsymbol{a}_{ijkl}^0\mathbf{n}_i\mathbf{n}_l$. Perturbation of the matrix $\bar{\boldsymbol{\Gamma}}_{jk}$ is generally caused by perturbation $\Delta\boldsymbol{a}_{ijkl}$ of the medium parameters and/or by perturbation $\Delta\mathbf{n}_i$ of the wave normal

$$\Delta\bar{\Gamma}_{jk} = \Delta\boldsymbol{a}_{ijkl}\mathbf{n}_i\mathbf{n}_l + \boldsymbol{a}_{ijkl}^0(\mathbf{n}_l\Delta\mathbf{n}_i + \mathbf{n}_i\Delta\mathbf{n}_l). \tag{15}$$

$\bar{G}_m$ can be expressed as

$$\bar{G}_m = \bar{G}_m^0 + \Delta\bar{G}_m. \tag{16}$$

Here $\bar{G}_m^0 = \alpha^2$ or β^2. $\Delta\bar{G}_m$ is the perturbation of the eigenvalue (of the square of phase velocity) $\bar{G}_m^0$.

Now we insert eqs (13), (14) and (16) into eq. (2), then the Christoffel equation yields

$$\left(\bar{\Gamma}_{jk}^0 - \bar{G}_m^0\delta_{jk}\right)\Delta\mathbf{g}_j^{(m)} + (\Delta\bar{\Gamma}_{jk} - \Delta\bar{G}_m\delta_{jk})\mathbf{g}_j^{(m)0} + (\Delta\bar{\Gamma}_{jk} - \Delta\bar{G}_m\delta_{jk})\Delta\mathbf{g}_j^{(m)} = 0. \tag{17}$$

3 FORWARD FORMULAE FOR *qP* WAVES

First, we consider a qP wave ($m = 3$) in eq. (17) and take vectors $\mathbf{g}_i^{(1)0}$, $\mathbf{g}_i^{(2)0}$ and $\mathbf{g}_i^{(3)0}$ to be identical with vectors $\mathbf{e}_i^{(1)}$, $\mathbf{e}_i^{(2)}$ and $\mathbf{e}_i^{(3)}$, respectively. From the requirement that $\mathbf{g}_i^{(3)}$ is a unit vector, we have

$$\mathbf{g}_i^{(3)0}\Delta\mathbf{g}_i^{(3)} = 0, \tag{18}$$

which means

$$\Delta\mathbf{g}_i^{(3)} = c_1\mathbf{g}_i^{(1)0} + c_2\mathbf{g}_i^{(2)0}, \tag{19}$$

where c_1 and c_2 are two coefficients which will be determined later.

Inserting eq. (19) into eq. (17) and keeping the first-order terms, we obtain

$$\left(\bar{\Gamma}_{jk}^0 - \bar{G}_3^0\delta_{jk}\right)\left(c_1\mathbf{g}_j^{(1)0} + c_2\mathbf{g}_j^{(2)0}\right) + (\Delta\bar{\Gamma}_{jk} - \Delta\bar{G}_3\delta_{jk})\mathbf{g}_j^{(3)0} = 0. \tag{20}$$

3.1 Slowness formula

Multiplying eq. (20) by $\mathbf{g}_k^{(3)0}$, we obtain

$$\Delta\bar{G}_3 = B_{33}, \tag{21}$$

where $\Delta\bar{G}_3$ is the perturbation of the square of the P-wave velocity. The symbol B_{33} denotes an element of the matrix of weak anisotropy

$$B_{mn} = \Delta\boldsymbol{a}_{ijkl}\mathbf{e}_i^{(m)}\mathbf{n}_j\mathbf{n}_k\mathbf{e}_l^{(n)}. \tag{22}$$

The explicit expressions for the elements of the weak anisotropy matrix B_{mn} are given in Appendix A (Zheng & Pšenčík 2002; Farra & Pšenčík 2003).

The square of the phase velocity of a qP wave in a weakly anisotropic medium v_3^2 can thus be written as

$$v_3^2 \sim \alpha^2 + \Delta\bar{G}_3 = \alpha^2 + B_{33}. \tag{23}$$

From eq. (23) we get

$$v_3^{-2} \sim \alpha^{-2}\left(1 - \frac{B_{33}}{\alpha^2}\right). \tag{24}$$

Eq. (24) is a formula for the square of the slowness of the qP wave in a weakly anisotropic medium (Pšenčík & Gajewski 1998; Pšenčík & Zheng 1998; Zheng & Pšenčík 2002).

3.2 Polarization formula

Multiplying eq. (20) by $\mathbf{g}_k^{(K)0}$ ($K = 1, 2$), and taking into account eq. (15), we obtain the coefficients c_K

$$c_K = \alpha\mathbf{p}_j\mathbf{e}_j^{(K)} + \frac{B_{K3}}{\alpha^2 - \beta^2}. \tag{25}$$

Combining eqs (13), (19) and (25) yields,

$$\mathbf{g}_i^{(3)} = \mathbf{n}_i + \alpha\left(\mathbf{p}_j \mathbf{e}_j^{(K)}\right)\mathbf{e}_i^{(K)} + \frac{B_{K3}}{\alpha^2 - \beta^2}\mathbf{e}_i^{(K)}. \tag{26}$$

Eq. (26) is a formula for the polarization vector of a qP wave in a weakly anisotropic medium (Pšenčík & Zheng 1998; Zheng & Pšenčík 2002). The second term on the right-hand side (RHS) of eq. (26) represents a correction of the slowness vector from its orientation in the reference isotropic medium (Pšenčík 1998). This correction is due to the perturbation of the wave front, which, in turn, is due to the perturbation $\Delta \boldsymbol{a}_{ijkl}$. The third term on the RHS of eq. (26) represents deviation of the polarization vector of the qP wave from the corresponding slowness vector (Jech & Pšenčík 1989; Pšenčík 1998).

The two terms mentioned above lead to deviation of the polarization vector from the direction $\mathbf{e}_i^{(3)} = \mathbf{n}_i$ of the polarization vector in the reference isotropic medium. The two terms in eq. (26) remain non-zero even in homogeneous anisotropic media.

4 FORWARD FORMULAE FOR qS WAVES

4.1 Slowness formulae

When an isotropic medium is taken as a reference medium, the perturbation procedure for qS and qP waves is different (see Jech & Pšenčík 1989; Červený 2001). For qS waves, we only know that $\mathbf{g}_i^{(1)0}$ and $\mathbf{g}_i^{(2)0}$ are mutually perpendicular and that they are situated in a plane perpendicular to $\mathbf{g}_i^{(3)0}$. Therefore, $\mathbf{g}_i^{(1)0}$ and $\mathbf{g}_i^{(2)0}$ can be expressed as

$$\mathbf{g}_i^{(1)0} = \cos\psi_1 \mathbf{e}_i^{(1)} + \sin\psi_1 \mathbf{e}_i^{(2)}, \tag{27a}$$

$$\mathbf{g}_i^{(2)0} = -\sin\psi_1 \mathbf{e}_i^{(1)} + \cos\psi_1 \mathbf{e}_i^{(2)}, \tag{27b}$$

where ψ_1 is an angle between $\mathbf{g}_i^{(1)0}$ and $\mathbf{e}_i^{(1)}$, which will be determined later.

Let us consider wave qS_1 ($m = 1$) in eq. (17) and keep the first-order terms, we then have

$$(\bar{\Gamma}_{jk}^0 - \bar{G}_1^0 \delta_{jk})\Delta\mathbf{g}_j^{(1)} + (\Delta\bar{\Gamma}_{jk} - \Delta\bar{G}_1 \delta_{jk})\mathbf{g}_j^{(1)0} = 0. \tag{28}$$

Multiplying eq. (28) by $\mathbf{g}_k^{(2)0}$ yields

$$\bar{B}_{12} = 0, \tag{29}$$

where $\bar{B}_{12}$ is an element of the matrix $\bar{B}_{mn}$

$$\bar{B}_{mn} = \Delta\boldsymbol{a}_{ijkl}\mathbf{g}_i^{(m)0}\mathbf{n}_j\mathbf{n}_k\mathbf{g}_l^{(n)0}. \tag{30}$$

The relationship between B_{mn} and $\bar{B}_{mn}$ is given in Appendix B (Pšenčík & Vavryčuk 2002).

Combining eqs (29) and (B2), we get

$$\tan 2\psi_1 = \frac{2B_{12}}{B_{11} - B_{22}}, \tag{31}$$

from which we obtain the angle ψ_1. Therefore, the polarization vectors $\mathbf{g}_i^{(1)0}$ and $\mathbf{g}_i^{(2)0}$ in the reference isotropic medium can be determined.

Multiplying eq. (28) by $\mathbf{g}_k^{(1)0}$, we get

$$\Delta\bar{G}_1 = \bar{B}_{11}, \tag{32}$$

where $\Delta\bar{G}_1$ is the perturbation of the square of the S-wave velocity. $\bar{B}_{11}$ is an element of the matrix $\bar{B}_{mn}$.

The square of the phase velocity of a qS_1 wave in a weakly anisotropic medium v_1^2 can thus be written as

$$v_1^2 \sim \beta^2 + \Delta\bar{G}_1 = \beta^2 + \bar{B}_{11}. \tag{33}$$

From eqs (33) and (B1), we have

$$v_1^{-2} \sim \beta^{-2}\left[1 - \beta^{-2}\left(\cos^2\psi_1 B_{11} + \sin^2\psi_1 B_{22} + 2\sin\psi_1\cos\psi_1 B_{12}\right)\right]. \tag{34}$$

Eq. (34) is the first-order approximation formula for the square of slowness of a qS_1 wave in weakly anisotropic media. Because $\psi_1 = \psi_1$ (B_{mn}), eq. (34) is a non-linear function of weak anisotropy parameters.

For a qS_2 wave, if we take $m = 2$ in eq. (17) and keep the first-order terms, we can get

$$v_2^{-2} \sim \beta^{-2}\left[1 - \beta^{-2}\left(\cos^2\psi_2 B_{11} + \sin^2\psi_2 B_{22} + 2\sin\psi_2\cos\psi_2 B_{12}\right)\right], \tag{35}$$

where v_2 is the phase velocity of the qS_2 wave and $\psi_2 = \psi_1 + \frac{\pi}{2}$ is the angle between $\mathbf{g}_i^{(2)0}$ and $\mathbf{e}_i^{(1)}$.

4.2 Polarization formulae

Because the polarization vectors $\mathbf{g}_i^{(1)0}$ and $\mathbf{g}_i^{(2)0}$ in the reference isotropic medium are known (see eqs 27 and 31), perturbation of the polarization vector $\mathbf{g}_i^{(1)0}$ can be expressed as

$$\Delta\mathbf{g}_i^{(1)} = c_3\mathbf{g}_i^{(2)0} + c_4\mathbf{g}_i^{(3)0}, \tag{36}$$

where c_3 and c_4 are two constants.

Inserting eq. (36) into eq. (28) and multiplying the result by $\mathbf{g}_k^{(3)0}$ and then considering eq. (15), we obtain the parameter c_4:

$$c_4 = -\beta \mathbf{p}_j \mathbf{g}_j^{(1)0} - \frac{\bar{B}_{13}}{\alpha^2 - \beta^2}. \tag{37}$$

Inserting eq. (36) into eq. (17) and multiplying it by $\mathbf{g}_k^{(2)0}$, we obtain

$$c_3(\bar{B}_{22} - \bar{B}_{11}) + c_4(\alpha^2 - \beta^2)\left(\beta \mathbf{p}_j \mathbf{g}_j^{(2)0} + \frac{\bar{B}_{23}}{\alpha^2 - \beta^2}\right) = 0. \tag{38}$$

Substituting eq. (37) into eq. (38) yields

$$c_3 = \frac{\alpha^2 - \beta^2}{\bar{B}_{22} - \bar{B}_{11}} \left(\beta \mathbf{p}_j \mathbf{g}_j^{(1)0} + \frac{\bar{B}_{13}}{\alpha^2 - \beta^2}\right) \left(\beta \mathbf{p}_j \mathbf{g}_j^{(2)0} + \frac{\bar{B}_{23}}{\alpha^2 - \beta^2}\right). \tag{39}$$

Combining eqs (36), (37) and (39), we obtain

$$\Delta \mathbf{g}_i^{(1)} = \frac{\alpha^2 - \beta^2}{\bar{B}_{22} - \bar{B}_{11}} \left(\beta \mathbf{p}_j \mathbf{g}_j^{(1)0} + \frac{\bar{B}_{13}}{\alpha^2 - \beta^2}\right) \left(\beta \mathbf{p}_j \mathbf{g}_j^{(2)0} + \frac{\bar{B}_{23}}{\alpha^2 - \beta^2}\right) \mathbf{g}_i^{(2)0} - \left(\beta \mathbf{p}_j \mathbf{g}_j^{(1)0} + \frac{\bar{B}_{13}}{\alpha^2 - \beta^2}\right) \mathbf{g}_i^{(3)0}. \tag{40}$$

The perturbation of polarization vector $\mathbf{g}_i^{(1)0}$ in weakly anisotropic media is a non-linear function of weak anisotropy parameters. Note that we obtain eq. (40) by taking eq. (29) into consideration.

For wave qS_2 we can obtain the perturbation $\Delta \mathbf{g}_i^{(2)}$ of polarization vector $\mathbf{g}_i^{(2)0}$

$$\Delta \mathbf{g}_i^{(2)} = \frac{\alpha^2 - \beta^2}{\bar{B}_{11} - \bar{B}_{22}} \left(\beta \mathbf{p}_j \mathbf{g}_j^{(2)0} + \frac{\bar{B}_{23}}{\alpha^2 - \beta^2}\right) \left(\beta \mathbf{p}_j \mathbf{g}_j^{(1)0} + \frac{\bar{B}_{13}}{\alpha^2 - \beta^2}\right) \mathbf{g}_i^{(1)0} - \left(\beta \mathbf{p}_j \mathbf{g}_j^{(2)0} + \frac{\bar{B}_{23}}{\alpha^2 - \beta^2}\right) \mathbf{g}_i^{(3)0}. \tag{41}$$

In the degenerate case, the two qS waves have the same velocity, $\bar{B}_{11} = \bar{B}_{22}$ and $\bar{B}_{12} = 0$, and ψ_1 in eq. (31) becomes undefined. The polarization formulae (40) and (41) are no longer valid. The polarization vectors of the two qS waves coincide and can be any unit vector in the plane perpendicular to $\mathbf{g}_i^{(3)}$.

5 INVERSION FORMULAE FOR *qP* AND *qS* WAVES

The formulae derived from the perturbation theory in Sections 3 and 4 are first-order approximations, which can be used to determine approximately the slowness and polarization vectors of qP and qS waves in weakly anisotropic media. The formulae can, however, also be used in a reverse way. When the polarization and slowness vectors are known, the formulae can be used for determining the weak anisotropy and elastic parameters of the medium. The formulae for such an inversion in general weakly anisotropic media are given in this section.

5.1 *qP* wave

Multiplying eq. (26) by $\mathbf{e}_i^{(K)}$ and using eq. (9), we get

$$B_{K3} = (\alpha^2 - \beta^2)\left(\mathbf{g}_j^{(3)} \mathbf{e}_j^{(K)} - \alpha \Delta\xi \mathbf{e}_1^{(K)} - \alpha \Delta\zeta \mathbf{e}_2^{(K)} - \alpha \Delta\eta \mathbf{e}_3^{(K)}\right), \tag{42}$$

where $K = 1, 2$.

Combining eqs (24) and (12) yields

$$B_{33} = -2\alpha^4 \xi \Delta\xi - 2\alpha^4 \zeta \Delta\zeta - 2\alpha^4 \eta \Delta\eta. \tag{43}$$

Eqs (42) and (43) are linear equations for 15 unknown weak anisotropy parameters (see eq. A7). The weak anisotropy parameters enter eqs (42) and (43) through the elements of the weak anisotropy matrix B_{mn} given in eqs (A1)–(A6) in Appendix A. If all three components, i.e. $\Delta\xi$, $\Delta\zeta$ and $\Delta\eta$, of the slowness vector are known, eqs (42) and (43) can be used to obtain the 15 weak anisotropy parameters by inversion.

If only two components of the slowness vector are known, say p_1 and p_3 determined from observation, and p_2 and thus the perturbation $\Delta\zeta$ is unknown, for example in the walkaway VSP with two neighbouring boreholes (White *et al.* 1983), we can obtain the equations for inversion by eliminating $\Delta\zeta$ from eqs (42) and (43). In such a way, we get

$$D(\alpha^2 - \beta^2)^{-1} B_{13} - \frac{1}{2}\alpha^{-1} \eta B_{33} = D \mathbf{g}_j^{(3)} \mathbf{e}_j^{(1)} + \alpha \Delta\eta, \tag{44}$$

$$D(\alpha^2 - \beta^2)^{-1} \zeta B_{23} - \frac{1}{2}\alpha^{-2} \xi B_{33} = D \zeta \mathbf{g}_j^{(3)} \mathbf{e}_j^{(2)} + D^2 \Delta\xi + \alpha^2 \xi \eta \Delta\eta. \tag{45}$$

When only one component of the slowness vector is known, which is the most common case, e.g. when only the deviation $\Delta\eta$ is determined from the observation, and the perturbations $\Delta\xi$ and $\Delta\zeta$ are unknown, we obtain the same equation as eq. (44) by eliminating $\Delta\xi$ and $\Delta\zeta$ from eqs (42) and (43). Eq. (44) is a simplification of eq. (22) of Zheng & Pšenčík (2002).

5.2 *qS* waves

In measurements, for example in walkaway VSP, the polarization vectors of qP and qS waves are observed quantities and can be determined by three-component seismograms. Let us take $\mathbf{g}_i^{(1)}$ identical to $\mathbf{g}_i^{(1)0}$, $\mathbf{g}_i^{(2)}$ identical to $\mathbf{g}_i^{(2)0}$, and $\mathbf{g}_i^{(3)0}$ perpendicular to the plane ($\mathbf{g}_i^{(1)0}$, $\mathbf{g}_i^{(2)0}$). In such a coordinate system, perturbations of $\mathbf{g}_i^{(1)0}$ and $\mathbf{g}_i^{(2)0}$ are zero. Since both qS wave normals are identical, from eqs (40) and (41), we can obtain the following equations:

$$\frac{\bar{B}_{13}}{\alpha^2-\beta^2} = -\beta \mathbf{p}_j \mathbf{g}_j^{(1)0}, \tag{46}$$

$$\frac{\bar{B}_{23}}{\alpha^2-\beta^2} = -\beta \mathbf{p}_j \mathbf{g}_j^{(2)0}. \tag{47}$$

Eqs (46) and (47) describe the deviation of qS wave normals in weakly anisotropic medium from the S-wave normal in the reference isotropic medium.

Inserting eqs (B4) and (9) into eq. (46) yields

$$\cos\psi_1 B_{13} + \sin\psi_1 B_{23} = -\beta(\alpha^2-\beta^2)\left(\mathbf{g}_1^{(1)0}\Delta\xi + \mathbf{g}_2^{(1)0}\Delta\zeta + \mathbf{g}_3^{(1)0}\Delta\eta\right). \tag{48}$$

Inserting eqs (B5) and (9) into eq. (47) yields

$$-\sin\psi_1 B_{13} + \cos\psi_1 B_{23} = -\beta(\alpha^2-\beta^2)\left(\mathbf{g}_1^{(2)0}\Delta\xi + \mathbf{g}_2^{(2)0}\Delta\zeta + \mathbf{g}_3^{(2)0}\Delta\eta\right). \tag{49}$$

Combining eqs (34) and (12), we get

$$\cos^2\psi_1 B_{11} + \sin^2\psi_1 B_{22} + 2\sin\psi_1\cos\psi_1 B_{12} = -2\beta^4(\xi\Delta\xi + \zeta\Delta\zeta + \eta\Delta\eta). \tag{50}$$

Here ψ_1 is the angle between $\mathbf{g}_i^{(1)}$ and $\mathbf{e}_i^{(1)}$. Eqs (48)–(50) are linear functions of weakly anisotropic parameters. They can be used for the inversion of the qS_1 wave if three components of the slowness vector are known.

When only two components of the slowness vector are known, i.e. $\Delta\xi$ and $\Delta\eta$ are determined from observations, and the perturbation $\Delta\zeta$ is unknown, we can obtain the equations for inversion by eliminating $\Delta\zeta$ in eqs (48)–(50). Thus we obtain

$$\beta^{-1}\left(\zeta\eta\cos\psi_1 + \beta^{-1}\xi\sin\psi_1\right)\left(\frac{1}{2}\cos^2\psi_1 B_{11} + \frac{1}{2}\sin^2\psi_1 B_{22} + \sin\psi_1\cos\psi_1 B_{12}\right) - D\zeta(\alpha^2-\beta^2)^{-1}(\cos\psi_1 B_{13} + \sin\psi_1 B_{23})$$
$$= -D^2\sin\psi_1\Delta\xi - \beta(\zeta\cos\psi_1 + \beta\xi\eta\sin\psi_1)\Delta\eta, \tag{51}$$

$$\beta^{-1}\left(\zeta\eta\sin\psi_1 - \beta^{-1}\xi\cos\psi_1\right)\left(\frac{1}{2}\cos^2\psi_1 B_{11} + \frac{1}{2}\sin^2\psi_1 B_{22} + \sin\psi_1\cos\psi_1 B_{12}\right) - D\zeta(\alpha^2-\beta^2)^{-1}(\sin\psi_1 B_{13} - \cos\psi_1 B_{23})$$
$$= D^2\cos\psi_1\Delta\xi - \beta(\zeta\sin\psi_1 - \beta\xi\eta\cos\psi_1)\Delta\eta. \tag{52}$$

When only one component of the slowness vector is known, i.e. $\Delta\eta$ is determined from observation, and the perturbations $\Delta\xi$ and $\Delta\zeta$ are unknown, we can obtain the equation for inversion by eliminating $\Delta\xi$ and $\Delta\zeta$ in eqs (48)–(50):

$$\eta\beta^{-1}\left(\frac{1}{2}\cos^2\psi_1 B_{11} + \frac{1}{2}\sin^2\psi_1 B_{22} + \sin\psi_1\cos\psi_1 B_{12}\right) - D\frac{B_{13}}{\alpha^2-\beta^2} = -\beta\Delta\eta. \tag{53}$$

In the same way as for the qS_1 wave, we can obtain similar formulae for the qS_2 wave. The difference between forward and inverse problems is that the polarization vectors in an inverse problem are observed quantities and they are determined by three-component seismograms. When polarization vectors are known, we can build a coordinate system in which the equations of qS waves are linear functions of weak anisotropy parameters. $\sin\psi_1$ and $\cos\psi_1$ in the equations are constants.

6 SYNTHETIC EXPERIMENT

The inversion formulae for the qP wave have been tested by a synthetic multi-azimuthal multiple-source walkaway VSP experiment and by real data (Zheng & Pšenčík 2002; Zheng 2003; Gomes *et al.* 2004). In this section we shall use a similar configuration (Fig. 1) of the

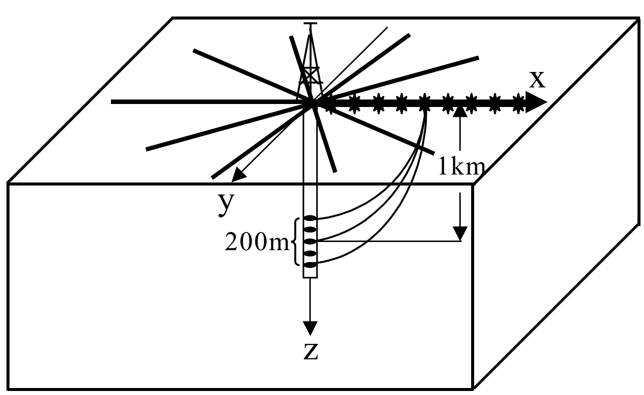

Figure 1. VSP configuration. Thirty-two shots at each side of the borehole, starting at 0.1 km with an interval 0.05 km. Five receivers in the borehole and separated by 0.05 km. The top receiver is at a depth of 0.9 km and the borehole at the centre of the box.

synthetic walkaway VSP experiment to deal with both *qP* and *qS* waves together. The medium in the experiment is a vertical inhomogeneous orthorhombic medium with tilted symmetry axes. The quantities used for inversion are slowness and polarization vectors that can be obtained by analysis of the three-component synthetic seismograms. The inverted parameters consist of 21 weak anisotropy parameters which linearly depend on 21 elastic parameters for a general weakly anisotropic medium. In the coupling case of the two *qS* waves only the first arrival is considered, and information from the polarization vector is indispensable for inversion. The formulae for qS_1 and qS_2 waves have the same form and they can be used without identifying which wave is considered, but when we determine the slowness and polarization vectors from the seismograms or visualize the inverted polarization vectors and phase velocities we have to identify the two *qS* waves. Instead of using fast and slow waves for the two *qS* waves, we identify them by continuity of the polarization vectors (Vavryčuk 2001). By comparing adjacent polarization vectors, we can find correct wave fronts for the qS_1 and qS_2 waves. Special attention should be paid to singular directions, where the polarization vectors of the two *qS* waves coincide.

6.1 Set-up of the experiment

In the following synthetic VSP experiment, we use a Cartesian coordinate system with an (x, y) plane which coincides with the Earth's surface and a z-axis which is positive downwards. The VSP configuration is shown schematically in Fig. 1. The whole model is confined in a model box whose horizontal dimensions are 10×10 km. The borehole is situated in the centre of the box.

We test the inversion formulae for *qP* and *qS* waves on a simple model of a vertically inhomogeneous orthorhombic medium with tilted symmetry axes. Density-normalized elastic parameters are used at the top and bottom surfaces of the model (see Table 1). The symmetry axes in both top and bottom matrices are rotated by 30° from the z-axis in the (x, z) plane, and then by 30° around the z-axis. The rotated elastic matrix can be taken as a special case of general anisotropy, since all the elements of the rotated elastic matrix are non-zero. The top and bottom surfaces of the model are horizontal planes situated at depths of 0 and 5 km. Between the top and bottom surfaces the elastic parameters are determined by linear interpolation. Since both surfaces are horizontal in the model, variation of elastic parameters with depth is the same everywhere and the medium is laterally homogeneous.

In the model, we consider five surface profiles with their centres at the mouth of the borehole (see Fig. 1). The five profiles are distributed uniformly with the step in the angle between them equal to 36°. This means that the five profiles can be associated successively with 0°, 36°, 72°, 108° and 144°, measured from the positive x-axis.

Each profile contains 32 sources to each side of the borehole, starting at 0.1 km from the mouth of the borehole and separated by 0.05 km. The vertical borehole contains five three-component receivers starting at a depth of 0.9 km and separated by 0.05 km (see Fig. 1). Three-component synthetic seismograms at each receiver are generated using the modified program package ANRAY (Gajewski & Pšenčík 1990).

Data sets for inversion consist of slowness and polarization vectors which can be obtained by analysis of the three-component seismograms. For synthetic experiments we use the following methods to determine slowness and polarization vectors. For real data, the methods of Esmersoy (1990) and Leaney (1990) should be used.

For the *qP* wave and uncoupled *qS* waves we use a method similar to the method used by Zheng (2003) to determine slowness and polarization vectors at each receiver. From the synthetic seismograms, traveltimes corresponding to the maximum amplitude are picked up and particle motion diagrams are constructed for each wave. If the synthetic seismograms are noise-free, the trace of the particle motion is a straight line for the *qP* wave and uncoupled *qS* waves, respectively. To determine the polarization vector from the three-component seismograms, a straight line is sought to fit the trace for each wave (Zheng 2003). Picked-up travel times at five receivers are used to fit a curve by the $x^2 - t^2$ method, from which partial derivatives of the traveltime with respect to the z-coordinate (coordinate along the borehole) are found. The partial derivatives represent vertical components of the slowness vectors. The above method causes inaccuracies when it is used to determine components of the slowness vector along the borehole at the shallowest and deepest receivers. Since the medium is laterally homogeneous, we can make use of traveltime reciprocity to determine horizontal components of the slowness vector from travel times between sources and the given receiver (Gaiser 1990).

In the coupling case of *qS* waves, slowness and polarization vectors for the first arrival are determined and used for inversion. To determine the slowness and polarization vector of the first arrival, we have to construct a particle motion diagram and to find a coordinate system in which the two *qS* waves can be separated. With the separated seismogram for the first arrival, the slowness and polarization vector can be obtained by the same method described above. In a singular direction, we need to identify the qS_1 and qS_2 waves in its vicinity and to find the correct wave front by considering continuity of the polarization vectors of adjacent receivers.

Table 1. Unrotated elastic matrices at the top and bottom surfaces in the model.

Elastic matrix at the top surface ($km^2\ s^{-2}$)	Elastic matrix at the bottom surface ($km^2\ s^{-2}$)
$\begin{pmatrix} 4.86 & 0.99 & 0.86 & 0 & 0 & 0 \\ & 5.09 & 0.77 & 0 & 0 & 0 \\ & & 3.82 & 0 & 0 & 0 \\ & & & 1.62 & 0 & 0 \\ & & & & 1.75 & 0 \\ & & & & & 1.93 \end{pmatrix}$	$\begin{pmatrix} 34.56 & 7.04 & 6.08 & 0 & 0 & 0 \\ & 36.16 & 5.44 & 0 & 0 & 0 \\ & & 27.20 & 0 & 0 & 0 \\ & & & 11.52 & 0 & 0 \\ & & & & 12.48 & 0 \\ & & & & & 13.76 \end{pmatrix}$

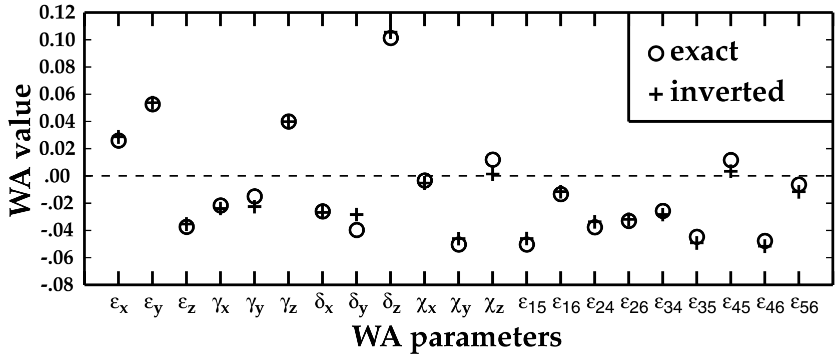

Figure 2. Exact (circles) and inverted (cross) weak anisotropy parameters.

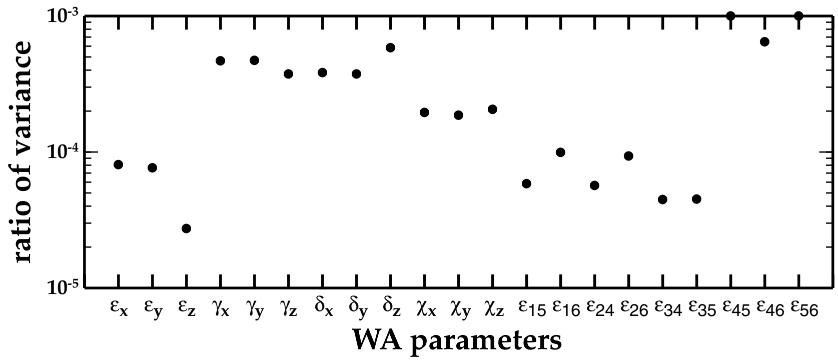

Figure 3. Variance ratio σ^2/σ_0^2 of 21 weak anisotropy parameters obtained from eq. (54).

We use the 'slowness' eqs (43) and (50) and 'polarization' eqs (42), (48) and (49) for *qP* and *qS* waves to invert all the 21 weak anisotropy parameters. We get reference velocities α and β, respectively, by averaging the slowness of *qP* and *qS* waves for all rays at the receiver. To determine the wave normal $\mathbf{n}_i$ in the reference isotropic medium, we have several choices. Zheng & Pšenčík (2002) constructed a reference isotropic model in the overburden through which they traced rays in order to determine $\mathbf{n}_i$. Zheng (2003) and Gomes *et al.* (2004) chose the wave normal $\mathbf{n}_i$ to be parallel to the observed polarization vector $\mathbf{g}_i$ of the *qP* wave. Here we use the same choice as Zheng (2003) and Gomes *et al.* (2004) did. This makes it unnecessary to perform the time-consuming ray tracing in the reference medium. $\sin\psi_1$ and $\cos\psi_1$ in eqs. (48)–(50) can be determined because the polarization vectors of *qS* waves are known. The singular value decomposition method is used to solve the linear equations.

6.2 Inversion results

Through inversion, we obtain all the 21 weak anisotropy parameters for each receiver. In the following, we will show the result for the receiver at a depth of 1 km. Fig. 2 shows the exact (circle) and inverted (cross) weak anisotropy parameters. The maximum absolute error of inverted weak anisotropy parameters appearing in δ_y is less than 0.012. The variance $\sigma^2(\epsilon_i)$ of weak anisotropy parameter ϵ_i is calculated from singular values λ_j ($j = 1, \ldots, 21$) and a 21×21 orthogonal matrix V_{ji} that spans the model space of the weak anisotropy parameters

$$\sigma^2(\epsilon_i) = \sigma_0^2 \Sigma (V_{ji}/\lambda_j)^2. \qquad (54)$$

The symbol σ_0^2 denotes an averaging variance from the 'observed' data. The variances depend on the number of profiles used, on the illumination of receivers and also on the chosen reference velocities. Instead of showing the variance $\sigma^2(\epsilon_i)$, we show the ratio of $\sigma^2(\epsilon_i)/\sigma_0^2$ which represents how random noise affects the inverted weak anisotropy parameters. In Fig. 3, the horizontal axis denotes the weak anisotropy parameters and the vertical axis shows the values of the ratio σ^2/σ_0^2. Some weak anisotropy parameters, e.g. δ_z, ϵ_{45}, ϵ_{46} and ϵ_{56} prevailingly connecting the horizontal propagation of waves, have large values of variance ratio. This is a consequence of insufficient horizontal illumination, since the maximum polar angle (angle of the ray from the vertical direction) is about 68°. If the data contain a high level of noise we can expect that the above weak anisotropy parameters have large errors. To improve the inversion results, it is necessary to increase the length of the profiles.

The inverted weak anisotropy parameters can be transformed into density-normalized elastic parameters shown in the right column in Table 2. Comparing the inverted elastic parameters with the exact ones (the left column in Table 2), we find that most of the elastic parameters are retrieved quite well. The maximum relative error of its diagonal elements is about 2 per cent. The parameter A_{23} has a maximum absolute error of 0.14 km^2 s^{-2} among all non-diagonal elements, the absolute error of the other non-diagonal elastic parameters is less than 0.12 km^2 s^{-2}.

Table 2. Exact and inverted elastic matrices of ORT anisotropic medium.

Exact elastic matrix ($km^2\ s^{-2}$)						Inverted elastic matrix ($km^2\ s^{-2}$)					
10.34	2.24	1.87	0.02	−0.49	−0.13	10.40	2.29	1.98	0.04	−0.45	−0.11
	10.87	1.84	−0.37	−0.11	−0.33		10.89	1.98	−0.33	−0.04	−0.31
		9.09	−0.25	−0.44	0.02			9.14	−0.28	−0.48	−0.01
			3.80	0.05	−0.19				3.78	0.01	−0.20
				3.85	−0.03					3.80	−0.05
					4.29						4.29

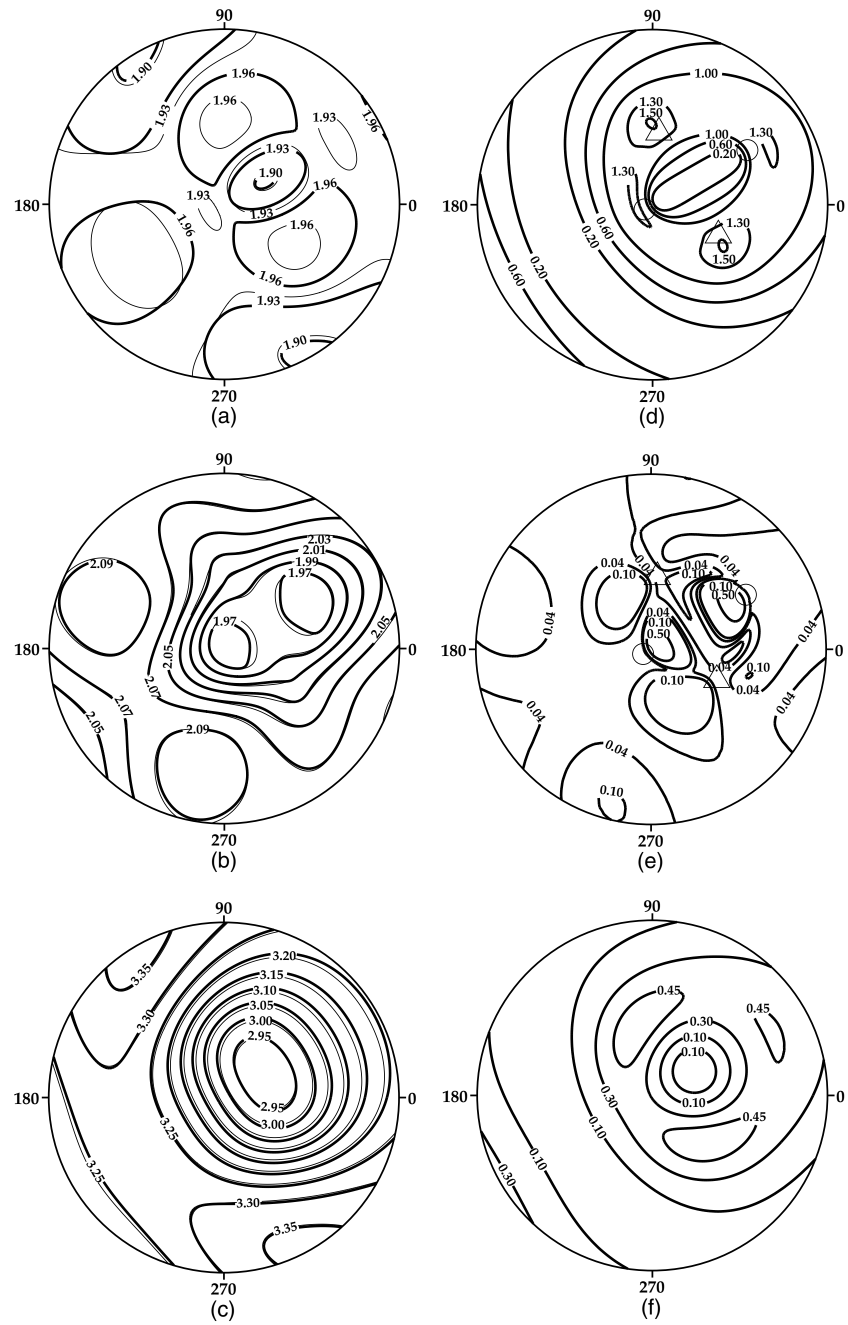

Figure 4. Equal area projection of phase velocity and relative errors (in per cent) of the inverted phase velocity. The left column shows the exact (thick line) and inverted (thin line) phase velocities of qS_1 (a), qS_2 (b) and qP (c) waves. The right column shows the relative error of the inverted phase velocities of qS_1 (d), qS_2 (e) and qP (f) waves. Two small circles in (d) and (e) denote the singular directions, and two small triangles denote the positions where two sheets of qS waves almost touch each other.

In order to find singular directions of the qS waves for the true model we calculate phase velocities of the two qS waves with an azimuthal angle interval and a polar angle interval 0.01°. In directions (30°, 64°) and (210°, 6°), the difference in the phase velocities of the two qS waves is less than 10^{-12} km s^{-1}. We take these two directions as singular directions of the qS waves. There are two other directions (85°, 45°) and (335°, 45°) in which the difference of phase velocities of the two qS waves is less than 3×10^{-2} km s^{-1}. Fig. 4 shows an equal area projection of phase velocity and relative error of phase velocity on the (x, y) plane. The left column shows the exact (thick line) and inverted (thin line) phase velocities (km s^{-1}), the right column shows the relative error (in per cent) of inverted phase velocities. The results for qS_1, qS_2 and qP waves are displayed from top to bottom. Small circles in Figs 4(d) and (e) denote the two singular directions for qS waves. The two triangles denote two directions in which the sheets of the two qS wave phase velocities almost touch each other. We find that the inverted phase velocities fit the exact ones quite well not only for the qP wave but also for the qS waves. The maximum relative errors of inverted phase velocities are 1.52, 1.06 and 0.53 per cent for qS_1, qS_2 and qP waves, respectively. Several contours of relative errors for the two qS waves converge on the singular points (Figs 4d and e), indicating that the phase velocities of qS waves around the singular directions change rapidly and fitting by the first-order approximate formulae varies with the local azimuth. The contours of relative errors for the qS_2 wave also have denser distribution around the two directions mentioned above.

Fig. 5 shows an equal area projection of polarization vectors and the errors of inverted polarization vectors of qS waves. The left column shows the exact (solid line) and inverted (dashed line) polarization vectors for qS_1 (a) and qS_2 (b) waves. The right column shows the errors (in degrees) of the inverted polarization vectors of qS_1 (c) and qS_2 (d) waves. The two small circles and two small triangles in the left column denote the four positions as shown in Fig. 4. The error of inverted polarization vectors is less than 10° except in the small areas (labelled 10 in Figs 5c and d) around the four directions. Because the first-order approximate formulae do not present enough accuracy, larger errors of the inverted polarization vectors appear around the singular directions. They also appear in the two areas mentioned above. Out of these areas, the errors of polarization vectors become negligible. In order to reduce these errors, higher-order perturbations should be considered (Farra 2001).

For the qP wave the difference between the inverted and exact polarization vectors is quite small, because there are no problems with singular directions. The maximum error of the inverted polarization vector is less than 1° (see Fig. 6).

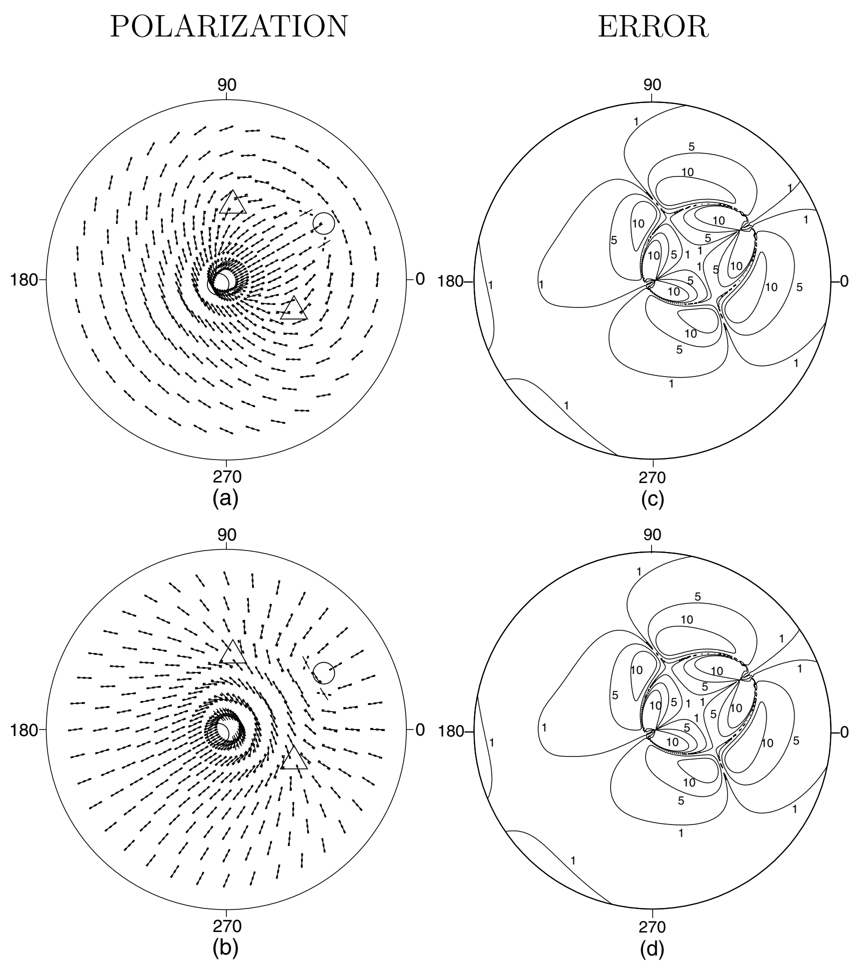

Figure 5. Equal area projection of polarization vectors and errors of inverted polarization vectors. The left column shows the exact (solid line) and inverted (dashed line) polarization vectors of qS_1 (a) and qS_2 (b) waves and the right column the errors (in degrees) of the inverted polarization vectors of qS_1 (c) and qS_2 (d) waves. Two small circles and two small triangles denote the same as in Fig. 4.

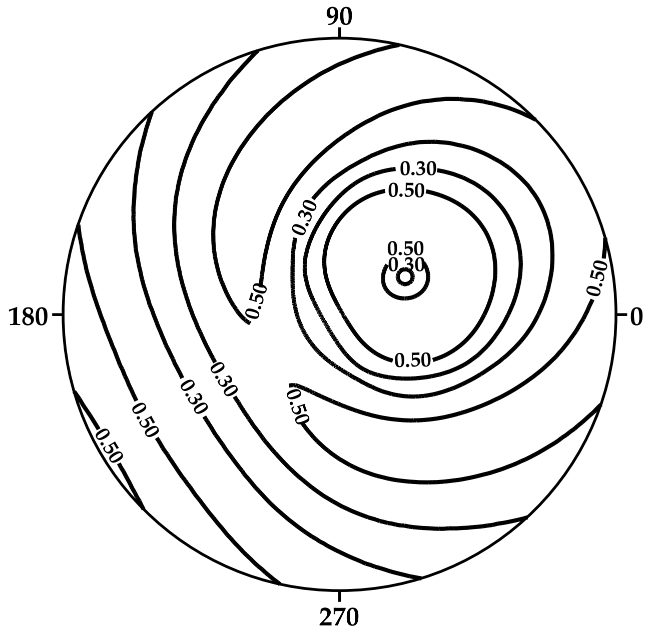

Figure 6. Equal area projection of errors (in degrees) of inverted polarization vectors for the *qP* wave.

7 CONCLUSIONS

The inversion equations have a very general sense with regard to local determination of weak anisotropy parameters in arbitrary weakly anisotropic inhomogeneous media. The equations hold not only in the regular regions of *qS* waves but also in the singularities where only one *qS* wave with a polarization controlled by a source is observed. In the coupling case, the *qS* waves form a complicated polarization pattern; however, the first arrival can be incorporated into the inversion.

Since the inversion equations hold for local determination of weak anisotropy parameters, variation of observed quantities at a receiver is considered to be caused by anisotropy alone. Therefore, the method has a potential to separate anisotropy from inhomogeneities.

The synthetic experiment shows that the inverted phase velocity and polarization vector for *qS* waves have a high accuracy in regular regions. The slightly larger error of the polarization vector for *qS* waves can be found in the vicinity of the singularities, which is considered to be the result of the first-order approximation.

Previous studies for the *qP* wave (Zheng & Pšenčík 2002; Zheng 2003; Gomes *et al.* 2004) have shown that the scheme used here is independent of structural complexities in the overburden. More complicated experiments will be carried out and real data will be processed for both *qP* and *qS* waves in the continued study. We can expect that the inversion method will find important applications in exploration geophysics and seismology.

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REFERENCES

Backus, G.E., 1965. Possible form of seismic anisotropy of the upper-most mantle under oceans, *J. geophys. Res.,* **70,** 3429–3439.

Červený, V., 2001. *Seismic Ray Theory,* Cambridge University Press, Cambridge.

Chapman, C.H. & Shearer, P.M., 1989. Ray tracing in azimuthally anisotropic media, II, Quasi-shear wave coupling, *Geophys. J. Int.,* **96,** 65–83.

Coates, R.T. & Chapman, C.H., 1990. Quasi-shear wave coupling in weakly anisotropic 3-D media, *Geophys. J. Int.,* **103,** 301–320.

Esmersoy, C., 1990. Inversion of P and SV waves from multi-component offset vertical seismic profiles, *Geophysics,* **55,** 39–50.

Farra, V., 2001. High order perturbations of the phase velocity and polarization of *qP* and *qS* waves in anisotropic media, *Geophys. J. Int.,* **147,** 93–104.

Farra, V. & Pšenčík, I., 2003. Properties of the zero-, first-, and higher- order approximations of attributes of elastic waves in weakly anisotropic media, *J. acoust. Soc. Am.,* **114,** 1366–1378.

Gaiser, J.E., 1990. Transversely isotropic phase velocity analysis from slowness estimates, *J. geophys. Res.,* **95,** 11 241–11 254.

Gajewski, D. & Pšenčík, I., 1990. Vertical seismic profile synthetics by dynamic ray tracing in laterally varying layered anisotropic structures, *J. geophys. Rec.,* **95,** 11 301–11 315.

Gomes, E., Zheng, X., Pšenčík, I., Horne, S. & Leaney, S., 2004. Local determination of weak anisotropy parameters from walkaway VSP *qP*-wave data in the Java Sea region, *Stud. Geophys. Geod.,* **48,** 215–231.

Jech, J. & Pšenčík, I., 1989. First-order perturbation method for anisotropic media, *Geophys. J. Int.,* **99,** 369–376.

Leaney, W.S., 1990. Parametric wavefield decomposition and application, in *Expanded Abstracts, 60th Annual International Meeting of the Society of Exploration Geophysicists, Tulsa, 1990,* pp. 1097–1100.

Mensch, T. & Rasolofosaon, P., 1997. Elastic wave velocities in anisotropic media of arbitrary symmetry—generalization of Thomsen's parameters ϵ, δ and γ, *Geophys. J. Int.,* **128,** 43–64.

Pšenčík, I., 1998. Green's functions for inhomogeneous weakly anisotropic media, *Geophys. J. Int.,* **135,** 279–288.

Pšenčík, I. & Gajewski, D., 1998. Polarization, phase velocity and NMO velocity of *qP* waves in arbitrary weakly anisotropic media, *Geophysics,* **69,** 1754–1766.

Pšenčík, I. & Vavryčuk, V., 2002. Approximate relation between the ray vector and the wave normal in weakly anisotropic media, *Stud. Geophys. Geod.,* **46,** 793–807.

Pšenčík, I. & Zheng, X., 1998. Determination of weak anisotropy parameters from the *qP* wave particle motion measurements, in *Seismic Waves in Complex 3-D Structures,* pp. 299–305, Report 7, Department of Geophysics, Charles University, Prague.

Sayers, C.M., 1994. *P*-wave propagation in weakly anisotropic media., *Geophys. J. Int.,* **116,** 799–805.

Thomsen, L., 1986. Weak elastic anisotropy, *Geophysics,* **51,** 1954–1966.

Vavryčuk, V., 1999. Applicability of higher-order ray theory for S wave propagation in inhomogeneous weakly anisotropic elastic media, *J. geophys. Res.,* **104**(B12), 28 829–28 840.

Vavryčuk, V., 2001. Ray tracing in anisotropic media with singularities, *Geophys. J. Int.,* **145,** 265–276.

White, J.E., Martineau-Nicoletis, L. & Monash, C., 1983. Measured anisotropy in Pierre shale, *Geophys. Prospect.,* **31,** 709–725.

Zheng, X., 2003. Determination of weak anisotropy parameters from *qP* wave slowness and polarization vectors, *PhD thesis*, Charles University, Prague.

Zheng, X. & Pšenčík, I., 2002. Local determination of weak anisotropy parameters from *qP*-wave slowness and particle motion measurements, *PAGEOPH,* **159,** 1881–1905.

Zillmer, M., Kashtan, B.M. & Gajewski, D., 1998. Quasi-isotropic approximation of ray theory for anisotropic media, *Geophys. J. Int.,* **132,** 643–653.

APPENDIX A: ELEMENTS OF THE WEAK ANISOTROPY MATRIX

Elements of the weak anisotropy matrix for a general medium:

$$\begin{aligned}B_{11} =\ & 2D^{-2}\{\alpha^2[\epsilon_x n_1^4 n_3^2 + \epsilon_y n_2^4 n_3^2 + \epsilon_z n_3^2\left(n_3^2-1\right)^2 + \delta_x n_1^2 n_3^2\left(n_3^2-1\right)\\ & + \delta_y n_2^2 n_3^2\left(n_3^2-1\right) + \delta_z n_1^2 n_2^2 n_3^2 + \chi_x n_1^2 n_2 n_3\left(2n_3^2-1\right) + \chi_y n_1 n_2^2 n_3\\ & \left(2n_3^2-1\right) + 2\chi_z n_1 n_2 n_3^2\left(n_3^2-1\right) + \epsilon_{15} n_1^3 n_3\left(2n_3^2-1\right) + 2\epsilon_{16} n_1^3 n_2 n_3^2\\ & + \epsilon_{24} n_2^3 n_3\left(2n_3^2-1\right) + 2\epsilon_{26} n_1 n_2^3 n_3^2 + \epsilon_{34} n_2 n_3\left(n_3^2-1\right)\left(2n_3^2-1\right)\\ & + \epsilon_{35}\ n_1 n_3\left(n_3^2-1\right)\left(2n_3^2-1\right)] + \beta^2\left[\gamma_x n_2^2 + \gamma_y n_1^2 + \epsilon_{45} n_1 n_2\right]\}\end{aligned} \quad \text{(A1)}$$

$$\begin{aligned}B_{12} =\ & D^{-2}\{\alpha^2[-2\epsilon_x n_1^3 n_2 n_3 + 2\epsilon_y n_1 n_2^3 n_3 - \delta_x n_1 n_2 n_3\left(n_3^2-1\right)\\ & + \delta_y n_1 n_2 n_3\left(n_3^2-1\right) + \delta_z n_1 n_2 n_3\left(n_1^2-n_2^2\right) + \chi_x n_1\left(n_1^2 n_3^2 + n_2^2 - 2n_2^2 n_3^2\right)\\ & - \chi_y n_2\left(n_2^2 n_3^2 + n_1^2 - 2n_1^2 n_3^2\right) + \chi_z n_3\left(n_3^2-1\right)\left(n_1^2-n_2^2\right) - \epsilon_{15} n_1^2 n_2\left(3n_3^2-1\right)\\ & - \epsilon_{16} n_1^2 n_3\left(3n_2^2-n_1^2\right) + \epsilon_{24} n_1 n_2^2\left(3n_3^2-1\right) + \epsilon_{26} n_2^2 n_3\left(3n_1^2-n_2^2\right)\\ & +\ \epsilon_{34} n_1 n_3^2\left(n_3^2-1\right) - \epsilon_{35} n_2 n_3^2\left(n_3^2-1\right)] + \beta^2[2\gamma_x n_1 n_2 n_3\\ & -\ 2\gamma_y n_1 n_2 n_3 + \epsilon_{45} n_3\left(n_1^2-n_2^2\right) - \epsilon_{46} n_2\left(n_3^2-1\right) + \epsilon_{56} n_1\left(n_3^2-1\right)]\}\end{aligned} \quad \text{(A2)}$$

$$\begin{aligned}B_{22} =\ & 2D^{-2}\{\alpha^2[\epsilon_x n_1^2 n_2^2 + \epsilon_y n_1^2 n_2^2 - \delta_z n_1^2 n_2^2 - \chi_x n_1^2 n_2 n_3 - \chi_y n_1 n_2^2 n_3\\ & \epsilon_{15} n_1 n_2^2 n_3 - \epsilon_{16} n_1 n_2\left(n_1^2-n_2^2\right) + \epsilon_{24} n_1^2 n_2 n_3 + \epsilon_{26} n_1 n_2\left(n_1^2-n_2^2\right)]\\ & +\beta^2[\gamma_x n_1^2 n_3^2 + \gamma_y n_2^2 n_3^2 + \gamma_z\left(n_3^2-1\right)^2 - \epsilon_{45} n_1 n_2 n_3^2\\ & -\ \epsilon_{46} n_1 n_3\left(n_3^2-1\right) - \epsilon_{56} n_2 n_3\left(n_3^2-1\right)]\}\end{aligned} \quad \text{(A3)}$$

$$\begin{aligned}B_{13} =\ & \alpha^2 D^{-1}[2\epsilon_z n_3^5 + 4(\epsilon_{34} n_2 + \epsilon_{35} n_1) n_3^4 + (-2\epsilon_z + 2\delta_x n_1^2 + 2\delta_y n_2^2\\ & +4\chi_z n_1 n_2) n_3^3 + (4\chi_x n_1^2 n_2 + 4\chi_y n_1 n_2^2 + 4\epsilon_{15} n_1^3 + 4\epsilon_{24} n_2^3 - 3\epsilon_{34} n_2\\ & -3\epsilon_{35} n_1) n_3^2 + (2\epsilon_x n_1^4 + 2\epsilon_y n_2^4 - \delta_x n_1^2 - \delta_y n_2^2 + 2\delta_z n_1^2 n_2^2 - 2\chi_z n_1 n_2\\ & +\ 4\epsilon_{16} n_1^3 n_2 + 4\epsilon_{26} n_1 n_2^3)\, n_3 - \chi_x n_1^2 n_2 - \chi_y n_1 n_2^2 - \epsilon_{15} n_1^3 - \epsilon_{24} n_2^3]\end{aligned} \quad \text{(A4)}$$

$$\begin{aligned}B_{23} =\ & \alpha^2 D^{-1}[(\epsilon_{34} n_1 - \epsilon_{35} n_2) n_3^3 + (-\delta_x n_1 n_2 + \delta_y n_1 n_2 + \chi_z n_1^2\\ & -\ \chi_z n_2^2)\, n_3^2 + (-2\chi_x n_1 n_2^2 + \chi_x n_1^3 + 2\chi_y n_1^2 n_2 - \chi_y n_2^3 - 3\epsilon_{15} n_1^2 n_2\\ & +\ 3\epsilon_{24} n_1 n_2^2)\, n_3 - 2\epsilon_x n_1^3 n_2 + 2\epsilon_y n_1 n_2^3 - \delta_z n_1 n_2^3 + \delta_z n_1^3 n_2\\ & -\ 3\epsilon_{16} n_1^2 n_2^2 + \epsilon_{16} n_1^4 + 3\epsilon_{26} n_1^2 n_2^2 - \epsilon_{26} n_2^4]\end{aligned} \quad \text{(A5)}$$

$$B_{33} = 2\alpha^2 \left[\epsilon_z n_3^4 + 2\left(\epsilon_{34} n_2 + \epsilon_{35} n_1\right) n_3^3 + \left(\delta_x n_1^2 + \delta_y n_2^2 + 2\chi_z n_1 n_2\right) n_3^2 \right.$$
$$+ 2\left(\chi_x n_1^2 n_2 + \chi_y n_1 n_2^2 + \epsilon_{15} n_1^3 + \epsilon_{24} n_2^3\right) n_3 \tag{A6}$$
$$\left. + \epsilon_x n_1^4 + \epsilon_y n_2^4 + \delta_z n_1^2 n_2^2 + 2\epsilon_{16} n_1^3 n_2 + 2\epsilon_{26} n_1 n_2^3 \right].$$

Weak anisotropy parameters for a general medium:

$$\epsilon_x = \frac{A_{11} - \alpha^2}{2\alpha^2}, \quad \epsilon_y = \frac{A_{22} - \alpha^2}{2\alpha^2}, \quad \epsilon_z = \frac{A_{33} - \alpha^2}{2\alpha^2},$$
$$\gamma_x = \frac{A_{44} - \beta^2}{2\beta^2}, \quad \gamma_y = \frac{A_{55} - \beta^2}{2\beta^2}, \quad \gamma_z = \frac{A_{66} - \beta^2}{2\beta^2},$$
$$\delta_x = \frac{A_{13} + 2A_{55} - \alpha^2}{\alpha^2}, \quad \delta_y = \frac{A_{23} + 2A_{44} - \alpha^2}{\alpha^2}, \quad \delta_z = \frac{A_{12} + 2A_{66} - \alpha^2}{\alpha^2},$$
$$\chi_x = \frac{A_{14} + 2A_{56}}{\alpha^2}, \quad \chi_y = \frac{A_{25} + 2A_{46}}{\alpha^2}, \quad \chi_z = \frac{A_{36} + 2A_{45}}{\alpha^2},$$
$$\epsilon_{15} = \frac{A_{15}}{\alpha^2}, \quad \epsilon_{16} = \frac{A_{16}}{\alpha^2}, \quad \epsilon_{24} = \frac{A_{24}}{\alpha^2},$$
$$\epsilon_{26} = \frac{A_{26}}{\alpha^2}, \quad \epsilon_{34} = \frac{A_{34}}{\alpha^2}, \quad \epsilon_{35} = \frac{A_{35}}{\alpha^2},$$
$$\epsilon_{45} = \frac{A_{45}}{\beta^2}, \quad \epsilon_{46} = \frac{A_{46}}{\beta^2}, \quad \epsilon_{56} = \frac{A_{56}}{\beta^2}. \tag{A7}$$

APPENDIX B:

The relationship between B_{mn} and $\bar{B}_{mn}$ in the two coordinate systems ($\mathbf{e}_i^{(1)}$, $\mathbf{e}_i^{(2)}$, $\mathbf{e}_i^{(3)}$) and ($\mathbf{g}_i^{(1)0}$, $\mathbf{g}_i^{(2)0}$, $\mathbf{g}_i^{(3)0}$):

$$\bar{B}_{11} = \cos^2\psi\, B_{11} + 2\sin\psi\cos\psi\, B_{12} + \sin^2\psi\, B_{22} \tag{B1}$$

$$\bar{B}_{12} = -\sin\psi\cos\psi\, B_{11} + (\cos^2\psi - \sin^2\psi) B_{12} + \sin\psi\cos\psi\, B_{22} \tag{B2}$$

$$\bar{B}_{22} = \sin^2\psi\, B_{11} - 2\sin\psi\cos\psi\, B_{12} + \cos^2\psi\, B_{22} \tag{B3}$$

$$\bar{B}_{13} = \cos\psi\, B_{13} + \sin\psi\, B_{23} \tag{B4}$$

$$\bar{B}_{23} = -\sin\psi\, B_{13} + \cos\psi\, B_{23} \tag{B5}$$

$$\bar{B}_{33} = B_{33}, \tag{B6}$$

where ψ is the angle between basic vectors $\mathbf{g}_i^{(1)0}$ and $\mathbf{e}_i^{(1)}$.