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On the non-uniqueness of main geomagnetic field determined by surface intensity measurements: the Backus problem

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SUMMARY

We revisit the problem of non-uniqueness of harmonic magnetic field models in a region outside a sphere containing the field sources, when only intensity values on the sphere surface are known. Using the angular momentum algebra and the Clebsch-Gordan coefficients, we are able to treat different aspects of this non-uniqueness following a unified line of reasoning. In this new framework, we first recover two Backus results, namely the proof of uniqueness in the case of a field generated by a finite number of harmonics and the recurrence relation that defines the well-known Backus series. This formalism allows us to extend previous studies in two ways: firstly, we show how to produce an harmonic series orthogonal on the sphere to some other arbitrary harmonic series; secondly, we outline a new method for computing magnetic field models starting from scalar intensity values alone.

Key words: Backus effect, Backus problem, geomagnetism, main field models, non-uniqueness.

1 INTRODUCTION AND MOTIVATION

The main magnetic field **B** of the Earth is believed to be generated in the liquid core via a non-linear magnetohydrodynamic dynamo effect, a complex phenomenon, which still is an open problem in geomagnetism. Nevertheless, many properties of the magnetic field of the Earth can be learned by measuring directly **B**. From the surface of the Earth up to the ionosphere and above, once the contribution of the rotational fields associated with external electric currents has been removed, the geomagnetic field can be viewed as an harmonic field, i.e. it can be derived from a scalar potential V satisfying the Laplace equation. If V satisfies certain boundary conditions, the determination of **B** and V are equivalent problems.

The question we want to address is the determination of the geomagnetic potential V from measurements of the field intensity $F = |\mathbf{B}|$ over a closed spherical surface S. For the case considered in this study, a field vanishing at infinity, the discussion was initiated long ago (Backus 1968, 1970, 1974). One of the most important results was that, if V is given by an infinite multipole expansion, the knowledge of F on S does not necessarily determine V uniquely. An example of this non-uniqueness was given in Backus (1970), where various fields with the same intensity over a unit sphere are built explicitly. However, if the expansion of V has a finite number of terms, Backus proved that it is completely determined by F up to a sign (Backus 1968).

The theoretical non-uniqueness has been related to an experimental effect concerning geomagnetic field models. Different models for a certain epoch, computed independently using scalar and vectorial data, reproduce nearly the same F values. However, the corresponding **B** components show large differences, particularly in the region of the dip equator (the Backus effect) not explained from experimental errors. Lowes (1975) explained that this was the result of the components of the fitting field, which are locally perpendicular to the observed field being only slightly constrained by the fitting process. He called the observed error a perpendicular error effect. Various numerical experiments were performed to quantify the magnitude of this effect and to understand how it can be reduced (Hurwitz & Knapp 1974; Stern & Bredekamp 1975; Barraclough & Nevitt 1976; Lowes & Martin 1987). These studies suggested that a minimum amount of vector data should be included in the fits to reduce the differences in the final vector field.

Other authors (Stern & Bredekamp 1975; Stern *et al.* 1980) tried to link more explicitly the experimental non-uniqueness of **B** to the theoretical problem studied by Backus. Direct attempts to observe the series defined in Backus (1970) seem to find some evidence of it in a sequence of coefficients with a fixed value of m and an even value of $l + m$, where l and m stand for the degree and order of the associated Legendre functions $P_{lm}(\cos\theta)$. They are however inconclusive in explaining the disparate behavior of the $m = 1$ sequence and the relatively important values of non-Backus $m = 0$ and odd $l + m$ series.

Recently, Khokhlov *et al.* (1997) demonstrated that scalar data provides a unique potential V if the field dip equator is known; see also Ultré-Guérard *et al.* (1998) for a numerical verification of this result and Khokhlov *et al.* (1999) for a discussion on the influence of knowledge of the dip equator on the uniqueness problem.

Another uniqueness problem for the geomagnetic field is the determination of this field when only its direction on a spherical surface is known. Recently, Hulot *et al.* (1997) derived conditions on the dimension of the solution space in terms of the number of loci in the spherical surface where the tangent components of the vector field vanish. Kaiser & Neudert (2004) have shown that, for dipole axisymmetric direction fields or general 2^N-pole axisymmetric direction fields and geomagnetic scalar potential defined in a truncated Hilbert space, the geomagnetic field is uniquely determined. Some of the results of this paper are related to the results we present here for the orthogonal field expansions in Sections 3.2 and 3.3.

Until the NASA Magsat satellite (1979–80), satellite surveys produced only intensity data. Even nowadays, as a result of the difficulty of accurately determining the attitude of the spacecrafts in orbit, vector data may not be available or otherwise it may be strongly contaminated by attitude errors. An example of the former situation concerns the Oersted satellite launched in 1999 February and still in orbit. As a result of radiation effects on the star imager that determines the attitude of the satellite, vector data is very sparse over the South Atlantic anomaly (see e.g. Olsen *et al.* 2000). In such cases, a Backus effect becomes evident in the resulting main field models, requiring inclusion of ground vectorial data (with poor geographical distribution) or synthetic data in order to be minimized. Holme & Bloxham (1995) and Holme (2000) studied the effect of attitude uncertainty on satellite vector data. They developed a formalism to model those uncertainties with a view to alleviating the Backus effect. In spite of the different solutions found to decrease the impact of the Backus effect, it continues to contribute significantly to the quality of main field models. A broader understanding of the Backus non-uniqueness theoretical problem and the way it can be related to the experimental Backus effect is thus still of practical relevance.

In this letter, we address the non-uniqueness problem exploring the properties of the harmonic functions. We first write a generalized form for the scalar product of two harmonic vectors as a series of complex spherical harmonics. The coefficients of the series are easily related to the Gauss coefficients of the two harmonic potentials. A general formula allowing the definition of Backus-like series beyond the dipole series given in Backus (1970) is derived. The quadrupole series is given explicitly. Our result allows us to provide another proof of uniqueness for finite expansions and to outline a new linear method to compute the magnetic field from measurements of F.

2 FIELD DEFINITIONS AND SERIES EXPANSIONS

Let $\mathbf{B}_1 = \nabla V_1$ and $\mathbf{B}_2 = \nabla V_2$ be two magnetic fields derived from the potentials V_1 and V_2, defined outside a spherical surface of unit radius S that contains the field sources. Then, V_1 and V_2 are harmonic functions vanishing at infinity. If V stands for either V_1 or V_2, we have, for $r \geq 1$,

$$\nabla^2 V(r, \theta, \phi) = 0, \tag{1}$$

$$V(r, \theta, \phi) = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \frac{a_{lm}}{r^{l+1}} Y_{lm}(\theta\,,\ \phi), \tag{2}$$

where $Y_{lm}(\theta, \phi)$ are the complex spherical harmonic functions and the monopole term $l = 0$ was excluded from the sum. The complex coefficients a_{lm} must have the property

$$a^*_{l\,m} = (-1)^m a_{l\,-m} \tag{3}$$

in order that V be real. The vector field associated with eq. (2) is (Jackson 1999)

$$\mathbf{B} = \sum_{l,m} \frac{a_{lm}}{r^{l+2}} \left[-(l+1)\,\frac{\mathbf{r}}{r}\,Y_{l\,m} - i\,\sqrt{l(l+1)}\,\frac{\mathbf{r}}{r} \times \mathbf{X}_{l\,m} \right], \tag{4}$$

where

$$\mathbf{X}_{l\,m} = \frac{\mathbf{L}\,Y_{l\,m}}{\sqrt{l(l+1)}}\,, \qquad \mathbf{L} = -i\,\mathbf{r} \times \nabla. \tag{5}$$

If $|\mathbf{B}_1| = |\mathbf{B}_2|$ on S $(r = 1)$, then the vectors

$$\mathbf{B}^{\pm} = \frac{1}{2}\,(\mathbf{B}_1 \pm \mathbf{B}_2)|_{r=1} \tag{6}$$

are orthogonal, i.e. $\mathbf{B}^+ \cdot \mathbf{B}^- = 0$. From eq. (4) and the definition of $\mathbf{X}_{lm}$ it follows

$$\mathbf{B}^+ \cdot \mathbf{B}^- = \sum_{l^+,m^+} \sum_{l^-,m^-} \left(a^+_{l^+m^+}\right)^* a^-_{l^-m^-} \times \left[(l^+ + 1)\,(l^- + 1)\,Y^*_{l^+m^+}\,Y_{l^-m^-} + \sqrt{l^+(l^+ + 1)\,l^-(l^- + 1)}\,\mathbf{X}^*_{l^+m^+} \cdot \mathbf{X}_{l^-m^-} \right] = 0. \tag{7}$$

The last term can be simplified using the relations

$$L_{\pm}\,Y_{lm} = \sqrt{l(l+1) - m(m \pm 1)}\,Y_{lm\pm1}, \tag{8}$$

$$L_z Y_{lm} = m Y_{lm}, \tag{9}$$

where $L_\pm = L_x \pm \mathrm{i}\, L_y$ and the property $Y^*_{lm} = (-1)^m\, Y_{l\,-m}$. The scalar product becomes

$$\begin{aligned}\mathbf{B}^+ \cdot \mathbf{B}^- = \sum_{l^+,m^+} \sum_{l^-,m^-} \left(a^+_{l^+m^+}\right)^* a^-_{l^-m^-} (-1)^{m^+} \Big[(l^+ + 1)\,(l^- + 1)\, Y_{l^+\,-m^+}\, Y_{l^-\,m^-} \\ - \frac{1}{2} \sqrt{l^+(l^+ + 1) - m^+(m^+ + 1)} \sqrt{l^-(l^- + 1) - m^-(m^- + 1)}\, Y_{l^+\,-m^+-1}\, Y_{l^-\,m^-+1} \\ - \frac{1}{2} \sqrt{l^+(l^+ + 1) - m^+(m^+ - 1)} \sqrt{l^-(l^- + 1) - m^-(m^- - 1)}\, Y_{l^+\,-m^++1}\, Y_{l^-\,m^--1} \\ + m^+ m^-\, Y_{l^+\,-m^+}\, Y_{l^-\,m^-} \Big]. \end{aligned} \tag{10}$$

The product of two spherical harmonic functions can be written as a linear combination of spherical harmonic functions (Edmonds 1996),

$$Y_{l_1\,m_1} Y_{l_2\,m_2} = \sum_{l=|l_1-l_2|}^{l_1+l_2} \sum_{m=-l}^{l} \sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2l+1)}} \langle\, l_1\, 0\,;\, l_2\, 0 \,|\, l\, 0\,\rangle \langle\, l_1\, m_1\,;\, l_2\, m_2 \,|\, l\, m\,\rangle Y_{lm}. \tag{11}$$

The Clebsch–Gordan coefficients $\langle\, l_1\, m_1;\, l_2\, m_2 \,|\, l\, m\,\rangle$ and $\langle\, l_1\, 0\,;\, l_2\, 0 \,|\, l\, 0\,\rangle$ are real coefficients coming from the theory of angular momentum coupling in quantum mechanics (Edmonds 1996). Inserting this decomposition into eq. (10) and using the properties of the Clebsch–Gordan coefficients (Edmonds 1996; see also Section 3.1), one gets

$$\begin{aligned}\mathbf{B}^+ \cdot \mathbf{B}^- = \frac{1}{2} \sum_{l^+,m^+} \sum_{l^-,m^-} \sum_{k=|l^+-l^-|}^{l^++l^-} \sum_{m_k=-k}^{k} \sqrt{\frac{(2l^++1)(2l^-+1)}{4\pi(2k+1)}} \langle\, l^+\, 0\,;\, l^-\, 0 \,|\, k\, 0\,\rangle \\ \times \big\langle\, l^+\, m^+\,;\, l^-\, m^- \,|\, k\, m_k \big\rangle (l^+ + l^- + k + 2)(l^+ + l^- - k + 1) a^+_{l^+\,m^+}\, a^-_{l^-\,m^-}\, Y_{k\,m_k}. \end{aligned} \tag{12}$$

Notice that, if we set $a^+_{lm} = a^-_{lm} = a_{lm}$ in eq. (12), we obtain the expression for $|\,\mathbf{B}|^2$ on S.

3 THE PROBLEM OF UNIQUENESS

The complex spherical harmonic functions define an orthogonal basis on a spherical surface, because one has

$$\int d\Omega\, Y^*_{l'm'}(\theta,\phi) Y_{lm}(\theta,\phi) = \delta_{l\,l'}\, \delta_{mm'}, \tag{13}$$

where $d\Omega \equiv \sin\theta\, d\theta\, d\phi$. This means that, for $\mathbf{B}^+ \cdot \mathbf{B}^- = 0$ to hold, the linear combination of coefficients that multiplies each spherical harmonic in eq. (12) must be zero:

$$\begin{aligned}\sum_{l^+,m^+} \sum_{l^-,m^-} a^+_{l^+\,m^+}\, a^-_{l^-\,m^-} \sqrt{(2l^++1)(2l^-+1)} (l^+ + l^- + k + 2)(l^+ + l^- - k + 1) \\ \times \langle\, l^+\, 0\,;\, l^-\, 0 \,|\, k\, 0\,\rangle \big\langle\, l^+\, m^+\,;\, l^-\, m^- \,|\, k\, m_k \big\rangle = 0, \end{aligned} \tag{14}$$

for every pair (k, m_k). We discuss now some of the consequences of eq. (14).

3.1 The case of finite expansions

Let $L^\pm$ be the maximum value of $l^\pm$ in the multipole expansions of $\mathbf{B}^\pm$. We assume that one of the fields, for instance $\mathbf{B}^+$, is not zero and every coefficient of its expansion $a^+_{l^+m^+}$ is different from zero for all possible values $l^+ = 1, \ldots, L^+$ and $m^+ = -l^+, \ldots, l^+$. We set out now to determine all the coefficients $a^-_{l^-m^-}$.

The Clebsch–Gordan coefficients $\langle\, l_1\, m_1\,; l_2\, m_2 \,|\, l\, m\,\rangle$ in which $-l_1 \le m_1 \le l_1$, $-\, l_2 \le m_2 \le l_2$ and $-l \le m \le l$, can be different from zero only if

$$m_1 + m_2 = m, \tag{15}$$

$$|l_2 - l_1| \le l \le l_1 + l_2 \qquad \text{(triangular condition)}. \tag{16}$$

Two other triangular conditions can be obtained by cyclic permutations of l_1, l_2 and l in eq. (16). The Clebsch–Gordan coefficients also satisfy some symmetry properties (Edmonds 1996), which imply that the coefficient $\langle l^+\, 0;\, l^-\, 0 \mid k\, 0\rangle$ in eq. (14) is zero when $k + l^+ + l^-$ is odd. From the conditions (15) and (16) we have

$$|L^+ - L^-| \le k \le L^+ + L^- \quad \text{and} \quad m_k = m^- + m^+. \tag{17}$$

On the other hand, for fixed values of k and m_k, these same conditions restrict the values which l^-, l^+, m^- and m^+ can take in the sums (14). For instance, if we set $k = L^+ + L^-$ and $m_k = L^+ + L^-$, i.e. their maximum values, eq. (14) reduces to just one term:

$$a^+_{L^+L^+} a^-_{L^-L^-} \langle\, L^+\, 0\,;\, L^-\, 0 \,|\, L^+ \,+\, L^-\, 0\,\rangle \sqrt{(2L^++1)(2L^-+1)} [2(L^+ + L^- + 1)] = 0, \tag{18}$$

because $\langle L^+ L^+;\, L^-\, L^- |\, L^+ + L^- L^+ + L^-\rangle = 1$. Because $a^+_{L^+L^+} \ne 0$ and the remaining Clebsch–Gordan coefficient is not zero, it follows that $a^-_{L^-L^-} = 0$.

If we set now $k = L^+ + L^-$ and $m_k = L^+ + L^- - 1$, only two terms are allowed in the sums (14), namely the combinations ($m^- = L^-$; $m^+ = L^+ - 1$) and ($m^- = L^- - 1$; $m^+ = L^+$). Replacing the coefficients $\langle L^+ m^+ ; L^- m^- \mid L^+ + L^- L^+ + L^- - 1\rangle$ by their values, we get

$$\sqrt{\frac{(2L^+ + 1)(2L^- + 1)}{L^+ + L^-}}[2(L^+ + L^- + 1)]\langle L^+ 0 ; L^- 0 \mid L^+ + L^- 0\rangle$$
$$\times \left(\sqrt{L^-} a^+_{L^+ L^+} a^-_{L^- L^- - 1} + \sqrt{L^+} a^+_{L^+ L^+ - 1} a^-_{L^- L^-}\right) = 0. \tag{19}$$

Because $a^-_{L^- L^-} = 0$ and $a^+_{L^+ L^+} \neq 0$, we have $a^-_{L^- L^- - 1} = 0$. In general, for $m_k = L^+ + L^- - n$, $n = 0, \ldots, L^-$ we have $n + 1$ terms in the sums (14) or $2L^+ + 1$, whichever is smaller. If we have found the previous $a^-_{L^- m^-}$, $m^- = L^-, \ldots, L^- - n + 1$ coefficients to be zero, then, because $a^+_{L^+ L^+} \neq 0$, $a^-_{L^- L^- - n} = 0$. So, by induction, we have proven that $a^-_{L^- m^-} = 0$, $m^- = L^-, L^- - 1, \ldots, 0$. However, then by the symmetry property (3), this means that $a^-_{L^- m^-}$ is zero for all m^- values. This result still holds if some of the coefficients $a^+_{L^+ m^+}$ are zero.

In turn, this means that $L^- - 1$ will be then the maximum value of l^- and the previous argument could be repeated all over again. Thus, we have proved that, if $\mathbf{B}^+$ and $\mathbf{B}^-$ are given by finite expansions and if $\mathbf{B}^+ \neq 0$, then $\mathbf{B}^- = 0$ on the surface S and thus for $r > 1$. Of course, we could have set $\mathbf{B}^- \neq 0$ and then concluded that $\mathbf{B}^+ = 0$. This is another proof of the Backus result (Backus 1968), which states that, if the intensity of the harmonic magnetic field, given by a finite multipole expansion, is known over a spherical surface, then the magnetic field in the space exterior to that surface is determined up to a sign.

3.2 The Backus series

Eq. (14) provides a general formula for deriving series of the Backus type where fields differ by any set of multipoles. Indeed, the series obtained in Backus (1970) follows immediately assuming that a^-_{10} is a constant and all other coefficients $a^-_{l^- m^-}$ are zero. Then, removing the constant factors, eq. (14) becomes

$$\sum_{l^+, m^+} \sqrt{(2l^+ + 1)}(3 + l^+ + k)(2 + l^+ - k)\langle l^+ 0 ; 1 0 \mid k 0\rangle \langle l^+ m^+ ; 1 0 \mid k m_k\rangle a^+_{l^+ m^+} = 0. \tag{20}$$

According to eq. (15) and (a permutation of) eq. (16), one has $m^+ = m_k$ and $|k - 1| \leq l^+ \leq k + 1$. Also, as stated before, the Clebsch–Gordan $\langle l^+ 0; 1 0 \mid k0\rangle$ is only non-zero if $1 + l^+ + k$ is even, which reduces the possible values of l^+ for a given k value to two: $l^+ = k - 1$ and $l^+ = k + 1$. With the requirement (see above) $m_k \leq k - 1$ the previous equation reads

$$2\sqrt{(2k - 1)}(k + 1)\langle k - 1 0 ; 1 0 \mid k 0\rangle \langle k - 1\, m_k ; 1 0 \mid k\, m_k\rangle a^+_{k-1\, m_k}$$
$$+ 6\sqrt{(2k + 3)}(k + 2)\langle k + 1 0 ; 1 0 \mid k 0\rangle \langle k + 1\, m_k ; 1 0 \mid k\, m_k\rangle a^+_{k+1\, m_k} = 0. \tag{21}$$

If we set $m_k = k$ in eq. (20) and because $m^+ \leq l^+$, we have, from the conditions stated before, $a^+_{k+1\,k} = 0$. It then follows from eq. (21) that $a^+_{k+2p+1\,k} = 0$, with p a positive integer. On the other hand, setting $n = k - 1$ and $m = m_k$ in eq. (21), with $n + m$ even and replacing the Clebsch–Gordan coefficients by their values, we get the following recursion relation

$$a^+_{n+2\,m} = -\frac{1}{3}\frac{(n + 2)}{(n + 3)}\sqrt{\frac{(2n + 5)(n - m + 1)(n + m + 1)}{(2n + 1)(n - m + 2)(n + m + 2)}}\, a^+_{nm}. \tag{22}$$

The series formed by these coefficients is precisely the Backus series given in Backus (1970), where it was derived using the recurrence relations for Legendre associated functions. Note that, because n must be greater or equal to 1, so must m. From eq. (20) and the fact that there is no monopole term in the geomagnetic field expansion ($l^+ \geq 1$) one sees that all the coefficients a^+_{n0} are zero, i.e. the axisymmetric $\mathbf{B}^+$ field does not exist.

3.3 Quadrupole and more involved series

Clearly, the Backus series is the simplest series one can derive from eq. (14). For example, if the field $\mathbf{B}^-$ is a zonal quadrupole, i.e. a^-_{20} is a constant and all other coefficients $a^-_{l^- m^-}$ are zero, one gets

$$\sum_{\ell^+, m^+} \sqrt{\frac{2\ell^+ + 1}{2k + 1}}\sqrt{(\ell^+ + k + 4)(\ell^+ + k + 5)(\ell^+ - k + 3)(\ell^+ - k + 4)}$$
$$\times\langle \ell + 1 0 ; 3 0 \mid k 0\rangle\langle \ell^+ m^+ ; 2 0 \mid k\, m_k\rangle a^+_{\ell^+ m^+} = 0, \tag{23}$$

which yields a three-term recursion relation

$$a^+_{n+4\,m} = -\frac{2(n + 4)}{5(n + 5)}\frac{[(n + 2)(n + 3) - 3m^2]\sqrt{(2n + 5)(2n + 9)}}{(2n + 3)\sqrt{(n - m + 3)(n - m + 4)(n + m + 3)(n + m + 4)}}\, a^+_{n+2\,m}$$
$$- \frac{1}{5}\frac{(n + 3)(2n + 7)}{(n + 5)(2n + 3)}\sqrt{\frac{(2n + 9)}{(2n + 1)}}$$
$$\times\sqrt{\frac{(n - m + 1)(n - m + 2)(n + m + 1)(n + m + 2)}{(n - m + 3)(n - m + 4)(n + m + 3)(n + m + 4)}}\, a^+_{n\,m}. \tag{24}$$

As in the dipole case, there are two independent series for $n+m$ even and $n+m$ odd, but now the $n+m$ odd sequence terms are not necessarily zero. These series are initiated by $a^+_{m\,m}$ and $a^+_{m+1\,m}$, respectively. In Appendix A we give a proof of the convergence of the series generated by eq. (24).

In a similar fashion as in the case of the dipole orthogonal field, one can show, using eq. (23) and the properties of the Clebsch–Gordan coefficients, that when $n+m$ is even, there is no axisymmetric ($m=0$) field, whereas when $n+m$ is odd such a field exists.

Interestingly, similar series and results for $m=0$ are obtained by Kaiser & Neudert (2004) for the solution of the problem of determining the external geomagnetic field from a direction field on the surface S, when this field is an axisymmetric dipole or quadrupole.

In general, if the difference between $\mathbf{B}_1$ and $\mathbf{B}_2$ is a multipole of order l (with $m^-=0$), then by the properties of Clebsch–Gordan coefficients it follows that there are two series associated to a fixed value of $m^+=m$. The recursion relation defining the series involves the $l+1$ coefficients $a^+_{n\,m}, a^+_{n+2\,m} \dots a^+_{n+2l\,m}$.

One further application tested by us was a linear combination of dipole and quadrupole terms for $\mathbf{B}^-$, by setting $a^-_{1\,0}=\lambda_1$ and $a^-_{2\,0}=\lambda_2$ as the only non-zero coefficients. This gives rise to a five-term recursion relation. However, in this case, the convergence of the resulting series is not guaranteed and we have verified that it depends on the ratios λ_1/λ_2 and $a^+_{m\,m}/a^+_{m+1\,m}$.

Naturally, one could think of designing more complicated series from eq. (14). It suffices to set arbitrarily some finite expansion for $\mathbf{B}^+$ (or $\mathbf{B}^-$) in eq. (14) and, after some algebra, a recurrence relation for the $\mathbf{B}^-$ (or $\mathbf{B}^+$) infinite expansion coefficients will be established. Notice that a similar generalization based upon the recurrence relations for Legendre associated functions is much harder to achieve.

4 METHOD TO CALCULATE THE MAGNETIC FIELD FROM ITS MAGNITUDE

The proof of uniqueness given at the end of Section 3.1 suggests a way to calculate the expansion coefficients a_{lm} of a magnetic field if the coefficients A_{LM} of the $|\mathbf{B}|^2$ expansion in a spherical surface S are known from a previous fitting procedure. Indeed, based on eq. (12) one can write

$$|\mathbf{B}|^2 = \sum_{L,M} A_{LM} Y_{LM}(\theta,\phi), \tag{25}$$

which, given the orthogonality property (13), means that we have

$$A_{L\,M} = \sum_{l',m'}\sum_{l,m} \frac{1}{2}\sqrt{\frac{(2l'+1)(2l+1)}{4\pi(2L+1)}}(l'+l+L+2)(l'+l-L+1) \times \langle l'\,0\,;\,l\,0\,|\,L\,0\,\rangle\,\langle l'\,m'\,;\,l\,m\,|\,L\,M\,\rangle a_{l'\,m'}\,a_{l\,m}. \tag{26}$$

If $\mathbf{B}$ can be described by a finite series ($l, l' \le l_{\max}$) then, in the expansion of $|\mathbf{B}|^2$, L would have also a maximum value $L_{\max} = 2\,l_{\max}$, and we would have

$$A_{L_{\max}\,L_{\max}} = C(L_{\max})\,a^2_{l_{\max}\,l_{\max}}, \tag{27}$$

where $C(L_{\max})$ is a positive constant. In this way, up to a sign, one can determine $a_{l_{\max}\,l_{\max}}$. Following a similar reasoning as in Section 3.1, we set again $l=l'=l_{\max}$ in eq. (26) with decreasing values of M and get a series of linear equations for the unknown $a_{l_{\max}\,m}$ coefficients, which are solved in succession. If we set then $L=L_{\max}-1$ and $M=L_{\max}-1$, eq. (26) will have two terms with the combinations ($l=l_{\max}$, $m=l_{\max}$); ($l'=l_{\max}-1$, $m'=l_{\max}-1$) and ($l=l_{\max}-1$, $m=l_{\max}-1$); ($l'=l_{\max}$, $m'=l_{\max}$) and become a linear equation for $a_{l_{\max}-1\,l_{\max}-1}$. We can then proceed to find all the other $a_{l_{\max}-1\,m}$ coefficients by decreasing M as above and then repeat all over again the whole procedure to find all the remaining coefficients $a_{l\,m}$.

5 CONCLUSIONS AND OUTLOOK

We have generalized the example of Backus (1970) to illustrate the non-uniqueness of the magnetic field when only its magnitude is known on a spherical surface, using the theory of angular momentum coupling borrowed from quantum mechanics. This approach allowed us to extend the Backus result for infinite series that differ by a dipole field, the Backus series, to any infinite series differing by an arbitrary l-pole form or combinations thereof. This means that, given some $\mathbf{B}^-$ field consisting of only a finite number of harmonics, it is possible to solve eq. (14) for a field $\mathbf{B}^+$ (an infinite expansion of harmonics, necessarily) such that $\mathbf{B}^+\cdot\mathbf{B}^-=0$ on each point of a spherical surface. The convergence of such expansion depends on the choice of parameters in the linear combinations and has to be studied separately for each case. However, if a physically meaningful $\mathbf{B}^+$ exists, then the field $\mathbf{B}_1=\mathbf{B}^++\mathbf{B}^-$ shares the same intensity values on the sphere as the field $\mathbf{B}_2=\mathbf{B}^+-\mathbf{B}^-$. In this way, as a result of the general setting in which we have cast the problem, we can improve our knowledge of the space of magnetic fields sharing a Backus-type non-uniqueness.

As stands out from Section 3.3, $m=0$ series and odd $m+n$ series will be present in the $\mathbf{B}^+$ field if a quadrupole term will be included in $\mathbf{B}^-$. A natural application of our study is then an extension of the Stern *et al.* (1980) study. In this way and benefitting from high-quality satellite data now available, it will be possible to achieve a better understanding of the relation between the practical perpendicular error effect and the formal non-uniqueness treated in this paper.

For magnetic fields $\mathbf{B}$ given by truncated expansions, we propose a new way of calculating every coefficient of these expansions directly from the experimental data of its magnitude on some points of a surface S. The method requires a linear fit, in principle more stable and accurate numerically than the quadratic fit that is currently used. More important is the fact that, contrary to classical non-linear methods, it does not require an initial ansatz for the field, sufficiently close so as to avoid secondary minima. An algorithm derived from this method is being implemented for field modelling, where we intend to use high-quality satellite data. Results will be reported in a future publication.

In recent satellite missions, the occasional inaccuracy in the estimated attitude of the spacecraft has been an important source of errors affecting vectorial data. In those and other cases, where use of scalar data is inevitable, we believe that the formalism developed here will help to shed some light on the fundamental non-uniqueness associated to these problems.

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APPENDIX A: PROOF OF THE CONVERGENCE OF THE QUADRUPOLE SERIES

The quadrupole series for the potential V^+ on the unit radius surface *S*, from which the field $\mathbf{B}^+$ is derived, is written, in the notation of Section 3.3, as

$$V^+(\theta, \phi) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} a^+_{n\,m}\, Y_{n\,m}(\theta\,,\,\phi), \tag{A1}$$

where the coefficients $a^+_{n\,m}$ satisfy the recurrence relation (24). From the spherical harmonic addition theorem (see, e.g. Edmonds 1996), one has

$$\sum_{m=-n}^{n} |Y_{n\,m}(\theta\,,\,\phi)|^2 = \frac{2n+1}{4\pi} P_n(1) = \frac{2n+1}{4\pi}, \tag{A2}$$

where $P_n(\cos\theta)$ denotes the Legendre polynomials of degree *n*. From this expression it follows that, for any value of θ and ϕ (Backus 1970),

$$|Y_{n\,m}(\theta\,,\,\phi)| \le \sqrt{\frac{2n+1}{4\pi}} \Rightarrow V^+(\theta, \phi) \le \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \sqrt{\frac{2n+1}{4\pi}}\, |a^+_{n\,m}|. \tag{A3}$$

According to the Weierstrass majorant test (see, e.g. Arkfen & Weber 2001), to prove that the series (A1) is uniformly convergent for all (θ, ϕ), it is sufficient to prove that the numerical series $\sum_{n=1}^{\infty} \sum_{m=-n}^{n} u_{n\,m}$ is convergent, where

$$u_{n\,m} = \sqrt{\frac{2n+1}{4\pi}}\, |a^+_{n\,m}|. \tag{A4}$$

Actually, as we have seen in Section 3.3, we are interested in a series with a fixed value of *m*, with $n+m$ even or odd, so that the series involves only the sum in *n*, from $n = |m|$ to infinity in steps of two.

Writing the recurrence relation (24) as $a^+_{n+4\,m} = A_{n\,m} a^+_{n+2m} + B_{n\,m} a^+_{n\,m}$, we see that the coefficients A_{nm} and B_{nm} have the asymptotic values $-\frac{2}{5}$ and $-\frac{1}{5}$, respectively, when $n \to \infty$. A Taylor series expansion of these coefficients in powers of $\frac{1}{n}$ reveals that the first order correction is positive, meaning that their absolute values $|A_{n\,m}|$ and $|B_{n\,m}|$ go to the limits $\frac{2}{5}$ and $\frac{1}{5}$ by lesser values, respectively. This means that there is an order N such that, for $n \geq N$ and a given m,

$$\left|a^+_{n+4\,m}\right| \leq |A_{n\,m}|\left|a^+_{n+2\,m}\right| + |B_{n\,m}|\left|a^+_{n\,m}\right| < a'_{n+4} = \frac{2}{5}\left|a^+_{n+2\,m}\right| + \frac{1}{5}\left|a^+_{n\,m}\right| \qquad n = N, N+2, \ldots, \tag{A5}$$

where the index m was dropped in the definition of the coefficients a', because, as we will see later, its limiting value is independent of m. We now show that the positive series generated by the recurrence relation $a'_{n+4} = \frac{2}{5}\, a'_{n+2} + \frac{1}{5}\, a'_n$, $\;n = N, N+2, \ldots,$ where $a'_N \equiv |a^+_{Nm}|$ and $a'_{N+2} \equiv |a^+_{N+2m}|$, is convergent.

Defining $x_0 = a'_N/(5\, a'_{N+2})$, it is easily shown that the relation

$$a'_{N+2i+4} = \left(\frac{2}{5} + x_i\right) a'_{N+2i+2} \qquad i = 0, 1, 2, \ldots, \tag{A6}$$

holds, where $x_{i+1} = 1/(2 + 5\,x_i) \quad i = 0, 1, \ldots$. The sequence of values x_i converges rapidly to a value $x^* = (\sqrt{6} - 1)/5$, which is the positive root of the equation $5x^{*\,2} + 2\,x^* - 1 = 0$. This limit value is independent of m. Considering now the series with general term $s_i = a'_{N+2i}\sqrt{(2N + 4i + 1)/(4\pi)}\; i = 0, 1, \ldots$ (see eq. A4), we have

$$\lim_{i\to\infty} \frac{s_{i+1}}{s_i} = \lim_{i\to\infty} \frac{a'_{N+2i+2}}{a'_{N+2i}} = \frac{\sqrt{6}+1}{5} < 1. \tag{A7}$$

Therefore, from the Cauchy ratio test (Arkfen & Weber 2001) the series $\sum_{i=0}^{\infty} s_i$ converges and, from eq. (A5), so does the series $\sum_{n=N}^{\infty} u_{n\,m}$, by the comparison test. In turn, this implies that the whole series $\sum_{n=|m|}^{\infty} u_{n\,m}$ converge and, as stated before, so does $V^+(\theta, \phi)$, which completes the proof. As shown in Backus (1970) this convergence implies the convergence of the corresponding magnetic field $\mathbf{B}^+$ on the surface S and in the space exterior to it.