Impact of inner core rotation on outer core flow: the role of outer core viscosity

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SUMMARY

The viscous flow in the outer core induced by inner core rotation relative to the mantle is computed under the geostrophic approximation for a simple Earth model having a homogeneous, incompressible and viscous outer core. The mantle, the inner core and their interfaces with the outer core are taken to be axially symmetric ellipsoids and the mantle is considered to be in uniform rotation about its symmetry axis. The inner core rotation relative to the mantle involves, in general, both a super-rotation and an inner core wobble (ICW). In the former, the component of the inner core angular velocity parallel to the mantle symmetry axis is higher than the angular velocity of rotation of the mantle itself; in the ICW, the symmetry axis of the inner core is inclined to that of the mantle and rotates around the latter. In both cases, the outer core flow outside the cylinder circumscribing the inner core and having its axis parallel to the mantle rotation axis is equivalent to a rigid rotation with the same angular velocity as that of the mantle. In the bulk of the region within the cylinder, the flow relative to the mantle is again a rotation around the mantle rotation axis, but with the angular rate varying continuously with distance from the rotation axis: from the rate of super-rotation itself at the surface of the cylinder, down to half that rate on the mantle rotation axis. The flows in the Ekman boundary layers at the inner core boundary (ICB) and at the core–mantle boundary (CMB) are also determined. The viscous torque exerted on the inner core is derived from the velocity field in the boundary layer, and its damping effects on the inner core super-rotation and the ICW are investigated. As various estimates found in the literature for the outer core viscosity differ by many orders of magnitude, numerical values of the Ekman depth and the damping time of inner core rotation are computed here for a few sample values.

Key words: inner core rotation, inner core wobble, outer core flow, super-rotations, viscosity.

1 INTRODUCTION

The theory of Greenspan (1969) is applied to compute the viscous flow of the outer core caused by possible rotation of the inner core relative to the mantle. Two cases of inner core rotation on which the theory may be applicable, the inner core super-rotation and the the inner core wobble (ICW), are studied. The viscous reaction of the outer core flow on the inner core rotation is also investigated.

The problems studied in this paper are different from the coupling problem of precession–nutation theory in the magnitudes of the periods involved. In the coupling problem of precession–nutation theory, the periods in Earth fixed frame being nearly diurnal, the flow in the bulk of the outer core is customarily assumed to be a rotation with an angular velocity slightly different from that of the mantle (Mathews *et al.* 1991a,b; Buffett *et al.* 2002; Mathews *et al.* 2002). However, in the problems studied in this paper, the periods in the Earth fixed frame being much longer than a day, the flow in the bulk of the outer core, which represents the balance between the Coriolis force and the gradient of the disturbing pressure, is practically governed by the boundary layer—bulk interaction (Greenspan 1969).

Evidence of super-rotation of $1^\circ\ \mathrm{yr}^{-1}$ has been claimed on the basis of results from the inversion of seismic wave data (Song & Richards 1996). Poupinet *et al.* (2000) and Souriau & Poupinet (2000) have argued that the presently available seismic data do not support it, but can neither exclude it, their estimates for the magnitude of the super-rotation being smaller than the uncertainties in the estimates. In brief, the existence and amplitude of the inner core super-rotation is quite controversial (Poupitet & Souriau 2001; Song 2001). Buffett (1996a,b) argued that if the inner core is rigid, it would be phase locked to the mantle by the gravitational torque arising from the density anomalies in the inner core and in the mantle, so that no super-rotation would be possible. Buffett (1997) concluded, however, that super-rotation can exist if the inner core is viscous and he estimated the viscosity of the inner core assuming the super-rotation rate given by Song & Richards (1996). Although the inner core super-rotation is present in some geodynamo models, more recent investigation shows that the gravitational coupling between the inner core and the mantle induces almost locking of the inner core with respect to the mantle (Buffett & Glatzmaier 2000).

The ICW is a rotational normal mode emerged from theoretical developments in the theory of nutations of the Earth. It is a mode in which the figure axis of the inner core is tilted and executes a regular precession relative to the mantle with a period of 6.6 yr. It is the result of the action of gravitation and pressure in the inner core—outer core—mantle system. Mathews *et al.* (1991a,b), Dehant *et al.* (1993) and Mathews *et al.* (2002) investigated the ICW for an elastic inner core by using the linearized equations for the coupled rotational motions of the inner core, outer core and mantle. Xu & Szeto (1998) studied the ICW by numerically integrating the equations of rotation in terms of the Eulerien angles. Greff-Lefftz *et al.* (2000) investigated the influence of inner core viscosity on the ICW. Guo & Ning (2002) studied the influence of inner core obliquity (i.e. the inclination of the symmetry axis of the inner core to the axis of the mantle, which may be understood as the amplitude of the ICW) and that of super-rotation, on the ICW itself, for a simple Earth model in which both the inner core and mantle are assumed to be rigid and the mantle is assumed to be rotating uniformly in the space. Dumberry & Bloxham (2002) studied another wobble of the inner core arising from the electromagnetic coupling at the inner core boundary (ICB). In this paper, the ICW with the influence of inner core obliquity and super-rotation is considered, as that in Guo & Ning (2002).

Busse (1970) studied the flow in a homogeneous, incompressible and inviscid outer core caused by the oscillatory rotation of the mantle and the inner core, and computed the inertial torque acting on the inner core as a result of outer core flow. This work was followed by Kakuta *et al.* (1975) and Smylie *et al.* (1984). In this paper, we emphasize the effect of viscosity. The influence of inner core rotation and outer core flow on mantle rotation is neglected, because it is shown to be small by Mathews *et al.* (1991a,b) using the angular momentum equations of the whole Earth, the outer core and the inner core. So the mantle is assumed to be rotating uniformly in the space.

The theory of Greenspan (1969) assumes the outer core to be homogeneous and incompressible. The real outer core is more complex. Its density varies from 9.9 to 12.2 g cm^{-3} from the core–mantle boundary (CMB) to the ICB (Dziewonski & Anderson 1981). The outer core viscosity is one of the most poorly determined of the geophysical parameters: estimates obtained by different methods range over more than ten orders of magnitude, the lower values being obtained from laboratory studies of liquid metals and the higher ones from geophysical observations. Lumb & Aldridge (1991) tried to reconcile the differences by introducing the effective or eddy viscosity, commonly considered in atmospheric dynamics, which may have different values for different physical phenomena. It is not unlikely that viscosity may vary to some extent in the outer core (Rutter *et al.* 2002), though no commonly accepted estimate is available at present. The outer core flow is also affected by the electromagnetic field. The consideration of any of these complexities may make the problem unsolvable analytically. So, as a first attempt, a simplified outer core is considered and the electromagnetic effect is not included. It is expected that the analytical solution for the simplified model would be of help in testing numerical codes for more realistic models. Although the gravitational effect was not included in the original formulation of Greenspan (1969), it is included in this paper by a suitable choice of variables.

In Section 2, we first build an outer core flow model assuming a more general inner core rotation composed of a regular precession of the inner core figure axis relative to the mantle and a rotation of the inner core around its own figure axis. The angular rates, relative to the mantle, of both these components of the inner core rotation are assumed to be small compared with the rotation rate of the mantle itself. We apply the results from this model to the inner core super-rotation and ICW. We first compute, in Section 3, the viscous torque exerted on the inner core by the boundary layer flow obtained in Section 2, and then investigate the viscous drag effect on the ICW and the inner core super-rotation. Finally, in Section 4, we discuss the results obtained in the preceding two sections.

In our formulation, both the ICB and the CMB are assumed to be ellipsoids of revolution. Numerical computation is performed using the preliminary reference Earth model (PREM) (Dziewonski & Anderson 1981), with several alternative values of viscosity, estimated from laboratory studies of liquid metals (Gans 1972) or from geophysical measurements (Lumb & Aldridge 1991; Smylie & McMillan 2000; Molodensky & Groten 2001).

2 THE FLOW IN THE OUTER CORE

2.1 Equations of motion

We denote the obliquity of the inner core by ϵ and the common centre of the inner core and the mantle by O. We introduce three reference frames: $Ox_0y_0z_0$, fixed to the mantle, having z_0 on the figure axis of the mantle; $Oxyz$, such that x is on the intersection of the equators of the inner core and the mantle and z is on the figure axis of the mantle; and $Ox'y'z'$, with x' is on the intersection line of the equators of the inner core and the mantle, and z' on the figure axis of the inner core. See Fig. 1. The unit vectors along the axes are denoted by: $\mathbf{e}_{x_0}$, $\mathbf{e}_{y_0}$, $\mathbf{e}_{z_0}$; $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$; and $\mathbf{e}_{x'}$, $\mathbf{e}_{y'}$, $\mathbf{e}_{z'}$.

We use $\Omega_0\,\mathbf{e}_z$ to denote the angular velocity of the mantle relative to inertial space, i.e. the angular velocity of the Earth. As ϵ is assumed to be constant at the present stage, we can express the angular velocity of the inner core relative to the mantle as $\omega_{\rm p}\mathbf{e}_z + \omega_0\mathbf{e}_{z'}$, the first term being the angular velocity of the frame $Ox'y'z'$ relative to the mantle and the second term being the angular velocity of the inner core relative

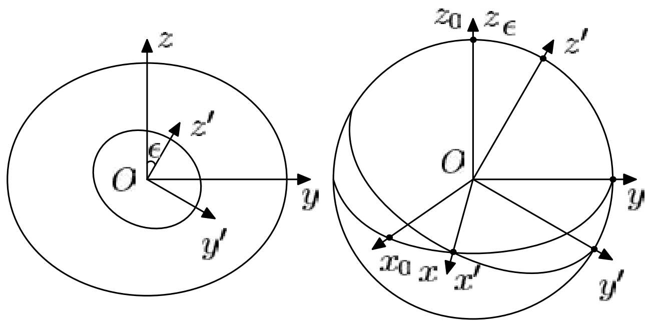

Figure 1. Coordinate systems.

to the frame *Ox′y′z′*. (It is evident that there is no relative rotation between *Oxyz* and *Ox′y′z′*.) It is clear that ω_p is the angular rate of precession of the figure axis of the inner core relative to the mantle and ω_0 is the angular rate of the inner core rotation around its own figure axis relative to the frame *Oxyz* or *Ox′y′z′*. Both ω_p and ω_0 are assumed constant in the theory. We can see that $\Omega = \Omega_0 + \omega_\mathrm{p}$ is the angular rate of precession of the figure axis of the inner core relative to inertial space and that the angular velocity of the frame *Oxyz* relative to inertial space is $\Omega\,\mathbf{e}_z$.

It is simplest to study the outer core flow in the frame *Oxyz*, because both the inner core and mantle axes do not move with respect to this frame and thus both the inner core and mantle rotation with respect to this frame are around their own axes. As consequences of this fact, the equations of both the ICB and the CMB, which are assumed to be ellipsoids of revolution, are independent of time in this frame, and the velocity of any point at the inner core surface with respect to this frame is tangential to the ICB and the velocity of any point at the mantle bottom with respect to this frame is tangential to the CMB.

The linearized equations of motion of an incompressible fluid in a rotating frame are well known:

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + 2\boldsymbol{\Omega}\times\mathbf{v}\right) = -\nabla p - \rho\nabla\phi + \eta\Delta\mathbf{v}, \tag{1}$$

$$\nabla\cdot\mathbf{v} = 0, \tag{2}$$

where $\boldsymbol{\Omega} = \Omega\,\mathbf{e}_z$ is the angular velocity of the frame *Oxyz* relative to inertial space, ρ is the fluid density, $\mathbf{v}$ is the velocity field describing the flow in the outer core in the frame *Oxyz*, p is the pressure, ϕ is the gravity potential and η is the dynamic viscosity. The gravity potential ϕ is the sum of the gravitational potential ϕ_g and centrifugal potential ϕ_c: $\phi = \phi_g + \phi_\mathrm{c}$. The outer core is assumed homogeneous and incompressible here, i.e. ρ and η are constant. So we can separate p into two parts as in Guo & Ning (2002):

$$p = p_s + p_v, \tag{3}$$

p_s being independent of $\mathbf{v}$ and defined by

$$\nabla p_s = -\rho\nabla\phi. \tag{4}$$

Eq. (1) can then be simplified to an equation for $\mathbf{v}$ that involves only the part p_v of the pressure:

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + 2\boldsymbol{\Omega}\times\mathbf{v}\right) = -\nabla p_v + \eta\Delta\mathbf{v}. \tag{5}$$

The equations governing the outer core flow are now eqs (2) and (5). If the assumptions of homogeneity and incompressibility are removed, our development from eqs (1) to (5) implies that the variations of ρ and ϕ apart from hydrostatic equilibrium state are neglected. Xu & Szeto (1998) discussed the general case where variation in ρ and hence in ϕ are considered.

As the outer core is considered to be viscous, the boundary values of $\mathbf{v}$ at the ICB and CMB are just equal to the velocities of the inner core and mantle at the respective boundaries. They are obtained from the fact that the inner core and the mantle are rotating relative to the frame *Oxyz* with the angular velocities $\omega_0\,\mathbf{e}_{z'}$ and $-\,\omega_p\,\mathbf{e}_z$:

$$\mathbf{v}_\mathrm{ICB} = \omega_0\mathbf{e}_{z'}\times\mathbf{r}_0, \quad \mathbf{v}_\mathrm{CMB} = -\omega_p\mathbf{e}_z\times\mathbf{r}_0, \tag{6}$$

where $\mathbf{r}_0$ is the position vector of some point on the CMB or ICB (for simplicity, a common notation $\mathbf{r}_0$ is used for $\mathbf{r}_\mathrm{ICB}$ and $\mathbf{r}_\mathrm{CMB}$). We see that the velocities of points at the boundaries are tangential to the boundaries themselves, as stated above.

2.2 Greenspan's theory on contained flow

Greenspan (1969) has elegantly solved the problem of steady fluid flow in a rotating container caused by the tangential velocity of the boundary. We give here a brief summary of his theory. Notations are chosen for our convenience. The equations of motion are eqs (2) and (5), with the inertial term $\partial\mathbf{v}/\partial t$ in the latter dropped because the theory deals only with steady flow. (It is presumed that the boundary velocity is kept steady despite the viscous drag.) As both ω_p and ω_0 are assumed constant in our treatment, the steady flow criterion is satisfied.

It is convenient to scale the variables in the following manner:

$$\mathbf{r} \to L\mathbf{r},\, t \to \Omega^{-1}t,\, \mathbf{v} \to U\mathbf{v},\, \boldsymbol{\Omega} \to \Omega\mathbf{e}_z,\, p_v \to \Omega U L \rho P. \tag{7}$$

The equations of motion then reduce to

$$2\mathbf{e}_z \times \mathbf{v} = -\nabla P + E\Delta\mathbf{v},\, \nabla\cdot\mathbf{v} = 0, \tag{8}$$

where E is the Ekman number defined by

$$E = \eta/(\Omega L^2 \rho). \tag{9}$$

We outline the solution of eq. (8) in following paragraphs, following section 2.17 of Greenspan (1969), presenting only the formulae needed for explaining the reasoning. The details may be found in Greenspan (1969) and Guo (1989).

Any line parallel to $\mathbf{e}_z$ intersects the boundary at two points, one on the upper boundary Σ_T and the other on the lower boundary Σ_B. The two boundaries are characterized as follows:

$$z = f(x, y) \quad \text{on} \quad \Sigma_\mathrm{T}; \quad z = -g(x, y) \quad \text{on} \quad \Sigma_\mathrm{B}. \tag{10}$$

For expressing the results, we need the normal vectors $\mathbf{n}_\mathrm{T}$ to Σ_T and $\mathbf{n}_\mathrm{B}$ to Σ_B (pointing outwards from the fluid core) at any given (x, y):

$$\mathbf{n}_\mathrm{T} = \mathbf{e}_z - \nabla f = [1 + (\nabla f)^2]^{1/2}\hat{\mathbf{n}}_\mathrm{T}, \quad \mathbf{n}_\mathrm{B} = -\mathbf{e}_z - \nabla g = [1 + (\nabla g)^2]^{1/2}\hat{\mathbf{n}}_\mathrm{B}, \tag{11}$$

where $\hat{\mathbf{n}}_\mathrm{T}$ and $\hat{\mathbf{n}}_\mathrm{B}$ are unit vectors.

The solution to eq. (8) is expressed in the form

$$\mathbf{v} = \mathbf{v}_0 + \tilde{\mathbf{v}}_0 + E^{1/2}\mathbf{v}_1 + E^{1/2}\tilde{\mathbf{v}}_1, \quad P = P_0 + \tilde{P}_0 + E^{1/2}P_1 + E^{1/2}\tilde{P}_1, \tag{12}$$

where the variables without a tilde represent the flow in the bulk and those with a tilde are boundary layer corrections, which are assumed to differ appreciably from zero only in the boundary layer.

As is customary in boundary layer analysis, we consider the derivatives of the tilde variables in directions tangential to the boundary to be negligibly small compared with the derivative in the normal direction and retain only the latter in the equations of motion. It is convenient to express the normal part of the gradient in terms of a variable ζ, which represents the scaled normal distance from the boundary, scaled also by the dimensionless Ekman depth $E^{1/2}$:

$$\zeta = E^{-1/2}\hat{\mathbf{n}}\cdot(\mathbf{r}_0 - \mathbf{r}), \tag{13}$$

where $\mathbf{r}_0$ is a point on the boundary, $\mathbf{r}$ is within the boundary layer and $(\mathbf{r} - \mathbf{r}_0)$ is in the same direction as $\hat{\mathbf{n}}$. Then we can write

$$\nabla = -\hat{\mathbf{n}}E^{-1/2}\frac{\partial}{\partial\zeta} \tag{14}$$

when applied to quantities with a tilde, which represent boundary layer flow. This expression implies that the derivative of the variables with a tilde, in the direction normal to the boundary, is expected to be of the order of $E^{-1/2}$. Eq. (14) is not applicable to the derivative of bulk flow quantities (without tilde), which vary slowly over much larger length scales.

Substituting eq. (12) into the equations of motion and the boundary conditions of the problem, we decompose them into parts that are of different orders in $E^{1/2}$. We obtain thus a sequence of problems for the flow in the bulk of the core, flow in the boundary layer and their interactions. The total flow is determined by solving this sequence of problems.

The equations involving terms of order $E^{-1/2}$ are

$$\frac{\partial}{\partial\zeta}\tilde{P}_0 = 0, \quad \frac{\partial}{\partial\zeta}(\hat{\mathbf{n}}\cdot\tilde{\mathbf{v}}_0) = 0, \tag{15}$$

with the boundary condition $\hat{\mathbf{n}}\cdot\tilde{\mathbf{v}}_0 = 0$. Because both $\tilde{P}_0$ and $\tilde{\mathbf{v}}_0$ should fall off to zero outside the boundary layer, it is evident that the solutions are

$$\tilde{P}_0 = 0,\, \hat{\mathbf{n}}\cdot\tilde{\mathbf{v}}_0 = 0. \tag{16}$$

The equations given by terms of order E^0 in the bulk are

$$2\mathbf{e}_z \times \mathbf{v}_0 = -\nabla P_0,\, \nabla\cdot\mathbf{v}_0 = 0, \tag{17}$$

with a boundary condition $\hat{\mathbf{n}}\cdot\mathbf{v}_0 = 0$. The solution for P_0 is, as shown by Greenspan (1969),

$$P_0 = P_0(h), \tag{18}$$

where $h = f + g$ is the height of the column of fluid at (x, y) from the lower surface, $z = -g$, to the upper, $z = f$, and the solution for $\mathbf{v}_0$ is

$$\mathbf{v}_0 = -\frac{1}{2}\left[\frac{\partial}{\partial h}P_0(h)\right]\mathbf{n}_\mathrm{T}\times\mathbf{n}_\mathrm{B}. \tag{19}$$

These solutions imply that the internal pressure, for given (x, y), is a function only of the height of the column at that point and is independent of z; the internal flow is columnar, the entire vertical column of fluid from the lower surface to the upper one moving as a unit, consistent with the Taylor–Proudman theorem.

Within the boundary layer, the equations of order E^0 are

$$2\mathbf{e}_z \times \tilde{\mathbf{v}}_0 - \hat{\mathbf{n}}\frac{\partial}{\partial \zeta}\tilde{P}_1 = \frac{\partial^2}{\partial \zeta^2}\tilde{\mathbf{v}}_0, \quad -\frac{\partial}{\partial \zeta}(\hat{\mathbf{n}} \cdot \tilde{\mathbf{v}}_1) + \nabla \times (\hat{\mathbf{n}} \times \tilde{\mathbf{v}}_0) = 0, \tag{20}$$

with the boundary condition $\mathbf{v}_0 + \tilde{\mathbf{v}}_0 = \mathbf{V}$, where $\mathbf{V}$ is the velocity of the solid side of the boundary scaled according to eq. (7). The solution given by Greenspan (1969) for $\tilde{\mathbf{v}}_0$ is

$$\hat{\mathbf{n}} \times \tilde{\mathbf{v}}_0 + i\tilde{\mathbf{v}}_0 = -[\hat{\mathbf{n}} \times (\mathbf{v}_0 - \mathbf{V}) + i(\mathbf{v}_0 - \mathbf{V})]_{\zeta=0} e^{-(2i\mathbf{e}_z \cdot \hat{\mathbf{n}})^{1/2}\zeta}, \tag{21}$$

where the root is taken with the positive sign for the real part:

$$(2i\mathbf{e}_z \cdot \hat{\mathbf{n}})^{1/2} = \left(1 + i\frac{\mathbf{e}_z \cdot \hat{\mathbf{n}}}{|\mathbf{e}_z \cdot \hat{\mathbf{n}}|}\right)|\mathbf{e}_z \cdot \hat{\mathbf{n}}|^{1/2}. \tag{22}$$

Thus, we obtain (Guo 1989)

$$\tilde{\mathbf{v}}_0 = e^{-|\mathbf{e}_z \cdot \hat{\mathbf{n}}|^{1/2}\zeta}\left[-(\mathbf{v}_0 - \mathbf{V})_{\zeta=0}\cos(|\mathbf{e}_z \cdot \hat{\mathbf{n}}|^{1/2}\zeta) + \frac{\mathbf{e}_z \cdot \hat{\mathbf{n}}}{|\mathbf{e}_z \cdot \hat{\mathbf{n}}|}\hat{\mathbf{n}} \times (\mathbf{v}_0 - \mathbf{V})_{\zeta=0}\sin(|\mathbf{e}_z \cdot \hat{\mathbf{n}}|^{1/2}\zeta)\right]. \tag{23}$$

We see that the thickness of the Ekman layer is proportional to $|\mathbf{e}_z \cdot \hat{\mathbf{n}}|^{-1/2}$. This property will be discussed further at a later stage when applying the theory to outer core flow. Eq. (23) is the solution of the first equation in eq. (20). The second one may be solved to obtain $(\hat{\mathbf{n}} \cdot \tilde{\mathbf{v}}_1)$ by using eq. (21). Its value at $\zeta = 0$, which is all we need here, is found to be

$$(\hat{\mathbf{n}} \cdot \tilde{\mathbf{v}}_1)_{\zeta=0} = \frac{1}{2}\hat{\mathbf{n}} \cdot \nabla \times \left\{\left[\hat{\mathbf{n}} \times (\mathbf{v}_0 - \mathbf{V}) + \frac{\mathbf{e}_z \cdot \hat{\mathbf{n}}}{|\mathbf{e}_z \cdot \hat{\mathbf{n}}|}(\mathbf{v}_0 - \mathbf{V})\right]|\mathbf{e}_z \cdot \hat{\mathbf{n}}|^{-1/2}\right\}_{\zeta=0}. \tag{24}$$

The equations for the terms of order $E^{1/2}$, in the bulk of the fluid, are

$$2\mathbf{e}_z \times \mathbf{v}_1 = -\nabla P_1, \nabla \cdot \mathbf{v}_1 = 0, \tag{25}$$

with the boundary condition $\hat{\mathbf{n}} \cdot (\mathbf{v}_1 + \tilde{\mathbf{v}}_1) = 0$. The presence of both bulk and boundary layer variables (with and without tilde, respectively) in the boundary condition produces, in effect, a bulk—boundary layer interaction. Greenspan (1969) used the eqs (19), (24) and (25) to obtain the following explicit expression for $\partial P_0(h)/\partial h$:

$$-\frac{1}{2}\left[\frac{\partial}{\partial h}P_0(h)\right] = \frac{I}{J} \tag{26}$$

with

$$I = \int_C |\mathbf{n}_\mathrm{T} \times \mathbf{n}_\mathrm{B}|\left(|\mathbf{e}_z \cdot \hat{\mathbf{n}}_\mathrm{T}|^{-1/2} + |\mathbf{e}_z \cdot \hat{\mathbf{n}}_\mathrm{B}|^{-1/2}\right)dC, \tag{27}$$

$$J = \int_C \frac{\mathbf{n}_\mathrm{T} \times \mathbf{n}_\mathrm{B}}{|\mathbf{n}_\mathrm{T} \times \mathbf{n}_\mathrm{B}|} \cdot \left[(\mathbf{V}_\mathrm{T} + \hat{\mathbf{n}}_\mathrm{T} \times \mathbf{V}_\mathrm{T})|\mathbf{e}_z \cdot \hat{\mathbf{n}}_\mathrm{T}|^{-1/2} + (\mathbf{V}_\mathrm{B} - \hat{\mathbf{n}}_\mathrm{B} \times \mathbf{V}_\mathrm{B})|\mathbf{e}_z \cdot \hat{\mathbf{n}}_\mathrm{B}|^{-1/2}\right]dC, \tag{28}$$

where $\mathbf{V}_\mathrm{T}$ and $\mathbf{V}_\mathrm{B}$ are the values of $\mathbf{V}$ at the top and bottom boundaries, respectively, and C is the isoheight contour of the container, i.e. the contour along which the height $h = f + g$ from the lower boundary to the upper boundary is constant.

The velocity $\mathbf{v}_0$ in the bulk is given by eq. (19) together with eqs (26), (27), and (28). The boundary layer correction $\tilde{\mathbf{v}}_0$ is given by eq. (23).

The bulk flow $\mathbf{v}_1$ of order $E^{1/2}$ is an example of the consequence of the Ekman pumping: the flow $\tilde{\mathbf{v}}_0$ of order E^0 in the Ekman layer induces the flow $\tilde{\mathbf{v}}_1$ of order $E^{1/2}$ in the layer, which in turn drives the flow $\mathbf{v}_1$ of the same order in the bulk of the fluid. One sees readily that $\mathbf{v}_1$ is determined by eq. (25) together with the boundary condition $\hat{\mathbf{n}} \cdot (\mathbf{v}_1 + \tilde{\mathbf{v}}_1) = 0$, which links it to $\tilde{\mathbf{v}}_1$. Furthermore, $\hat{\mathbf{n}} \cdot \tilde{\mathbf{v}}_1$ is itself determined by the difference $\mathbf{v}_0 - \mathbf{V}$ at the boundary through eq. (24). The Ekman pumping is a process of boundary layer—bulk interaction. The resulting bulk flow $\mathbf{v}_1$ cannot be a rigid rotation as the angular velocities of the upper and lower boundaries (for example, the CMB and ICB discussed in detail in the next subsection) are different.

2.3 Application to outer core flow

In the theory of Greenspan (1969), any line parallel to the rotation axis should intersect the container at just two points, one at the upper boundary Σ_T and one at the lower boundary Σ_B. Nevertheless, we can use the theory to determine the core flow even when the inner core is present, by dividing the outer core into three regions by the inner core surface and the cylindrical surface parallel to $\mathbf{e}_z$ touching the inner core, as shown in Fig. 2, and then computing the velocity fields in these regions separately. The nature of the flow in the bulk of each region is such that any column of the fluid parallel to the rotation axis moves only along the isoheight contour (within that region) on which it is located, as may be seen from the formulae for $\mathbf{v}_0$.

The following parameters will appear in our expressions for the velocity fields: a and c, the semi-major and semi-minor axes, respectively, of the inner core; $f = (a - c)/a$, the ellipticity (we use geometrical flattening as measure of ellipticity) of the surface of the ICB; a_c and c_c, the semi-major and semi-minor axes, respectively, of the CMB; and $f_c = (a_c - c_c)/a_c$, the flattening of the CMB. We use a, Ω^{-1} and $a\bar{\omega} = \max(a\omega_0, a\omega_\mathrm{p})$ to scale length, time and velocity, respectively. The expression of the Ekman number is then

$$E = \eta/(\Omega a^2 \rho) = \nu/(\Omega a^2), \tag{29}$$

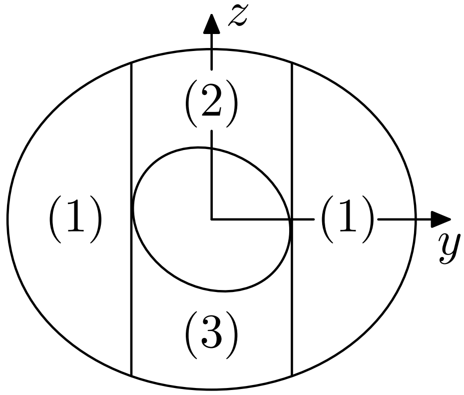

Figure 2. Three regions of the outer core.

where ν is the kinematic viscosity. In the computation of the internal flow $\mathbf{v}_0$, the terms of the first order in the ellipticities are taken into account in $\mathbf{n}_\mathrm{T} \times \mathbf{n}_\mathrm{T}$, but they are neglected when computing I and J, as the integrations in the expressions for I and J are quite complicated. Thus, the direction of the velocity is kept accurate to the first order in the ellipticities, but no ellipticity term is retained in the magnitude of the velocity. Ellipticity terms are neglected also in computing the viscosity correction $\tilde{\mathbf{v}}_0$ in the Ekman boundary layers. We are able, in this manner, to take into account the most relevant aspects of the physics of the process while obtaining analytical expressions for the quantities of interest.

We write out now the results obtained, in terms of the real (unscaled) physical variables. Further details may be found in Appendix A. In region (1), i.e. outside the cylindrical surface, the fluid rotates rigidly with the mantle. Within the region enclosed by the cylindrical surface, the velocity is

$$\mathbf{v}_0 = \frac{L_c L\left[\omega_p (a_c L)^{1/2} - \omega_0 \cos\epsilon (a L_c)^{1/2}\right]}{(L_c - L)\left[(a_c L)^{1/2} + (a L_c)^{1/2}\right]} \left[\left\{ -\frac{y}{L_c}(1-f_c) + \frac{y}{L'}\left[1 + 2f\left(1 - \frac{3}{2}\cos^2\epsilon\right)\right] + 2f(\mathrm{sign}z)\sin\epsilon\cos\epsilon \right\}\mathbf{e}_x + \left[\frac{x}{L_c}(1-f_c) - \frac{x}{L'}(1 - f\cos^2\epsilon)\right]\mathbf{e}_y + \left[-2f\frac{x}{L_c}\sin\epsilon\cos\epsilon - 2f\frac{xy}{L_c L'}(\mathrm{sign}z)\sin^2\epsilon\right]\mathbf{e}_z \right], \tag{30}$$

where

$$L_c = \left[a_c^2 - (x^2 + y^2)\right]^{1/2}, \tag{31}$$

$$L = [a^2 - (x^2 + y^2)]^{1/2}, \tag{32}$$

$$L' = \{a^2 - [x^2 + y^2(1 + 2f\sin^2\epsilon)]\}^{1/2}. \tag{33}$$

When all elliptic terms are neglected, eq. (30) simplifies to

$$\mathbf{v}_0 = -\frac{\omega_p (a_c L)^{1/2} - \omega_0 \cos\epsilon (a L_c)^{1/2}}{(a_c L)^{1/2} + (a L_c)^{1/2}}(-y\mathbf{e}_x + x\mathbf{e}_y). \tag{34}$$

As for the corrections as a result of viscosity in the boundary layer, we write them out in terms of spherical polar coordinates associated with the reference frame *Oxyz*. At the ICB, the boundary layer correction is

$$\tilde{\mathbf{v}}_0 = a e^{-r-a/\alpha}\left\{\left[\omega_1 \cos\left(\frac{r-a}{\alpha}\right) + (\mathrm{sign}z)\omega_2 \sin\left(\frac{r-a}{\alpha}\right)\right]\mathbf{e}_\theta + \left[\omega_2 \cos\left(\frac{r-a}{\alpha}\right) - (\mathrm{sign}z)\omega_1 \sin\left(\frac{r-a}{\alpha}\right)\right]\mathbf{e}_\lambda\right\}, \tag{35}$$

where

$$\alpha = a(E/|\cos\theta|)^{1/2} \tag{36}$$

is the Ekman depth, which is in fact independent of a according to the definition of E in eq. (29), and

$$\omega_1 = \omega_0 \sin\epsilon\cos\lambda, \tag{37}$$

$$\omega_2 = \frac{(\omega_p + \omega_0\cos\epsilon)\sin\theta|\cos\theta|^{1/2}}{|\cos\theta|^{1/2} + \left[1 - (a/a_c)^2\sin^2\theta\right]^{1/4}} - \omega_0\sin\epsilon\cos\theta\sin\lambda. \tag{38}$$

At the CMB, the boundary layer correction is

$$\tilde{\mathbf{v}}_0 = a_c e^{-r-a_c/\alpha}\gamma\left[-(\mathrm{sign}z)\sin\left(\frac{r-a_c}{\alpha}\right)\mathbf{e}_\theta - \cos\left(\frac{r-a_c}{\alpha}\right)\mathbf{e}_\lambda\right], \tag{39}$$

where

$$\gamma = \frac{(\omega_\mathrm{p} + \omega_0\cos\epsilon)\sin\theta|\cos\theta|^{1/2}}{|\cos\theta|^{1/2} + \left[1 - (a_c/a)^2\sin^2\theta\right]^{1/4}}. \tag{40}$$

Note that θ varies from *0* to arcsin(a/a_c) in region (2) and from $\pi - $ arcsin(a/a_c) to π in the region (3). It is evident that the Ekman depth $a(E/|\cos\theta|)^{1/2}$ is not constant over the ICB or the CMB. At the poles, where the ICB (or CMB) is perpendicular to the rotation axis, it is $aE^{1/2}$; but at the equator where the boundaries becomes parallel to the rotation axis, it becomes infinitely large. This is a reflection of the fact that the boundary layer at a surface parallel to the rotation axis is really a Stewartson layer (Stewartson 1966), which is characterized by a thickness of order $E^{1/4}$, thicker than the Eckman layer thickness of order $E^{1/2}$. Hence, our expression for boundary layer flow is not applicable in a small area near the equator.

In all the previous discussions, the velocity is expressed relative to the reference frame *Oxyz*. In the mantle fixed frame, $\omega_p \mathbf{e}_z \times \mathbf{r}$ has to be added to the expressions given above for the velocity $\mathbf{v}$ in the bulk of the fluid, while the boundary layer correction remains the same. Thus, the fluid velocity relative to the mantle in regions (2) and (3) inside the cylindrical surface mentioned becomes

$$\mathbf{v}_m = \omega_p \mathbf{e}_z \times \mathbf{r} + \mathbf{v}_0. \tag{41}$$

If all the ellipticity terms are neglected, it simplifies to

$$\mathbf{v}_m = \omega_m(-y\mathbf{e}_x + x\mathbf{e}_y), \tag{42}$$

where

$$\omega_m = \beta\delta\Omega, \tag{43}$$

$$\beta = (aL_c)^{1/2}/\left[(a_c L)^{1/2} + (aL_c)^{1/2}\right], \tag{44}$$

$$\delta\Omega = \omega_p + \omega_0 \cos\epsilon. \tag{45}$$

This clearly represents a rotation around the *Oz*-axis with the angular rate ω_m. It may be noted that $\delta\Omega$ is the projection of the inner core angular velocity relative to the mantle along the figure axis of the mantle, which is in fact the super-rotation rate of the inner core.

2.4 Numerical results for the outer core flow

The main result concerning the outer core flow is the velocity field in the bulk, which is given by the formulae (41) to (44) in the approximation neglecting ellipticity; they show the flow to be a rotation around the figure axis of the mantle. The angular velocity given by eq. (42) is a function of the distance $(x^2 + y^2)^{1/2}$ from the figure axis. It is determined by $\delta\Omega$ and β. The function β is plotted in the Fig. 3. It varies from 0.5 to 1 as the distance from the figure axis increases from 0 to a. However, in the most part of its definition domain it remains close to 0.5. Its overall average is 0.54.

For the pure super-rotation, the obliquity ϵ of the inner core is zero. Then $\omega_{\rm p}$ becomes irrelevant and may also be set equal to zero; and the super-rotation rate $\delta\Omega$ given in eq. (45) becomes ω_0. If we take the rate to be $\delta\Omega = 1^\circ$ yr^{-1} (Song & Richards 1996), the angular rate of the outer core flow would be continuously distributed in the interval between 0.5° and 1° yr^{-1}, the average being 0.54° yr^{-1}. Because the flow in the bulk depends only on the super-rotation rate $\delta\Omega$, the flow pattern would remain the same for all combinations of $\omega_{\rm p}$, ω_0 and ϵ that lead to the same super-rotation rate $\delta\Omega$ (which is equal to $\omega_p + \omega_0 \cos\epsilon$).

When inner core super-rotation is accompanying the ICW, the period of the ICW depends also on the super-rotation rate. Here we give the results for the particular values of super-rotation rate $\delta\Omega$ given in Guo & Ning (2002).

(i) For the ICW in the sense of Mathews *et al.* (1991a,b), which is not accompanied by a super-rotation, there is no flow in the bulk of the outer core.

(ii) If the figure axis of the inner core is assumed to be in the direction of the geomagnetic pole (Guo 1989), so that $\epsilon = 11.5^\circ$ and $\omega_{\rm p} = -0.05^\circ$ yr^{-1}, the corresponding numerical value of $\delta\Omega$ indicates a retrograde rotation of the inner core relative to the mantle with the period of 6.90 yr, as one sees on using the value 4.11×10^{-4} for $\delta N/(C\Omega_0^2)$ in eq. (63) of the next section. In this case, the flow in the interior of the outer core is also retrograde, the period being continuously distributed in the interval between 6.90 and 13.80 yr with an average of 12.78 yr.

(iii) Greiner-Mai & Barthelmes (2001) recovered a mean obliquity $\epsilon = 1^\circ$ and a mean eastward drift $\omega_{\rm p} = 0.7^\circ$ yr^{-1} of the inner core axis relative to the mantle by analysing polar motion data. If we interpret this inner core rotation as the ICW, the numerical value of $\delta\Omega$ indicates

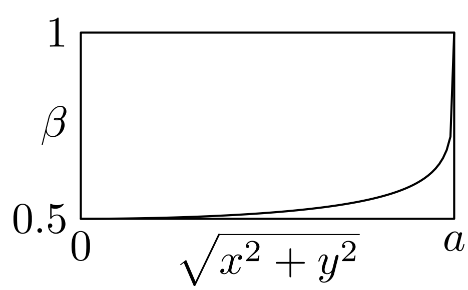

Figure 3. The function β depending on $(x^2 + y^2)^{1/2}$.

Table 1. Ekman number and Ekman depth at the ICB.

ν ($m^2\ s^{-1}$)	Author (s)	E	Ekman depth (m)
1.1×10^{-6}	Gans (1972)	1.0×10^{-14}	1.2×10^{-1}
3.0×10^{-3}	Molodensky & Groten (2001)	2.8×10^{-11}	6.4
2.0×10^{2}	Lumb & Aldridge (1991)	1.8×10^{-6}	1.6×10^{3}
1.0×10^{7}	Smylie & McMillan (2000)	9.2×10^{-2}	3.7×10^{5}

a retrograde rotation of the inner core relative to the mantle with the period of 6.73 yr, as may be seen also from eq. (63) in next section. The flow within the outer core is also retrograde, the period being continuously distributed in the interval between 6.73 and 13.46 yr with an average of 12.46 yr.

The boundary layer correction is confined to the interior of the thin Ekman layer, the thickness of the layer being of order $E^{1/2}$ in scaled length. We have evaluated E and the thickness $aE^{1/2}$ of the Ekman layer in physical units (dropping the factor $|\cos\theta|^{-1/2}$) using various estimates of viscosity near the ICB. The results are listed in Table 1 (when computing E, we take Ω to be Ω_0, their difference $\omega_{\rm p}$ being much smaller than Ω). We note that the viscosity estimate of Smylie & McMillan (2000) is very high, leading to a value of approximately 0.3 for $E^{1/2}$, which makes our boundary layer analysis inaccurate for this viscosity, because we neglect quantities that are smaller than of order $E^{1/2}$.

The viscous boundary layer flow is dissipative: so it causes damping of the inner core rotation. This is the subject of the next section.

2.5 Comparison with existing work

Busse (1970) studied the quasi-steady flow of a homogeneous, incompressible and inviscid outer core caused by oscillatory rotations of the inner core and the mantle, where the period of the flow is assumed to be much longer than 1 d. The inertial term being then much smaller than the Coriolis term, it was neglected in the momentum equation. We follow here the reformulated version of this problem from Kakuta *et al.* (1975). The frame ($\mathbf{I}_1$, $\mathbf{I}_2$, $\mathbf{I}_3$) is assumed to be rotating uniformly in space with angular velocity $\Omega_0\ \mathbf{I}_3$. The directions of the figure axes of the inner core and mantle are denoted by $\mathbf{I}_3 + \mathbf{i}'_r$ and $\mathbf{I}_3 + \mathbf{i}'_m$. The oscillatory rotations of the inner core and mantle are represented by the variations of $\mathbf{i}'_r$ and $\mathbf{i}'_m$ with time. The solution of the outer core flow velocity for the problem is (in the notation of Kakuta *et al.* 1975)

$$\mathbf{u} = \frac{1}{2\Omega_0}\mathbf{I}_3 \times \nabla\Pi + w\mathbf{I}_3 \tag{46}$$

with:

$$\Pi = \begin{cases} 4\Omega_0 \frac{z_m z_r}{z_m - z_r}(\mathbf{I}_3 \times \mathbf{r}) \cdot \left(e_m \frac{d\mathbf{i}'_m}{dt} - e_r \frac{d\mathbf{i}'_r}{dt}\right), & (0 \le s \le r_r); \\ 0, \quad (r_r \le s \le r_m); \end{cases} \tag{47}$$

$$w = \begin{cases} -\frac{2}{z_m - z_r}\mathbf{r} \cdot \left(z_m e_m \frac{d\mathbf{i}'_m}{dt} - z_r e_r \frac{d\mathbf{i}'_r}{dt}\right), & (0 \le s \le r_r); \\ -2e_m \mathbf{r} \cdot \frac{d\mathbf{i}'_m}{dt}, \quad (r_r \le s \le r_m); \end{cases} \tag{48}$$

where e_m and e_r are the ellipicities (geometrical flattenings) of the CMB and the ICB, respectively, r_m is the equatorial radius of the CMB and r_r is the equatorial radius of the ICB, which is also approximately the radius of the cylinder circumscribing the inner core, with its axis parallel to that of the mantle. s is the distance from the $\mathbf{I}_3$-axis and $z_m = (\operatorname{sign} z)(r_m^2 - s^2)^{1/2}$, $z_r = (\operatorname{sign} z)(r_r^2 - s^2)^{1/2}$.

Now we write out the inviscid flow given by eqs (46), (47) and (48) for the case of inner core rotation in our work using our notation. In our work, we assumed the mantle to be rotating uniformly in space. So the frame ($\mathbf{I}_1$, $\mathbf{I}_2$, $\mathbf{I}_3$) of Kakuta *et al.* (1975) is equivalent to our mantle-fixed frame, $\mathbf{I}_3$ being along the figure axis of the mantle. Keeping this in mind, we can write out the correspondence between the variables of Kakuta *et al.* (1975) and those of this paper: $r_r \Leftrightarrow a$, $r_m \Leftrightarrow a_c$, $e_r \Leftrightarrow f$, $e_m \Leftrightarrow f_c$, $\mathbf{I}_1 \Leftrightarrow \mathbf{e}_{x0}$, $\mathbf{I}_2 \Leftrightarrow \mathbf{e}_{y0}$, $\mathbf{I}_3 \Leftrightarrow \mathbf{e}_{z0}$ (or $\mathbf{e}_z$), $\mathbf{i}'_m \Leftrightarrow 0$, $\mathbf{i}'_r \Leftrightarrow \mathbf{e}_{z'} - \mathbf{e}_z$. It is evident that

$$\frac{d\mathbf{i}'_m}{dt} = 0. \tag{49}$$

In view of the fact that the frame $Ox'y'z'$ rotates with angular velocity $\omega_{\rm p}\ \mathbf{e}_z + \omega_0\ \mathbf{e}_{z'}$ with respect to the mantle, we have

$$\frac{d\mathbf{i}'_r}{dt} = \frac{d}{dt}(\mathbf{e}_{z'} - \mathbf{e}_z) = -\omega_{\rm p} \sin\epsilon\, \mathbf{e}_x. \tag{50}$$

Substituting eqs (49) and (50) into eqs (46), (47) and (48), we obtain the inviscid flow within the cylindrical region $s < r_r$ of the outer core:

$$\mathbf{u} = -(\operatorname{sign} z)\frac{2f\omega_{\rm p}\sin\epsilon}{L_c - L}\left\{\left[y^2\left(\frac{L}{L_c} + \frac{L_c}{L} + 1\right) - L_c L\right]\mathbf{e}_x - xy\left(\frac{L}{L_c} + \frac{L_c}{L} + 1\right)\mathbf{e}_y + (\operatorname{sign} z)xL\mathbf{e}_z\right\}. \tag{51}$$

This is the velocity of the outer core with respect to the mantle for the inviscid case according to Busse (1970) expressed using our notation. The fluid outside the cylindrical surface mentioned above rotates rigidly with the mantle. We see that the inviscid flow $\mathbf{u}$ is proportional to the inner core ellipticity and so there would be no flow at all if the inner core were spherical. This is easy to understand. For inviscid flow, the

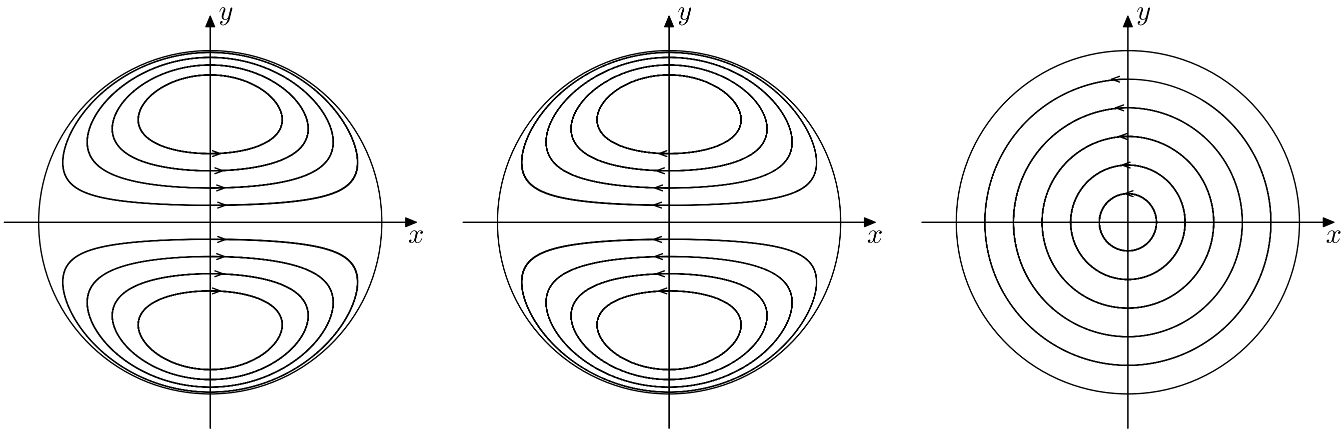

Figure 4. Patterns of flow in the outer core in the inviscid and viscous cases.

boundary condition is that the fluid velocity normal to the boundary is equal to the velocity of the boundary normal to itself and there is no constraint on the fluid velocity tangential to the boundary. For a spherical inner core, the surface velocity is always tangential to the surface, in whichever manner the inner core rotates around its centre. Thus, in this case, the inner core rotation does not influence the outer core flow at all. In fact, eq. (51) represents the outer core flow induced by the inner core surface velocity normal to the surface itself, which is caused by rotation and deviation from sphericity of the inner core. However, the presence of viscosity changes the pattern of the flow completely, as may be seen by a comparison of $\mathbf{v}_m$ of eqs (41) and (30) of the viscous case with $\mathbf{u}$ of eq. (51). Unlike $\mathbf{u}$, $\mathbf{v}_m$ does not vanish if the flattening f becomes zero. In fact, with f_c also set equal to zero, the flow becomes a rotation about the z-axis, as given by eqs (42) to (45). The difference between the flows in the inviscid and viscous cases is illustrated in Fig. 4. The flow patterns shown are in a plane z = constant. The first two are for the inviscid flow given by eq. (51) obtained according to Busse (1970) at a positive and a negative z, respectively; the third is for the viscous case (our eqs 42 to 45) for both a positive and a negative z. Notice that the inviscid flow velocity is proportional to $f\omega_{\mathrm{p}}\sin\epsilon$ and the viscous flow velocity is proportional to $\delta\Omega$. The viscous flow $\mathbf{v}_0$ is a consequence of the viscous drag through boundary layer—bulk interaction.

3 INFLUENCE OF THE FLOW ON INNER CORE ROTATION

3.1 Influence of the viscous torque on the inner core

For studying the influence of the outer core flow given in last section on inner core rotation, we first derive an expression for the viscous torque exerted by the outer core flow on the inner core. This torque can be readily written out (Guo & Ning 2002):

$$\mathbf{N}^\eta = \eta \int_S \mathbf{r} \times \mathbf{n} \cdot [\nabla\mathbf{v} + (\nabla\mathbf{v})^T]\, dS, \tag{52}$$

where $\mathbf{v}$ is the velocity field in the outer core with respect to the mantle fixed frame, S is the ICB and $\mathbf{n}$ is the outward unit normal of S. Using the spherical approximation to the ICB, which is sufficient for the present purposes, we have $\mathbf{n} = \mathbf{e}_r$ and $\mathbf{r} = a\,\mathbf{e}_r$. The total fluid velocity $\mathbf{v}$ in the outer core relative to the rotating mantle is $\mathbf{v} = \mathbf{v}_m + \tilde{\mathbf{v}}_0$, where $\tilde{\mathbf{v}}_0$ is given in eq. (35) on neglecting ellipticity terms. Both $\mathbf{v}_m$ and $\tilde{\mathbf{v}}_0$ are the same order of magnitude within the boundary layer. However, as $\tilde{\mathbf{v}}_0$ has a factor $e^{-\frac{r-a}{\alpha}} = e^{-\frac{E^{-1/2}}{a}|\cos\theta|^{1/2}(r-a)}$, $\partial\tilde{\mathbf{v}}_0/\partial r$ has a factor of order $E^{-1/2}$, which is quite large. So when computing $\mathbf{n}\cdot[\delta\mathbf{v} + (\delta\mathbf{v})^T]$, the contribution of $\mathbf{v}_m$ may be neglected as $\mathbf{v}_m$ varies significantly only over distances of the order of the dimension of the core. Ellipticity is ignored in the computation of $\mathbf{N}^\eta$: its contribution is only of order f, which is much smaller than the contribution of order $E^{-1/2}$ from $\tilde{\mathbf{v}}_0$. Thus, only $\tilde{\mathbf{v}}_0$ needs to be retained when computing $\mathbf{N}^\eta$ from eq. (52) and only the terms originating from $\partial\mathbf{v}/\partial r$ need to be considered. Then, expressing the gradient operator and the velocity in terms of their spherical components and retaining only the radial derivative terms, we obtain

$$\mathbf{N}^\eta = a\eta \int_S \left(\frac{\partial v_\theta}{\partial r}\mathbf{e}_\lambda - \frac{\partial v_\lambda}{\partial r}\mathbf{e}_\theta \right) dS. \tag{53}$$

Substituting for $\tilde{\mathbf{v}}_0$ from eq. (35) into eq. (53), retaining only the terms of order $E^{-1/2}$ (the other terms being much smaller) and then expressing $\mathbf{e}_\theta$, $\mathbf{e}_\lambda$ in terms of $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$, we obtain the Cartesian components of the viscous torque. Details of the computations are deferred to Appendix B. Here, we write out the results:

$$\begin{cases} N_x^\eta = \gamma_1(\omega_0/\Omega_0)\sin\epsilon, \\ N_y^\eta = -\gamma_2(\omega_0/\Omega_0)\sin\epsilon, \\ N_z^\eta = -\gamma_3(\omega_{\mathrm{p}} + \omega_0\cos\epsilon)/\Omega_0, \end{cases} \tag{54}$$

where

$$\begin{cases} \gamma_1 = \frac{8}{5}\pi\Omega_0^{3/2}a^4\rho\nu^{1/2} = \frac{8}{5}\pi\Omega_0^2 a^5\rho E^{1/2}, \\ \gamma_2 = \frac{40}{21}\pi\Omega_0^{3/2}a^4\rho\nu^{1/2} = \frac{40}{21}\pi\Omega_0^2 a^5\rho E^{1/2}, \\ \gamma_3 = 4Q\pi\Omega_0^{3/2}a^4\rho\nu^{1/2} = 4Q\pi\Omega_0^2 a^5\rho E^{1/2}, \end{cases} \tag{55}$$

with

$$Q = \int_0^{\pi/2} \frac{\sin^3\theta\cos\theta}{\cos^{1/2}\theta + \left[1-(a/a_c)^2\sin^2\theta\right]^{1/4}}d\theta = 0.15, \tag{56}$$

the numerical value being obtained by numerical integration. We see that γ_1, γ_2 and γ_3 have the dimensions of torque. Their numerical values, when various estimated values as shown in in Table 1 are used for the viscosity, are listed in Table 2.

The Cartesian components of the viscous torque along $\mathbf{e}_{x'}$, $\mathbf{e}_{y'}$, $\mathbf{e}_{z'}$ are readily found:

$$\begin{cases} N_{x'}^{\eta} = \gamma_1(\omega_0/\Omega_0)\sin\epsilon, \\ N_{y'}^{\eta} = -\gamma_2(\omega_0/\Omega_0)\sin\epsilon\cos\epsilon + \gamma_3[(\omega_{\rm p}+\omega_0\cos\epsilon)/\Omega_0]\sin\epsilon, \\ N_{z'}^{\eta} = -\gamma_2(\omega_0/\Omega_0)\sin^2\epsilon - \gamma_3[(\omega_{\rm p}+\omega_0\cos\epsilon)/\Omega_0]\cos\epsilon. \end{cases} \tag{57}$$

3.2 Influence on inner core rotation

Following the approach of Guo & Ning (2002), we use an inertial reference frame $O\xi\eta\zeta$ to express the equations governing the inner core rotation, ζ being defined in the direction of mantle axis. (Note the different meaning of ζ here from that of the last section.) The angle between the ξ and x axes is denoted by ψ. It is easy to see that ψ is related to Ω, Ω_0 and $\omega_{\rm p}$ in the following way:

$$\dot\psi = \Omega = \Omega_0 + \omega_{\rm p}. \tag{58}$$

According to eqs (5) and (6) of Guo & Ning (2002), the equations governing the inner core rotation in terms of ψ, ϵ, ω_0 are

$$\begin{cases} -A\frac{d\dot\epsilon}{dt} - [C(\omega_0+\dot\psi\cos\epsilon) - A\dot\psi\cos\epsilon]\dot\psi\sin\epsilon = N_{x'}, \\ -A\frac{d}{dt}(\dot\psi\sin\epsilon) + [C(\omega_0+\dot\psi\cos\epsilon) - A\dot\psi\cos\epsilon]\dot\epsilon = N_{y'}, \\ C\frac{d}{dt}(\omega_0+\dot\psi\cos\epsilon) = N_{z'}, \end{cases} \tag{59}$$

where $N_{x'}$, $N_{y'}$, $N_{z'}$ are the components of the total torque $\mathbf{N}$ acting on the inner core. It is to be noted that the coordinate system denoted by $Ox'y'z'$ here is called $Oxyz$ by Guo & Ning (2002). A and C are the principal moments of inertial of the inner core around its equatorial and polar axes, respectively.

Guo & Ning (2002) separated $\mathbf{N}$ into three parts: $\mathbf{N}^s$, $\mathbf{N}^v$, and $\mathbf{N}^\eta$, where $\mathbf{N}^s$ is the contribution of gravitation and the pressure p_s (defined in eq. 4), which is independent of the outer core flow, $\mathbf{N}^v$ is the contribution of the pressure p_v defined in eqs (3) and (4), which is related to the outer core flow, and $\mathbf{N}^\eta$ is the contribution of viscous friction related to the outer core flow. As $\mathbf{N}^s$ is predominant, Guo & Ning (2002) neglected the effect of $\mathbf{N}^v$ and $\mathbf{N}^\eta$ on the ICW.

The expression for $\mathbf{N}^s$ is (Guo & Ning 2002)

$$\mathbf{N}^s = \left[-\Omega_0^2(C-A) + \delta N\right]\sin\epsilon\cos\epsilon\,\mathbf{e}_{x'}, \tag{60}$$

$$\delta N = \frac{2}{5}[(C-A)-(C_0-A_0)]\left[\frac{GM}{a^3}(2f+af') + \frac{GM_0}{a^3}(3f)\right]. \tag{61}$$

Here, M and a are respectively the mass and semi-major axis of the inner core. f and f' are respectively the inner core ellipticity and its derivative with respect to the radius. A_0, C_0 and M_0 are respectively the principal moments of inertia and mass that the inner core would have if its density were equal to the outer core density at the ICB. δN represents the part of the torque on the inner core proportional to the density jump at the ICB. Its numerical value is 1.28×10^{23} Nm. The viscous torque $\mathbf{N}^\eta$ was computed in the last subsection. Here, we make some comments on the pressure torque $\mathbf{N}^v$ related to the pressure p_v caused by outer core flow (Guo & Ning 2002):

$$\mathbf{N}^v = -\oint_S \mathbf{r}\times(p_v\mathbf{n})\,dS, \tag{62}$$

Table 2. Viscous torque at the ICB.

ν (m^2 s^{-1})	γ_1 (Nm)	γ_2 (Nm)	γ_3 (Nm)
1.1×10^{-6}	8.8×10^{19}	1.1×10^{20}	3.3×10^{19}
3.0×10^{-3}	4.6×10^{21}	5.5×10^{21}	1.7×10^{21}
2.0×10^{2}	1.2×10^{24}	1.4×10^{24}	4.5×10^{23}
1.0×10^{7}	2.7×10^{26}	3.2×10^{26}	1.0×10^{26}

where S is the ICB and $\mathbf{n}$ is the unit normal of S, pointing outward from the inner core, as before. In the theory of Greenspan (1969), p_v is denoted by P in scaled units, as in eq. (7). We base our discussion here on the scaled variable. The dominant part of P in the solution of Greenspan (1969) is P_0. We recall from eq. (18) that P_0 at any point (x, y, z) is a function only of the total height h of the column of fluid from the lower boundary $z = -g$ to the upper boundary $z = f$ at the given (x, y): $h = f + g$, $P_0 = P_0(h)$. In the context of the outer core divided into three regions as shown in Fig. 2, P_0 is a function of the height from the ICB to the CMB in region (2) and of the height from the CMB to the ICB in region (3). Considering the fact that both the ICB and the CMB are assumed axially symmetric, this property of P_0 implies that the value of P_0 also has certain symmetries. For example, its values at two points having the same coordinates y and z but having opposite signs for x are equal, its values at two points having the same value of x but having opposite signs for y and z are equal, and its values at two points having opposite signs for x, y and z are equal. From these symmetries it can be inferred that the direction of the torque $\mathbf{N}^v$ should be along the axis x, the same as the dominant torque $\mathbf{N}^s$. Furthermore, from the same property of $P_0 = P_0(h)$, it can be inferred that, if the inner core is spherical, or if the obliquity ϵ is equal to zero, the pressure torque $\mathbf{N}^v$ should vanish. So the magnitude of $\mathbf{N}^v$ should be proportional to e, $\sin\epsilon$ and P_0; and P_0 is proportional to ω_0 (or ω_p), in view of eq. (17). Thus, $\mathbf{N}^v$ should be of much smaller magnitude than $\mathbf{N}^s$, which is proportional to e and $\sin\epsilon$. It is also in the same direction as $\mathbf{N}^s$. Therefore, we do not expect any new feature of the inner core rotation to emerge by taking account of $\mathbf{N}^v$ besides $\mathbf{N}^s$.

As will be shown, the viscous torque $\mathbf{N}^\eta$ is also small compared with $\mathbf{N}^s$ in magnitude. However, as the direction of $\mathbf{N}^\eta$ is different from that of $\mathbf{N}^s$, its influence on inner core rotation is expected to be different from that of $\mathbf{N}^s$. Thus, it is worthwhile to estimate the influence of $\mathbf{N}^\eta$ on inner core rotation in addition to that of $\mathbf{N}^s$. The only effect of the x component of $\mathbf{N}^\eta$ (the component parallel to the much larger $\mathbf{N}^s$) is to add a small correction to the angular rate of the ICW (see eq. 71 below) without affecting the ICW otherwise.

It is well established that in the absence of viscosity, ϵ, ω_p and $\delta\Omega$ are all constant (Mathews *et al.* 1991a,b; Guo & Ning 2002; Mathews *et al.* 2002). The dependence of the constant angular rate ω_p of the ICW on the other two quantities when $\mathbf{N}^v$ and $\mathbf{N}^\eta$ are neglected has been determined by (Guo & Ning 2002):

$$\frac{\omega_\mathrm{p}}{\Omega_0} = \frac{\delta N}{C\Omega_0^2}\cos^2\epsilon + \frac{\delta\Omega}{\Omega_0}, \tag{63}$$

where terms of higher orders in $\delta N/(C\Omega_0^2)$, $\delta\Omega/\Omega_0$ and the inner core dynamic flattening

$$e = (C - A)/C \tag{64}$$

are neglected. On introducing the definition of $\delta\Omega$ from eq. (45) into eq.(63), we find that

$$\frac{\omega_0}{\Omega_0} = -\frac{\delta N}{C\Omega_0^2}\cos\epsilon. \tag{65}$$

Now we incorporate the viscous torque given in eq. (57) into the theory of ICW. In this case, the torque on the right hand side of eq. (59) is

$$N_{x'} = N^s_{x'} + N^\eta_{x'}, \quad N_{y'} = N^\eta_{y'}, \quad N_{z'} = N^\eta_{z'}. \tag{66}$$

In order to see how the introduction of viscosity influences the result, we seek an approximate solution of the problem. For the inner core, $C = 5.85 \times 10^{34}$, and hence $C\Omega_0^2 = 3.11 \times 10^{26}$ and $\delta N/(C\Omega_0^2) \approx 4.11 \times 10^{-4} \approx 0.170\,e$. In the following analysis, we assume that the dimensionless quantities $\omega_\mathrm{p}/\Omega_0$, $\delta\Omega/\Omega_0$, $\gamma_1/(C\Omega_0^2)$, $\gamma_2/(C\Omega_0^2)$ and $\gamma_3/(C\Omega_0^2)$ are of the same order of magnitude as $\delta N/(C\Omega_0^2)$ or smaller and are therefore small compared with unity. Actually this assumption is not valid for the the values of the γ_i shown in the last row of Table 2, which correspond to the viscosity estimate of Smylie & McMillan (2000); $\gamma_1/(C\Omega_0^2)$, $\gamma_2/(C\Omega_0^2)$ and $\gamma_3/(C\Omega_0^2)$ are practically the order of unity in this case, which has to be therefore excluded from further consideration. For the viscosity values to which the first three rows of the table pertain, the magnitude of these quantities are the same order as or several orders smaller than $\delta N/(C\Omega_0^2)$ and the treatment that follows will be applicable to these cases. Viscosity enters into the equations of motion only through these quantities, which are very small: yet they must not be neglected when one seeks to determine the effects of viscosity on the inner core rotation. Our objective here is indeed to present an analytical treatment that leads to a solution displaying the dissipative effects (in particular, the decay of the ICW and the super-rotation) as a result of viscosity. In proceeding to obtain such a solution, we shall take advantage of the fact that the viscous torque given by eq. (54) is of second order in the small quantities mentioned. Naturally, the solution needs to be carried to the second order. It is to be expected that $(\dot\epsilon/\Omega_0)$, $(1/\Omega_0)(d\omega_\mathrm{p}/dt)$ and $(1/\Omega_0)(d\delta\Omega/dt)$, which represent the time variation, are also of second order. This expectation will be borne out by the solution obtained.

On substituting eqs (45), (58), (66), (57) and (64) into eq. (59), we have a system of ordinary differential equations for ω_p, ϵ and $\delta\Omega$, which will also be implicitly referred to as eq. (59) in the following development. Our task now is to solve these equations accurate to the second order terms. First, we look for the solution for ω_p accurate only to first order terms from the first equation of eq. (59). Predictably, it is the same as eq. (63), but with the various quantities no longer time independent. Secondly, we differentiate this expression to obtain

$$\frac{1}{\Omega_0^2}\frac{d\omega_\mathrm{p}}{dt} = \frac{1}{\Omega_0^2}\frac{d\delta\Omega}{dt}. \tag{67}$$

Note that the derivative of the first term from the right hand side of eq. (63) has been dropped, because it is proportional to both $\sin\epsilon$ and $\dot\epsilon$ besides δN and is therefore bound to be of the third or higher order. Thirdly, we substitute the expressions (63) and (67) for ω_p and

$(1/\Omega_0^2)(d\omega_\mathrm{p}/dt)$ into the second and third equations of eq. (59). The result is a pair of algebraic equations for $\dot\epsilon/\Omega_0$ and $(1/\Omega_0^2)(d\delta\Omega/dt)$. With the neglect of third and higher order terms, these equations may be written in the following form:

$$\begin{cases} X\cos\epsilon = -Y\sin\epsilon, \qquad X\sin\epsilon = Y\cos\epsilon, \\ \text{where} \\ X = \frac{\dot\epsilon}{\Omega_0} + \frac{\gamma_2}{C\Omega_0^2}\frac{\delta N}{C\Omega_0^2}\cos\epsilon\sin\epsilon, \qquad Y = \frac{1}{\Omega_0^2}\frac{d\delta\Omega}{dt}\sin\epsilon + \frac{\gamma_3}{C\Omega_0^2}\frac{\delta\Omega}{\Omega_0}\cos\epsilon. \end{cases} \tag{68}$$

It is evident that the solution of these equations is $X = Y = 0$ and hence that

$$\frac{\dot\epsilon}{\Omega_0} = -\frac{\gamma_2}{C\Omega_0^2}\frac{\delta N}{C\Omega_0^2}\sin\epsilon\cos\epsilon, \qquad \frac{1}{\Omega_0^2}\frac{d\delta\Omega}{dt} = -\frac{\gamma_3}{C\Omega_0^2}\frac{\delta\Omega}{\Omega_0}. \tag{69}$$

Thus, ϵ and $\delta\Omega$ are not really mutually coupled: they satisfy separate differential equations of the first order (of which the first one is non-linear). The time evolutions of the obliquity and the super-rotation are independent of each other. It is worth noting also that both the quantities in eq. (69) are of the second order, as was conjectured earlier.

Now we integrate eq. (69) to obtain ϵ and $\delta\Omega$ as functions of time t depending on their initial values $\epsilon = \epsilon_0$ and $\delta\Omega = \delta\Omega_0$ at $t = 0$. We obtain

$$\tan\epsilon = \tan\epsilon_0 e^{-(\gamma_2\delta N/C^2\Omega_0^3)t}, \qquad \delta\Omega = \delta\Omega_0 e^{-(\gamma_3/C\Omega_0)t}. \tag{70}$$

It is of considerable interest to observe that the viscous damping of the obliquity ϵ (or, more precisely, of $\tan\epsilon$) and that of the super-rotation $\delta\Omega$ take place on very different timescales. The ratio of the two decay times, $\tau_\epsilon \equiv [(\gamma_2\delta N)/(C^2\Omega_0^3)]^{-1}$ and $\tau_{\delta\Omega} \equiv [(\gamma_3)/(C\Omega_0)]^{-1}$, is independent of the viscosity. On evaluating the ratio using eqs (55) and (61), one finds that $\tau_{\delta\Omega}/\tau_\epsilon$ is approximately 1.4×10^{-3}, i.e. the super-rotation decays 700 times as fast as the obliquity. For the first three values of viscosity given in Table 1, the numerical values of the decay time τ_ϵ of $\tan\epsilon$ are 3.1×10^6, 6.0×10^4 and 2.3×10^2 yr respectively, and those of $\delta\Omega_0$ are 4.1×10^3, 7.8×10^1 and 3.0×10^{-1} yr respectively. Our treatment is not expected to be valid for the last value of viscosity in the table, as stated earlier.

We observe that according to the solution (70), the super-rotation decays exponentially. The interpretation of observations is however in terms of a more or less steady super-rotation, at least over a period of a few years. It is pertinent in this context to ask how a steady super-rotation could be sustained despite the damping resulting from viscosity. In seeking an answer, one needs to recognize that the viscosity effect is actually a damping of the relative motion between the fluid outer core and the solid inner core. Thus, if the outer core itself happens to be rotating faster than the inner core, the effect of viscosity would be to speed up the inner core till its speed of rotation eventually matches that of the outer core; and if the outer core rotation itself is faster than that of the mantle, the inner core would end up in a state of super-rotation relative to the mantle. The outer core rotating faster than the mantle is a feature of some models of the geodynamo driven by thermocompositional convection (Aurnou *et al.* 1996; Glatzmaier & Roberts 1996; Buffett & Glatzmaier 2000); speeding up of the inner core in these examples is the result of electromagnetic coupling to the outer core, but viscous coupling could well play the same role.

Finally, we substitute the expressions obtained for $\omega_\mathrm{p}/\Omega_0$, $(1/\Omega_0^2)(d\omega_\mathrm{p}/dt)$, $\dot\epsilon/\Omega_0$ and $(1/\Omega_0^2)(d\delta\Omega/dt)$ given by eqs (63), (67) and (69) into the first of the equations of eq. (59) to obtain $\omega_\mathrm{p}/\Omega_0$ in terms of ϵ and $\delta\Omega/\Omega_0$, accurate to second order terms:

$$\frac{\omega_\mathrm{p}}{\Omega_0} = \left(\frac{\delta N}{C\Omega_0^2}\cos^2\epsilon + \frac{\delta\Omega}{\Omega_0}\right)\left(1 + 2e\cos^2\epsilon - \frac{\delta N}{C\Omega_0^2}\cos^2\epsilon\right) - \frac{\gamma_1}{C\Omega_0^2}\frac{\delta N}{C\Omega_0^2}\cos^2\epsilon. \tag{71}$$

One readily obtains ω_0 now by combining eqs (45) and (71):

$$\frac{\omega_0}{\Omega_0} = -\frac{\delta N}{C\Omega_0^2}\cos\epsilon\left(1 + 2e\cos^2\epsilon - \frac{\delta N}{C\Omega_0^2}\cos^2\epsilon - \frac{\gamma_1}{C\Omega_0^2}\right) - \frac{\delta\Omega}{\Omega_0}\cos\epsilon\left(2e - \frac{\delta N}{C\Omega_0^2}\right). \tag{72}$$

Eqs (70), (71) and (72) constitute the complete solution of the problem.

It may be seen from the solutions (69) and (71) that our expectations regarding the orders of magnitude of the various terms are justified, thus confirming the correctness of our solutions to the accuracy sought. Furthermore, differentiation of the leading terms in eq. (72) shows that $(\dot\omega_0/\Omega_0^2)$ is of at least one order higher in the small quantities than $(\dot\epsilon/\Omega_0)$ and $(d\delta\Omega/dt)/\Omega_0^2$ and hence negligible in comparison with the latter. It may also be noted that, as mentioned earlier, γ_1 appears only in the expression for $\omega_\mathrm{p}/\Omega_0$ and plays no role in the variation of ϵ and $\delta\Omega$ with time.

Finally, we discuss briefly the validity of our computation of the outer core flow, which ignores the damping of ϵ and $\delta\Omega$ resulting from viscous drag. If the damping of $\delta\Omega$ and ϵ were taken into account, the outer core flow caused by the ICW or the super-rotation of the inner core would no longer be steady as in our treatment and the inertial term in the momentum equation for the flow, $\partial\mathbf{v}/\partial t$, would no longer vanish. Nevertheless, as long as the magnitude of this term is much smaller than that of the Coriolis term $2\boldsymbol{\Omega}\times\mathbf{v}$, our results should still be approximately valid. Thus, for giving an error estimate of our results for the outer core flow, we only need to compare the magnitudes of the inertial and Coriolis terms. To make this comparison, we observe first from eqs (30)–(40) that $\mathbf{v}_0$ and $\tilde{\mathbf{v}}_0$ depend linearly on ω_p and $\omega_0\cos\epsilon$ or, equivalently, on $\delta\Omega$ and $\omega_0\cos\epsilon$, in view of eq. (45):

$$\mathbf{v}_0 \sim \mathbf{L}_1\delta\Omega + \mathbf{L}_2\omega_0\cos\epsilon, \qquad \tilde{\mathbf{v}}_0 \sim \tilde{\mathbf{L}}_1\delta\Omega + \tilde{\mathbf{L}}_2\omega_0\cos\epsilon, \tag{73}$$

where $\mathbf{L}_1$, $\mathbf{L}_2$ and $\tilde{\mathbf{L}}_1$, $\tilde{\mathbf{L}}_2$ are factors having the dimension of length. In view of the observation made below eq. (72), the derivative of $\omega_0\cos\epsilon$ is negligibly small compared with that of $\delta\Omega$. Therefore,

$$\frac{d\mathbf{v}_0}{dt} \sim \mathbf{L}_1\frac{d\delta\Omega}{dt} = \left(-\frac{\gamma_3}{C\Omega_0}\right)\mathbf{L}_1\delta\Omega, \qquad \frac{d\tilde{\mathbf{v}}_0}{dt} \sim \tilde{\mathbf{L}}_1\frac{d\delta\Omega}{dt} = \left(-\frac{\gamma_3}{C\Omega_0}\right)\tilde{\mathbf{L}}_1\delta\Omega, \tag{74}$$

where we have made use of the second of the equations in eq. (69).

The order of magnitude of the Coriolis term, on the other hand, may evidently be written as

$$2\mathbf{\Omega} \times \mathbf{v}_0 \sim 2\Omega_0 \mathbf{L}_1 \delta\Omega + 2\Omega_0 \mathbf{L}_2 \omega_0 \cos\epsilon, \quad 2\mathbf{\Omega} \times \tilde{\mathbf{v}}_0 \sim 2\Omega_0 \tilde{\mathbf{L}}_1 \delta\Omega + 2\Omega_0 \tilde{\mathbf{L}}_2 \omega_0 \cos\epsilon. \tag{75}$$

The numerical value of the factor $2\Omega_0$ is 1.5×10^{-4} s^{-1}. For the first three values of viscosity given in Table 1, the values of $\gamma_3/(C\Omega_0)$ are respectively 7.8×10^{-12}, 4.1×10^{-10} and 1.0×10^{-7} s^{-1}. It is evident then that the inertial term is smaller than the Coriolis term by at least 10^{-3}. Therefore, the errors resulting from the neglect of the inertial term even while allowing the decay resulting from viscous drag is only of this order of magnitude.

4 CONCLUDING REMARKS

We have investigated the fluid flow in the outer core induced by two kinds of slow rotation of the inner core relative to the mantle: the ICW and the inner core super-rotation. For using the theory of Greenspan (1969) to treat the problem analytically, the outer core is assumed to be homogeneous and incompressible, and the effect of the electromagnetic fields within and at the boundaries of the fluid core is not included. We have also computed the viscous torque exerted on the inner core by the induced outer core flow, and studied its influence on the ICW and the inner core super-rotation.

The variation of density and viscosity over the outer core as well as the presence of electromagnetic fields would no doubt affect the results obtained. The density decreases by approximately 20 per cent from the ICB to the CMB (Dziewonski & Anderson 1981). It enters our equation of motion via the scaled pressure P and the Ekman number E. The variation in density influences not only the boundary layer flow but also the flow in the bulk, directly as a result of the appearance of the density in P as well as indirectly through bulk—boundary layer interaction as a result of its appearance in E. Considering the complexity of the theory of Greenspan (1969), it is difficult to quantify the level of influence of this density variation on the outer core flow obtained. The viscosity of the outer core is very poorly determined. Estimates found in the literature, which depend strongly on the assumptions and approaches employed, differ by over ten orders of magnitude. There is no generally accepted picture, either, of the variation of the viscosity over the fluid core and we can not assess its level of influence. The influence of electromagnetic fields generated by the geodynamo mechanism, which is quite complicated in itself, would be even more difficult to assess. It is not at all obvious as to how the electromagnetic fields could be integrated into the theory of Greenspan (1969) and the problem treated analytically, even in an approximate manner. All modern geodynamo models are based on numerical simulation; they differ considerably from one another, but all involve complicated temporal and spatial variations of the magnetic field. It would therefore be very difficult to draw any useful conclusions about the effect of the magnetic fields on the details of the core flow, even if some way could be found to tackle the problem analytically.

Despite the deficiencies discussed in the last paragraphs, we believe that our results still provide a useful demonstration as to how a slow inner core rotation may induce fluid flow in the outer core and in which manner the induced outer core flow may react on the inner core rotation. Furthermore, the analytical solution for the simplified model should serve a useful purpose by providing a means of testing the numerical codes for more realistic models.

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REFERENCES

Aurnou, J.M., Brito, D. & Olson, P.L., 1996. Mechanics of inner core super-rotation, *Geophys. Res. Lett.,* **23,** 3401–3404.

Buffett, B.A., 1996a. Gravitational oscillations in the length of day, *Geophys. Res. Lett.,* **23,** 2279–2282.

Buffett, B.A., 1996b. A mechanism for decade fluctuations in length of day, *Geophys. Res. Lett.,* **23,** 3803–3806.

Buffett, B.A., 1997. Geodynamic estimates of the viscosity of the Earth's inner core, *Nature,* **388,** 571–573.

Buffett, B.A. & Glatzmaier, G.A., 2000. Gravitational braking of inner-core rotation in geodynamo simulations, *Geophys. Res. Lett.,* **27,** 3125–3128.

Buffett, B.A., Mathews P. M. & Herring, T.A., 2002. Modeling of nutation and precession: effect of electromagnetic coupling, *J. geophys. Res.,* **107**(B4), 10.1029/2000JB000056.

Busse, F.H., 1970. The dynamical coupling between inner core and mantle of the Earth and the 24-year libration of the pole, in *Earthquake displacement field and the rotation of the Earth,* pp. 88–98, eds Masinha, L., Smylie, D.E. & Beck, A.E., Reidel, Dordrecht.

Dehant, V., Hinderer, J., Legros, H. & Leffz, M., 1993. Analytical approach to the computation of the Earth, the outer core and the inner core rotational motions, *Phys. Earth planet. Int.,* **76,** 259–282.

De Wijs, G.A., Kresse, G., Vočadlo, L., Dobson, D., Alfè, D., Gillan, M.J. & Price, G.D., 2000. The viscosity of liquid iron at the physical conditions of the Earth's core, *Nature,* **393,** 805–807.

Dumberry, M. & Buffett, B.A., 1999. On the validity of the geostrophic approximation for calculating the changes in the angular momentum of the core, *Phys. Earth planet. Int.,* **112,** 81–99.

Dumberry, M. & Bloxham, J., 2002. Inner core tilt and polar motion, *Geophys. J. Int.,* **151,** 377–392.

Dziewonski, A.M. & Anderson, D.L., 1981. Preliminary reference Earth model, *Phys. Earth planet. Int.,* **25,** 297–356.

Gans, R.F., 1972. Viscosity of the Earth's core, *J. geophys. Res.,* **77,** 360–366.

Glatzmaier, G.A. & Roberts, P.H., 1996. Rotation and magnetism of Earth's inner Core, *Science,* **274,** 1887–1891.

Greenspan, H.P., 1969. *The Theory of Rotating Fluids,* Cambridge University Press, Cambridge, UK, p. 328.

Greff-Lefftz, M., Legros, H. & Dehant, V., 2000. Influence of the inner core viscosity on the rotational eigenmodes of the Earth, *Phys. Earth planet. Int.,* **122,** 187–204.

Greiner-Mai, H. & Barthelmes, F., 2001. Relative wobble of the Earth's inner core derived from polar motion and associated gravity variation, *Geophys. J. Int.,* **144,** 27–36.

Guo, J.Y., 1989. On the Coupling Dynamics of the Inner core-Outer Core-Mantle System, *PhD thesis*, Université Catholique de Louvain, Louvain-La-Neuve.

Guo, J.Y. & Ning, J.S., 2002. Influence of inner core rotation and obliquity on the inner core wobble and the free inner core nutation, *Geophys. Res. Lett.,* **29**(8), 10.1029/2001GL014058.

Kakuta, C., Okamoto, I., & Sasao, T., 1975. Is the nutation of thesolid inner core responsable for the 24-year libration of the pole?, *Publ. Astron. Soc. Japan,* **27,** 357–365.

Lumb, L.A. & Aldridge, K.D., 1991. On viscosity estimates for Earth's fluid outer core and core-mantle coupling, *J. Geomag. Geoelectr.,* **43,** 93–110.

Mathews, P.M., Buffett, B.A., Herring, T.A. & Shapiro, I.I., 1991a. Forced Nutations of the Earth: Influence of inner core Dynamics 1. Theory, *J. geophys. Res.,* **96,** 8219–8242.

Mathews, P.M., Buffett, B.A., Herring, T.A. & Shapiro, I.I., 1991b. Forced Nutations of the Earth: Influence of inner core Dynamics 2. Numerical results and comparisons, *J. geophys. Res.,* **96,** 8243–8257.

Mathews, P.M., Herring, T.A. & Buffett, B.A., 2002. Modeling of nutation-precession: New nutation series for nonrigid Earth, and Insights into the Earth's Interior, *J. geophys. Res.,* **107**(B4), 10.1029/2001JB000390.

Molodensky S.M. & Groten E., 2001. On the upper bound of the liquid core viscosity, *Studia Geophysica et Geodaetica,* **45**(1), 12–36.

Poupinet, G., Souriau, A. & Coutant, O., 2000. The existence of an inner core super-rotation questioned by teleseismic doublets, *Phys. Earth planet. Int.,* **118,** 77–88.

Poupinet, G., & Souriau, A., 2001. Reply to Xiadong Songs comment on the existence of an inner core super-rotation questioned by teleseismic doublets, *Phys. Earth planet. Int.,* **124,** 275–279.

Rutter, M.D., Secco, R.A., Uchida, T., Liu, H., Wang, Y., Rivers, M.L., & Sutton, S.R., 2002. Towards evaluating the viscosity of the Earth's outer core: An experimental high pressure study of liquid Fe-S (8.5 wt. per cent S), *Geophys. Res. Lett.,* **29**(8), doi:10.1029/2001GL014392.

Smylie, D.E., 1999. Viscosity near Earth's solid inner core, *Science,* **284,** 461–463.

Smylie, D.E. & McMillan, D.G., 2000. The inner core as a dynamic viscometer, *Phys. Earth planet. Int.,* **117,** 71–79.

Smylie, D.E., Szeto, A.M.K. & Rochester, M.G., 1984. The dynamics of the Earth's inner and outer core, *Rep. Prog. Phys.,* **47,** 855–906.

Song, X., 2001. Comment on "The existence of an inner core super-rotation questioned by teleseismic doublets", by Georges Poupinet, Annie Souriau and Olivier Coutant, *Phys. Earth planet. Inter.,* **124,** 269–273.

Song, X. & Richards, P.G., 1996. Seismological evidence for differential rotation of the Earth's inner core, *Nature,* **382,** 221–224.

Souriau, A. & Poupinet, G., 2000. Inner core rotation: a test at the worldwide scale, *Phys. Earth planet. Int.,* **118,** 13–27.

Stewartson, K., 1966. On almost rigid rotation., *J. Fluid Mech.,* **26,** 131–144.

Szeto, A.M.K. & Smylie, D.E., 1984. The rotation of the Earth's inner core, *Phil. Trans. R. Soc. Lond.,* A., **313,** 171–184.

Szeto, A.M.K. & Xu, S., 1997. Gravitational coupling in a triaxial ellipsoidal Earth, *J. geophys. Res.,* **102,** 27 651–27 657.

Xu, S. & Szeto, A.M.K., 1998. The coupled motion of the inner core, *Geophys. J. Int.,* **133,** 279–297.

Xu, S., Crossley, D. & Szeto, A.M.K., 2000. Variation in length of day and inner core differential rotation from gravitational coupling, *Phys. Earth planet. Int.,* **117,** 95–110.

APPENDIX A: COMPUTATION OF OUTER CORE FLOW

The derivation of an expression for the velocity in the outer core using the theory of Greenspan (1969) involves very heavy algebra. In this appendix, we shall present the major steps of the derivation for the case of region (2) as an illustration. More details may be found in Guo (1989).

As stated in Section 2.3, we use a, Ω^{-1} and $a\bar{\omega} = \max(a\omega_0, a\omega_\mathrm{p})$ to scale length, time and velocity respectively. We shall use the same notations for the unscaled and scaled values of the quantities, as is customarily done. In the following development, all variables stand for their scaled dimensionless versions unless otherwise stated.

The equation for the ICB, expressed in the coordinate system $Ox'y'z'$ and accurate to the first order in the ellipticity, is

$$x'^2 + y'^2 + (1 + 2f)z'^2 = 1. \tag{A1}$$

The relation between coordinates in $Oxyz$ and $Ox'y'z'$ is given by

$$\begin{cases} x' = x, \\ y' = y\cos\epsilon - z\sin\epsilon, \\ z' = y\sin\epsilon + z\cos\epsilon; \end{cases} \qquad \begin{cases} \mathbf{e}_{x'} = \mathbf{e}_x, \\ \mathbf{e}_{y'} = \mathbf{e}_y\cos\epsilon - \mathbf{e}_z\sin\epsilon, \\ \mathbf{e}_{z'} = \mathbf{e}_y\sin\epsilon + \mathbf{e}_z\cos\epsilon. \end{cases} \tag{A2}$$

Thus, in the coordinate system $Oxyz$, the equation for the ICB becomes, accurate to the first order in the ellipticity,

$$x^2 + (1 + 2f\sin^2\epsilon)y^2 + (1 + 2f\cos^2\epsilon)z^2 + 4fyz\sin\epsilon\cos\epsilon = 1. \tag{A3}$$

The equation of the CMB is, to the first order in the ellipticity,

$$(x^2 + y^2)/a_c^2 + (1 + 2f_c)z^2\big/a_c^2 = 1. \tag{A4}$$

The functions $f(x, y)$ and $g(x, y)$, which characterize the top and bottom surfaces Σ_T and Σ_B, are, in view of eqs (A3) and (A4), given by

$$f(x, y) = (1 - f_c)\big[a_c^2 - (x^2 + y^2)\big]^{1/2}, \quad g(x, y) = 2fy\sin\epsilon\cos\epsilon - (1 - f\cos^2\epsilon)\{1 - [x^2 + y^2(1 + 2f\sin^2\epsilon)]\}^{1/2}. \tag{A5}$$

The height from the lower boundary to the upper boundary is then

$$h = (1 - f)\big[a_c^2 - (x^2 + y^2)\big]^{1/2} + 2fy\sin\epsilon\cos\epsilon - (1 - f\cos^2\epsilon)\{1 - [x^2 + y^2(1 + 2f\sin^2\epsilon)]\}^{1/2}. \tag{A6}$$

We can compute $\mathbf{n}_\mathrm{T}$, $\mathbf{n}_\mathrm{B}$, $\hat{\mathbf{n}}_\mathrm{T}$, $\hat{\mathbf{n}}_\mathrm{B}$, $\mathbf{n}_\mathrm{T} \times \mathbf{n}_\mathrm{B}$, I, J using the above formulae, and hence also $\mathbf{v}_0$ and $\tilde{\mathbf{v}}_0$ using the formulae given in Section 2.2. As mentioned in Section 2.3, for obtaining an analytical expression for these velocities, we need to neglect terms of the order of ellipticity when computing the integrals I and J, which determine the magnitude of the velocity $\mathbf{v}_0$, while keeping the direction of the velocity accurate to the first order in ellipticity by retaining terms of this order in $\mathbf{n}_\mathrm{T} \times \mathbf{n}_\mathrm{B}$.

The vectors $\mathbf{n}_\mathrm{T}$ and $\mathbf{n}_\mathrm{B}$ defined in the eq. (11) are computed from the expressions given above for $f(x, y)$ and $g(x, y)$. The results, accurate to the first order in the ellipticities, are:

$$\begin{cases} \mathbf{n}_\mathrm{T} = \mathbf{e}_x \frac{(1-f_c)x}{[a_c^2-(x^2+y^2)]^{1/2}} + \mathbf{e}_y \frac{(1-f_c)y}{[a_c^2-(x^2+y^2)]^{1/2}} + \mathbf{e}_z, \\ \mathbf{n}_\mathrm{B} = -\mathbf{e}_x \frac{x(1-f\cos^2\epsilon)}{\{1-[x^2+y^2(1+2f\sin^2\epsilon)]\}^{1/2}} - \mathbf{e}_y \left\{ 2f\sin\epsilon\cos\epsilon + \frac{y\left[1+2f\left(1-\frac{3}{2}\cos^2\epsilon\right)\right]}{\{1-[x^2+y^2(1+2f\sin^2\epsilon)]\}^{1/2}} \right\} - \mathbf{e}_z. \end{cases} \tag{A7}$$

The expression $\mathbf{n}_\mathrm{T} \times \mathbf{n}_\mathrm{B}$ is also computed to the same accuracy:

$$\begin{aligned} \mathbf{n}_\mathrm{T} \times \mathbf{n}_\mathrm{B} = {} & \mathbf{e}_x \left\{ -\frac{y(1-f_c)}{[a_c^2-(x^2+y^2)]^{1/2}} + 2f\sin\epsilon\cos\epsilon + \frac{y\left[1+2f\left(1-\frac{3}{2}\cos^2\epsilon\right)\right]}{\{1-[x^2+y^2(1+2f\sin^2\epsilon)]\}^{1/2}} \right\} \\ & + \mathbf{e}_y \left\{ -\frac{x(1-f\cos^2\epsilon)}{\{1-[x^2+y^2(1+2f\sin^2\epsilon)]\}^{1/2}} + \frac{x(1-f_c)}{[a_c^2-(x^2+y^2)]^{1/2}} \right\} \\ & + \mathbf{e}_z \left\{ -\frac{2fx\sin\epsilon\cos\epsilon}{\left[a_c^2-(x^2+y^2)\right]^{1/2}} - \frac{2fxy\sin^2\epsilon}{\left[a_c^2-(x^2+y^2)\right]^{1/2}\{1-[x^2+y^2(1+2f\sin^2\epsilon)]\}^{1/2}} \right\}. \end{aligned} \tag{A8}$$

As mentioned earlier, all ellipticity terms are neglected when computing I and J as given by eqs (27) and (28). With this approximation, the height h given by eq. (A6) depends only on $(x^2 + y^2)$ and so the equation of isoheight contour C, in the formulae for I and J in eqs (27) and (28), is simply

$$x^2 + y^2 = \text{Constant}. \tag{A9}$$

The expressions (A7) and (A8) for $\hat{\mathbf{n}}_\mathrm{T}$ and $\hat{\mathbf{n}}_\mathrm{B}$ are also to be taken with the neglect of terms involving the flattenings f and f_c when computing I and J from eqs (27) and (28). We then get

$$\hat{\mathbf{n}}_\mathrm{T} = \mathbf{e}_x \frac{x}{a_c} + \mathbf{e}_y \frac{y}{a_c} + \mathbf{e}_z \left(1 - \frac{x^2+y^2}{a_c^2}\right)^{1/2}, \quad \hat{\mathbf{n}}_\mathrm{B} = -\mathbf{e}_x x - \mathbf{e}_y y - \mathbf{e}_z[1-(x^2+y^2)]^{1/2}. \tag{A10}$$

With the use of these approximations,

$$|\mathbf{e}_z \cdot \hat{\mathbf{n}}_\mathrm{T}|^{-1/2} = \left[\frac{a_c^2}{a_c^2-(x^2+y^2)}\right]^{1/4}, |\mathbf{e}_z \cdot \hat{\mathbf{n}}_\mathrm{B}|^{-1/2} = \left[\frac{1}{1-(x^2+y^2)}\right]^{1/4}, \tag{A11}$$

$$|\mathbf{n}_\mathrm{T} \times \mathbf{n}_\mathrm{B}| = \frac{(x^2+y^2)^{1/2}}{1-(x^2+y^2)} - \frac{(x^2+y^2)^{1/2}}{a_c^2-(x^2+y^2)} \tag{A12}$$

and

$$\frac{\mathbf{n}_\mathrm{T} \times \mathbf{n}_\mathrm{B}}{|\mathbf{n}_\mathrm{T} \times \mathbf{n}_\mathrm{B}|} = \mathbf{e}_x \frac{y}{(x^2+y^2)^{1/2}} - \mathbf{e}_y \frac{x}{(x^2+y^2)^{1/2}}. \tag{A13}$$

The fluid velocities $\mathbf{V}_\mathrm{T}$ and $\mathbf{V}_\mathrm{B}$ at the upper and lower boundaries have to be equal, respectively, to the velocities $\mathbf{V}_\mathrm{CMB}$ and $\mathbf{V}_\mathrm{ICB}$ on the solid side of the respective boundaries:

$$\mathbf{V}_\mathrm{T} = \mathbf{V}_\mathrm{CMB}, \mathbf{V}_\mathrm{B} = \mathbf{V}_\mathrm{ICB}. \tag{A14}$$

The velocities on the solid side, given by eq. (6), can be expressed in terms of the scaled variables as

$$\mathbf{V}_\mathrm{CMB} = \mathbf{e}_x(\omega_\mathrm{p}/\bar{\omega})y - \mathbf{e}_y(\omega_\mathrm{p}/\bar{\omega})x, \mathbf{V}_\mathrm{ICB} = -\mathbf{e}_x(\omega_0/\bar{\omega})(y\cos\epsilon - z\sin\epsilon) + \mathbf{e}_y(\omega_0/\bar{\omega})x\cos\epsilon - \mathbf{e}_z(\omega_0/\bar{\omega})x\sin\epsilon. \tag{A15}$$

In $\mathbf{V}_\mathrm{ICB}$, z is to be replaced, with the neglect of ellipticity, by $[1-(x^2+y^2)]^{1/2}$. We have then

$$\hat{\mathbf{n}}_\mathrm{T} \times \mathbf{V}_\mathrm{T} = \mathbf{e}_x(\omega_\mathrm{p}/\bar{\omega})x\left[a_c^2-(x^2+y^2)\right]^{1/2}/a_c + \mathbf{e}_y(\omega_\mathrm{p}/\bar{\omega})y\left[a_c^2-(x^2+y^2)\right]^{1/2}/a_c - \mathbf{e}_z(\omega_\mathrm{p}/\bar{\omega})(x^2+y^2)/a_c, \tag{A16}$$

$$\begin{aligned} \hat{\mathbf{n}}_\mathrm{B} \times \mathbf{V}_\mathrm{B} = {} & \mathbf{e}_x(\omega_0/\bar{\omega})\{xy\sin\epsilon + x\cos\epsilon[1-(x^2+y^2)]^{1/2}\} + \mathbf{e}_y(\omega_0/\bar{\omega})\{y\cos\epsilon[1-(x^2+y^2)]^{1/2} - \sin\epsilon(1-y^2)\} \\ & + \mathbf{e}_z(\omega_0/\bar{\omega})\{y\sin\epsilon[1-(x^2+y^2)]^{1/2} - \cos\epsilon(x^2+y^2)\}. \end{aligned} \tag{A17}$$

The integrals I and J defined by eq. (27) may now be readily evaluated using the above expressions, noting that the contours C are circles $(x^2 + y^2) = $ constant as already mentioned:

$$\begin{aligned} I &= \int_C \left[\left\{\frac{(x^2+y^2)^{1/2}}{[1-(x^2+y^2)]^{1/2}} - \frac{(x^2+y^2)^{1/2}}{\left[a_c^2-(x^2+y^2)\right]^{1/2}}\right\}\left\{\left[\frac{a_c^2}{a_c^2-(x^2+y^2)}\right]^{1/4} + \left[\frac{1}{1-(x^2+y^2)}\right]^{1/4}\right\}\right] dC \\ &= 2\pi(x^2+y^2)\frac{\left[a_c^2-(x^2+y^2)\right]^{1/2} - [1-(x^2+y^2)]^{1/2}}{\left[a_c^2-(x^2+y^2)\right]^{1/2}[1-(x^2+y^2)]^{1/2}}\left\{\left[\frac{a_c^2}{a_c^2-(x^2+y^2)}\right]^{1/4} + \left[\frac{1}{1-(x^2+y^2)}\right]^{1/4}\right\}, \end{aligned} \tag{A18}$$

$$J = \int_C \frac{\omega_{\mathrm{p}}}{\bar{\omega}}(x^2+y^2)^{1/2}\left[\frac{a_c^2}{a_c^2-(x^2+y^2)}\right]^{1/4} dC$$

$$- \int_C \frac{\omega_0}{\bar{\omega}}\left\{\cos\epsilon(x^2+y^2)^{1/2} - \frac{y\sin\epsilon[1-(x^2+y^2)]^{1/2}}{(x^2+y^2)^{1/2}} - \frac{x\sin\epsilon}{(x^2+y^2)^{1/2}}\right\}\left[\frac{1}{1-(x^2+y^2)}\right]^{1/4} dC \tag{A19}$$

$$= \frac{2\pi(\omega_{\mathrm{p}}/\bar{\omega})a_c^{1/2}(x^2+y^2)}{\left[a_c^2-(x^2+y^2)\right]^{1/4}} - \frac{2\pi(\omega_0/\bar{\omega})\cos\epsilon(x^2+y^2)}{[1-(x^2+y^2)]^{1/4}}.$$

The internal flow velocity in region (2) can finally be computed using the fact that $\mathbf{v}_0 = (I/J)(\mathbf{n}_{\mathrm{T}} \times \mathbf{n}_{\mathrm{B}})$, according to eqs (19) and (26):

$$\mathbf{v}_0 = \frac{\left[a_c^2-(x^2+y^2)\right]^{1/2}[1-(x^2+y^2)]^{1/2}}{\left[a_c^2-(x^2+y^2)\right]^{1/2}-[1-(x^2+y^2)]^{1/2}}\,\frac{(\omega_{\mathrm{p}}/\bar{\omega})a_c^{1/2}[1-(x^2+y^2)]^{1/4}-(\omega_0/\bar{\omega})\left[a_c^2-(x^2+y^2)\right]^{1/4}\cos\epsilon}{a_c^{1/2}[1-(x^2+y^2)]^{1/4}+\left[a_c^2-(x^2+y^2)\right]^{1/4}}$$

$$\cdot\left\langle \mathbf{e}_x\left[-\frac{y(1-f_c)}{[a_c^2-(x^2+y^2)]^{1/2}} + 2f\sin\epsilon\cos\epsilon + \frac{y\left[1+2f\left(1-\frac{3}{2}\cos^2\epsilon\right)\right]}{\{1-[x^2+y^2(1+2f\sin^2\epsilon)]\}^{1/2}}\right]\right.$$

$$+\mathbf{e}_y\left[-\frac{x(1-f\cos^2\epsilon)}{\{1-[x^2+y^2(1+2f\sin^2\epsilon)]\}^{1/2}} + \frac{x(1-f_c)}{\left[a_c^2-(x^2+y^2)\right]^{1/2}}\right]$$

$$\left.+\mathbf{e}_z\left[-\frac{2fx\sin\epsilon\cos\epsilon}{[a_c^2-(x^2+y^2)]^{1/2}} - \frac{2fxy\sin^2\epsilon}{[a_c^2-(x^2+y^2)]^{1/2}\{1-[x^2+y^2(1+2f\sin^2\epsilon)]\}^{1/2}}\right]\right\rangle. \tag{A20}$$

This is the internal flow velocity in scaled dimensionless variables. The results in eq. (30) are obtained on transforming to real physical variables [in region (2), sign$z = 1$]. The corresponding expression for the region (3) (wherein sign$z = -1$), may be obtained in a similar fashion.

We proceed now to calculate the boundary layer correction $\tilde{\mathbf{v}}_0$ near the ICB by introducing the above results in eq. (23), neglecting all ellipticity terms; the calculation for the CMB is similar.

For region (2), the the lower boundary Σ_{B} is the ICB. We have then, from eq. (A10),

$$\mathbf{e}_z\cdot\hat{\mathbf{n}}_{\mathrm{B}} = -[1-(x^2+y^2)]^{1/2}, \qquad \frac{\mathbf{e}_z\cdot\hat{\mathbf{n}}_{\mathrm{B}}}{|\,\mathbf{e}_z\cdot\hat{\mathbf{n}}_{\mathrm{B}}\,|} = -1. \tag{A21}$$

We also have

$$\zeta = E^{-1/2}\hat{\mathbf{n}}_{\mathrm{B}}\cdot(\mathbf{r}_0-\mathbf{r}) = E^{-1/2}(r-1), \tag{A22}$$

remembering that, by definition, $\mathbf{r}_0 - \mathbf{r}$ should be in the same direction as $\hat{\mathbf{n}}_{\mathrm{B}}$.

On neglecting ellipticity, the internal flow velocity (eq. A20) reduces to

$$\mathbf{v}_0 = \frac{(\omega_{\mathrm{p}}/\bar{\omega})a_c^{1/2}[1-(x^2+y^2)]^{1/4}-(\omega_0/\bar{\omega})\left[a_c^2-(x^2+y^2)\right]^{1/4}\cos\epsilon}{a_c^{1/2}[1-(x^2+y^2)]^{1/4}+\left[a_c^2-(x^2+y^2)\right]^{1/4}}(\mathbf{e}_x y - \mathbf{e}_y x). \tag{A23}$$

Also, the inner core becomes a sphere whose radius is unity in the scaled units. So one has

$$x = \sin\theta\cos\lambda, \quad y = \sin\theta\sin\lambda \tag{A24}$$

for any point on the ICB.

From eqs (A23), (A14), (A15) and (A24), we now obtain

$$(\mathbf{v}_0-\mathbf{V}_{\mathrm{B}})_{\zeta=0} = \mathbf{e}_x\left[\frac{\{[\omega_{\mathrm{p}}/\bar{\omega}+(\omega_0/\bar{\omega})\cos\epsilon]\sin\theta\sin\lambda-(\omega_0/\bar{\omega})\sin\epsilon\cos\theta\}a_c^{1/2}\cos^{1/2}\theta}{a_c^{1/2}\cos^{1/2}\theta+(a_c^2-\sin^2\theta)^{1/4}} - \frac{(\omega_0/\bar{\omega})\sin\epsilon\cos\theta\left(a_c^2-\sin^2\theta\right)^{1/4}}{a_c^{1/2}\cos^{1/2}\theta+\left(a_c^2-\sin^2\theta\right)^{1/4}}\right]$$

$$-\mathbf{e}_y\frac{[\omega_{\mathrm{p}}/\bar{\omega}+(\omega_0/\bar{\omega})\cos\epsilon]\sin\theta\cos\lambda a_c^{1/2}\cos^{1/2}\theta}{a_c^{1/2}\cos^{1/2}\theta+\left(a_c^2-\sin^2\theta\right)^{1/4}} + \mathbf{e}_z(\omega_0/\bar{\omega})\sin\epsilon\sin\theta\cos\lambda \tag{A25}$$

and hence, with the use of eq. (A10),

$$[\hat{\mathbf{n}}_{\rm B}\times(\mathbf{v}_0-\mathbf{V}_{\rm B})]_{\zeta=0}=\mathbf{e}_x\left\{-(\omega_0/\bar{\omega})\sin\epsilon\sin^2\theta\sin\lambda\cos\lambda-\frac{[\omega_{\rm p}/\bar{\omega}+(\omega_0/\bar{\omega})\cos\epsilon]\sin\theta\cos\lambda a_c^{1/2}\cos^{3/2}\theta}{a_c^{1/2}\cos^{1/2}\theta+\left(a_c^2-\sin^2\theta\right)^{1/4}}\right\}$$
$$+\mathbf{e}_y\left[(\omega_0/\bar{\omega})\sin\epsilon\sin^2\theta\cos^2\lambda-\frac{\{[\omega_{\rm p}/\bar{\omega}+(\omega_0/\bar{\omega})\cos\epsilon]\sin\theta\sin\lambda-(\omega_0/\bar{\omega})\sin\epsilon\cos\theta\}a_c^{1/2}\cos^{3/2}\theta}{a_c^{1/2}\cos^{1/2}\theta+\left(a_c^2-\sin^2\theta\right)^{1/4}}\right.$$
$$\left.+\frac{(\omega_0/\bar{\omega})\sin\epsilon\cos^2\theta\left(a_c^2-\sin^2\theta\right)^{1/4}}{a_c^{1/2}\cos^{1/2}\theta+(a_c^2-\sin^2\theta)^{1/4}}\right]$$
$$+\mathbf{e}_z\left[\frac{\{[\omega_{\rm p}/\bar{\omega}+(\omega_0/\bar{\omega})\cos\epsilon]\sin\theta-(\omega_0/\bar{\omega})\sin\epsilon\cos\theta\cos\lambda\}\sin\theta a_c^{1/2}\cos^{1/2}\theta}{a_c^{1/2}\cos^{1/2}\theta+\left(a_c^2-\sin^2\theta\right)^{1/4}}\right.$$
$$\left.-\frac{(\omega_0/\bar{\omega})\sin\epsilon\sin\theta\cos\theta\sin\lambda\left(a_c^2-\sin^2\theta\right)^{1/4}}{a_c^{1/2}\cos^{1/2}\theta+\left(a_c^2-\sin^2\theta\right)^{1/4}}\right].\qquad\text{(A26)}$$

These expressions, and those which follow, are written for the region (2) where $\cos\theta$ is positive. In region (3) where $\cos\theta$ is negative, $(-\cos\theta)^{1/2}$ should appear instead of $\cos^{1/2}\theta$.

Finally the expression (23) for the boundary layer correction may be evaluated by introducing in it eqs (A21)–(A22), (A25) and (A26):

$$\tilde{\mathbf{v}}_0=e^{-E^{-1/2}(r-1)\cos^{1/2}\theta}\left\{\mathbf{e}_x\left\langle\left[-\frac{\{[\omega_{\rm p}/\bar{\omega}+(\omega_0/\bar{\omega})\cos\epsilon]\sin\theta\sin\lambda-(\omega_0/\bar{\omega})\sin\epsilon\cos\theta\}a_c^{1/2}\cos^{1/2}\theta}{a_c^{1/2}\cos^{1/2}\theta+\left(a_c^2-\sin^2\theta\right)^{1/4}}\right.\right.\right.$$
$$\left.+\frac{(\omega_0/\bar{\omega})\sin\epsilon\cos\theta\left(a_c^2-\sin^2\theta\right)^{1/4}}{a_c^{1/2}\cos^{1/2}\theta+(a_c^2-\sin^2\theta)^{1/4}}\right]\cos[E^{-1/2}\cos^{1/2}\theta(r-1)]$$
$$\left.+\left\{(\omega_0/\bar{\omega})\sin\epsilon\sin^2\theta\sin\lambda\cos\lambda+\frac{[\omega_{\rm p}/\bar{\omega}+(\omega_0/\bar{\omega})\cos\epsilon]\sin\theta\cos\lambda a_c^{1/2}\cos^{3/2}\theta}{a_c^{1/2}\cos^{1/2}\theta+\left(a_c^2-\sin^2\theta\right)^{1/4}}\right\}\sin[E^{-1/2}\cos^{1/2}\theta(r-1)]\right\rangle$$
$$+\mathbf{e}_y\left\langle\frac{[\omega_{\rm p}/\bar{\omega}+(\omega_0/\bar{\omega})\cos\epsilon]\sin\theta\cos\lambda a_c^{1/2}\cos^{1/2}\theta}{a_c^{1/2}\cos^{1/2}\theta+\left(a_c^2-\sin^2\theta\right)^{1/4}}\cos[E^{-1/2}\cos^{1/2}\theta(r-1)]\right.$$
$$+\left[-(\omega_0/\bar{\omega})\sin\epsilon\sin^2\theta\cos^2\lambda+\frac{\{[\omega_{\rm p}/\bar{\omega}+(\omega_0/\bar{\omega})\cos\epsilon]\sin\theta\sin\lambda-(\omega_0/\bar{\omega})\sin\epsilon\cos\theta\}a_c^{1/2}\cos^{3/2}\theta}{a_c^{1/2}\cos^{1/2}\theta+\left(a_c^2-\sin^2\theta\right)^{1/4}}\right.$$
$$\left.\left.-\frac{(\omega_0/\bar{\omega})\sin\epsilon\cos^2\theta(a_c^2-\sin^2\theta)^{1/4}}{a_c^{1/2}\cos^{1/2}\theta+(a_c^2-\sin^2\theta)^{1/4}}\right]\sin[E^{-1/2}\cos^{1/2}\theta(r-1)]\right\rangle$$
$$+\mathbf{e}_z\left\langle-(\omega_0/\bar{\omega})\sin\epsilon\sin\theta\cos\lambda\cos[E^{-1/2}\cos^{1/2}\theta(r-1)]\right.$$
$$-\left[\frac{\{[\omega_{\rm p}/\bar{\omega}+(\omega_0/\bar{\omega})\cos\epsilon]\sin\theta-(\omega_0/\bar{\omega})\sin\epsilon\cos\theta\sin\lambda\}\sin\theta a_c^{1/2}\cos^{1/2}\theta}{a_c^{1/2}\cos^{1/2}\theta+\left(a_c^2-\sin^2\theta\right)^{1/4}}\right.$$
$$\left.\left.\left.-\frac{(\omega_0/\bar{\omega})\sin\epsilon\sin\theta\cos\theta\sin\lambda\left(a_c^2-\sin^2\theta\right)^{1/4}}{a_c^{1/2}\cos^{1/2}\theta+\left(a_c^2-\sin^2\theta\right)^{1/4}}\right]\sin[E^{-1/2}\cos^{1/2}\theta(r-1)]\right\rangle\right\}.\qquad\text{(A27)}$$

It is a straightforward matter to compute the spherical components of $\tilde{\mathbf{v}}_0$ in real physical variables from the Cartesian components above, which are expressed in terms of dimensionless variables. The result is shown as eq. (35).

APPENDIX B: COMPUTATION OF THE TORQUE RESULTING FROM VISCOSITY

As mentioned in the main text, only the boundary layer flow $\tilde{\mathbf{v}}_0$ contributes significantly to the viscous torque. Substituting for the boundary value (at $r=a$) of $\tilde{\mathbf{v}}_0$ from eq. (35) into eq. (53), retaining only the terms of order $E^{-1/2}$ (the other terms being much smaller) and recalling that $r=a$ at the ICB, we obtain

$$\mathbf{N}^\eta=a^3\eta E^{-1/2}\int_S|\cos\theta|^{1/2}\sin\theta\{[-\omega_1+(\text{sign}z)\omega_2]\mathbf{e}_\lambda+[(\text{sign}z)\omega_1+\omega_2]\mathbf{e}_\theta\}\,d\theta\,d\lambda.\qquad\text{(B1)}$$

We now replace $\mathbf{e}_\theta$, $\mathbf{e}_\lambda$ with their expressions in terms of $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$, and substitute for ω_1 and ω_2 from eqs (37) and (38), to obtain the Cartesian components of the torque:

$$\begin{cases} N_x^\eta = \eta a^3 E^{-1/2} \int_0^{2\pi}\int_0^\pi |\cos\theta|^{1/2} \sin\theta \big\{ \chi [\cos\theta\cos\lambda - (\mathrm{sign}z)\sin\lambda] \\ \qquad\qquad + \omega_0 \sin^2\theta \sin\lambda\cos\lambda\sin\epsilon + (\mathrm{sign}z)\omega_0 \sin\epsilon\cos\theta \big\} d\theta d\lambda, \\ N_y^\eta = \eta a^3 E^{-1/2} \int_0^{2\pi}\int_0^\pi |\cos\theta|^{1/2} \sin\theta \{ \chi [\cos\theta\sin\lambda + (\mathrm{sign}z)\cos\lambda] - \omega_0 (1 - \sin^2\theta\sin^2\lambda)\sin\epsilon \} \, d\theta d\lambda, \\ N_z^\eta = \eta a^3 E^{-1/2} \int_0^{2\pi}\int_0^\pi |\cos\theta|^{1/2} \sin\theta [-\chi\sin\theta + \omega_0 \cos\theta\sin\theta\sin\lambda\sin\epsilon - (\mathrm{sign}z)\omega_0 \cos\lambda\sin\theta\sin\epsilon] \, d\theta d\lambda, \end{cases} \tag{B2}$$

where χ is defined as

$$\chi = \frac{(\omega_p + \omega_0 \cos\epsilon)\sin\theta |\cos\theta|^{1/2}}{|\cos\theta|^{1/2} + [1 - (a/a_c)^2 \sin^2\theta]^{1/4}}. \tag{B3}$$

We see that the integrations over λ and θ in eq. (B2) can be carried out in succession and that most of the integrals over λ are equal to zero. The expressions (54) and (55) may thus be obtained in a straightforward fashion.