Narrow scale flow and a weak field by the top of Earth's core: evidence from Ørsted, Magsat and secular variation

C. V. Voorhies, Geodynamics Branch, Code 921, NASA Goddard Space Flight Center, Greenbelt, MD 20771 USA Coerte.V.Voorhies@nasa.gov Revision 12/11/03
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Abstract: To test two hypotheses against seismology, the Ørsted Initial Field Model is used to estimate the radius of Earth's core by spectral methods. The model coefficients are used to compute the mean square magnetic flux density in spherical harmonics of degree n on the reference sphere of radius $a = 6371.2$ km, which is an observational spectrum R_n. The theoretical spectrum tested, $\{R_n^c\} = K(n+1/2)[(n(n+1)]^{-1}(c/a)^{2n+4}$, is obtained from the hypotheses of narrow scale flow and a dynamically weak magnetic field near the top of Earth's core; it describes a low degree, core-source magnetic energy range. Core radius c and amplitude K are estimated by fitting log-theoretical to log-observational spectra at low degrees. Estimates of c from R_n of degrees 1 through N vary between 3441 and 3542 km as N increases from 4 to 12. None of these estimates differ significantly from the seismologic core radius of 3480 km. Significant differences do occur if N exceeds 12, which is consistent with appreciable non-core, crustal source fields at degrees 13 and above, or if other spectral forms are assumed. Similar results are obtained from the 1980 epoch Magsat model CM3. One way to deduce $\{R_n^c\}$ uses an expectation spectrum for low degree secular variation (SV) induced by narrow scale flow near the top of Earth's core, $\{F_n^c\} = Cn(n+1/2)(n+1)(c/a)^{2n+4}$. The value of c obtained by fitting this form to the mean observational SV spectrum from model GSFC 9/80 is 3470 ± 91 km, also in accord with seismologic estimates. This test of the narrow scale flow hypothesis is independent of the weak field hypothesis. The agreement between SV, Magsat, Ørsted and seismologic estimates of core radius means the hypotheses pass these tests. Additional tests are described.

1. Introduction

Among many physical hypotheses about Earth's interior, let us develop and test two of geomagnetic and geodynamic interest. The first is the hypothesis of narrow scale fluid flow by the top of Earth's core, which concerns core kinematics and geomagnetic secular variation (SV; see, *e.g.*, *Roberts & Scott* [1965], *Backus* [1968]). "Narrow scale" here includes horizontal length scales less than about 10^3 km, such as the 4.2 km scale of *Benton* [1992]. Many inversions of global geomagnetic change for core surface flow estimate only a broad-scale flow with features over 10^3 km across (see, *e.g.*, *Voorhies* [1986, 1993, 1995]).

The second hypothesis is that of a dynamically weak Lorentz force, or weak magnetic field, by the top of Earth's core. This concerns core dynamics and the likelihood that the deep mantle is too poor a conductor to allow much electric current across the core-mantle boundary (CMB; see, *e.g., LeMouël* [1984], *Gubbins & Roberts* [1987], *Voorhies* [1991]). "Dynamically weak" here includes force magnitudes much less than those of gravity, pressure and the Coriolis effect, albeit not necessarily as weak as the net force driving broad-scale accelerations of about 2×10^{-13} m/s^2 [*Voorhies*, 1995]. This surficially weak field hypothesis is, however, compatible with a strong toroidal magnetic field deeper in the core (see, *e.g., Backus* [1986], *Roberts & Gubbins* [1987], *Benton* [1992]). A test might confirm that the potential field on Earth's surface mainly represents a poloidal field from within the core, instead of from a strong toroidal core field threading CMB topography – hence a slight bending of otherwise toroidal field and poloidal coupling currents at a somewhat smaller jump in conductivity across the CMB.

There are alternative hypotheses which have been tested; yet close fits to geomagnetic change obtained with broad-scale, non-geostrophic, core surface flows do not necessarily imply that narrow scale flow and/or weak field hypotheses are in error (see, *e.g., Voorhies* [1995]). Similarly, presence of some narrow scale flow does not imply absence of all broad-scale flow.

Our hypotheses are embedded in a larger theory and a vast body of geomagnetic observations which ease their conversion into quantitatively testable forms. To this end, let us establish notation and a CMB layer model in sections 2 & 3; develop and test the narrow scale flow hypothesis in sections 4 & 5; develop and test the weak field hypothesis in sections 6 & 7; and offer a constraint on either the conductivity or the magnetic Reynolds number of Earth's core.

2. Notation and Background

Let $\mathbf{B}(\mathbf{r}, t)$ denote magnetic flux density at time t and position $\mathbf{r}$ in geocentric spherical polar coordinates (r, θ, ϕ) caused by electric current of density $\mathbf{J}$ and magnetization $\mathbf{M}$ within the earth. Above Earth's surface, solenoidal $\mathbf{B}$ equals the negative gradient of the zero-mean scalar internal magnetic potential V. This potential satisfies Laplace's equation and has a Schmidt-normalized spherical harmonic expansion with Gauss coefficients of degree n and order m, denoted $[g_n^m(t), h_n^m(t)]$ on a reference sphere of radius $a = 6371.2$ km. Coefficients through finite degree N_F can be estimated by spherical harmonic analysis of the measured field, which also enables separation of the internal-source field considered here from external-source fields (see, *e.g., Langel* [1987]).

As is also well-known, the mean square field represented by harmonics of degree n, averaged over a sphere of radius r containing the sources, is given by

$$R_n(r, t) = (n+1)(a/r)^{2n+4} \sum_{m=0}^{n} [g_n^m(t)]^2 + [h_n^m(t)]^2 \tag{1}$$

(see, *e.g., Lowes* [1966], *Lucke* [1957], *Mauersberger* [1956]; *Meyer* [1985]). The R_n form a discrete geomagnetic spectrum with units of T^2. The integral of magnetic energy density over the sphere has a related spectrum, $2\pi r^2 R_n/\mu_o$ in vacuum permeability μ_o, with units of spatial power or force (J/m). A spherical harmonic expansion is equivalent to a centered, $2n$-pole moment expansion, so each R_n represents a multipole power.

The mean square rate of change of the magnetic field represented by harmonics of degree n, averaged over the same sphere, is given by the SV spectrum

$$F_n(r, t) = (n+1)(a/r)^{2n+4} \sum_{m=0}^{n} [(\partial_t g_n^m)^2 + (\partial_t h_n^m)^2] . \tag{2}$$

Note that F_n is not $\partial_t R_n$; moreover, the time average of magnetic spectrum (1), denoted $\langle R_n \rangle$, is not generally the spectrum of the time averaged field. Similarly, when SV itself varies in time, the time average of SV spectrum (2), denoted $\langle F_n \rangle$, is not the spectrum of net magnetic change.

Spectra computed from coefficients determined by weighted least squares fits to geomagnetic measurements, with no assumptions about a core field, are here called "observational spectra".

The accuracy of an observational spectrum depends on geomagnetic field modeling procedure, such as the choice of N_F, as well as data selection, distribution and accuracy. Spectra obtained from physical hypotheses about sources are here called "theoretical spectra" (see, *e.g., McLeod* [1996]; *Voorhies* [1998]; *Voorhies, Sabaka & Purucker* [2002]). Hypotheses about averages over physical processes may yield a theoretical spectrum that is also a mean, or "expectation spectrum", denoted $\{R_n\}$ for the main field or $\{F_n\}$ for the SV.

Observational spectra R_n determined by analyses of satellite magnetic data have been interpreted in terms of a mainly core-source field for $n < 14$ and a dominantly crustal-source field for $n > 14$ (see, *e.g., Langel & Estes* [1982], *Cain et al.* [1989], *Voorhies, Sabaka & Purucker* [2002]). A review of publications bearing on a core-source interpretation for portions of R_n and F_n is omitted for brevity (see, *e.g., Booker* [1969]; *Verosub & Cox* [1971]; *Lowes* [1974]; *McLeod & Coleman* [1980]; *Hide & Malin* [1981]; *Langel & Estes* [1982]; *Benton et al.* [1982]; *Shure, Parker & Backus* [1982]; *Voorhies & Benton* [1982]; *Gubbins* [1983]; *Stevenson* [1983]; *Voorhies* [1984]; *Gubbins & Bloxham* [1985]; *McLeod* [1985, 1996]; *Meyer* [1985]; *Benton & Alldredge* [1987]; *Benton & Voorhies* [1987]; *Backus* [1988]; *Constable & Parker* [1988]; *Cain et al.* [1989]; *Hulot, LeMouël & Wahr* [1992]; *Harrison* [1994]; *Hulot & LeMouël* [1994]; *Voorhies & Conrad* [1996]; *Walker & Backus* [1997]; *Voorhies et al.* [2002]; *De Santis, Barraclough & Tozzi* [2003]).

Clearly, an observational spectrum may be used to help test a theoretical spectrum, hence its underlying physical hypotheses. There is, however, a third class of spectra, here called "constrained spectra", computed from coefficients constrained by one or more assumptions about a core field (see, *e.g., Shure, Parker & Backus* [1982]; *Gubbins* [1983]; *Backus* [1988]). The independence, hence utility, of a constrained spectrum can be compromised by constraints that force it to either agree or disagree with a theoretical spectrum. In special circumstances, however, the statistical significance of increased misfit to measured data caused by a constraint could provide a test of physical hypotheses underlying the constraint.

Theoretical spectra from sources in a roughly spherical core of radius c are here denoted R_n^c. For example, *Gubbins'* [1975] expression for the minimum value of the finite Ohmic dissipation in the core implies that, for degrees in a magnetic dissipation range defined by $n \geq N_D$,

$$R_n^c(a; n \geq N_D) \;\leq\; K_G\, n^{-2-\delta}\, (c/a)^{2n}, \tag{3a}$$

where constant $K_G > 0$ and $\delta > 0$. Though finite, N_D may be much larger than 12. So there may be a low degree magnetic energy range, defined by $n \leq N_E < N_D$, and perhaps an intermediate, if not inertial, sub-range defined by $N_E < n < N_D$. This is the case for theoretical spectra of *Stevenson* [1983] and *McLeod* [1985, 1996], as shown in Appendix A. A scale analysis illustrates the physical plausibility of a generalized Stevenson - McLeod relation,

$$\{R_n^c(r > c; n \leq N_E)\} \;\approx\; K\,(n + 1/2)[n(n+1)]^{-1}\,(c/r)^{2n+4}, \tag{3b}$$

as an expectation spectrum for low degrees, as shown in Appendix B.

Both the scale analysis and the empirical approach in section 6 yield spectrum (3b) as an expected consequence of narrow scale flow and a weak field by the top of the core. Neither specify expected spectral variance, denoted $\{(R_n^c - \{R_n^c\})^2\}$; therefore, tests of spectrum (3b) are here limited to comparisons between magnetic estimates of c and independent estimates. Seismologic estimates of c differ by but a few km and are here denoted c_s = 3480 km (see, *e.g.*, *Dziewonski & Anderson* [1981]; *Kennett, Engdahl & Bulland* [1995]).

3. CMB Layer and Narrow Scale Flow

The transition from fluid conducting core to rigid resistive mantle has long been modeled simply by a sharp, impenetrable interface: a fixed jump in material properties across the CMB. Finite Lorentz and viscous forces are ensured by continuity of both magnetic and hydrodynamic stress tensors across this interface. Then **B** is also continuous and the relative velocity **u** of a Newtonian fluid that wets the mantle satisfies both kinematic and no-slip CMB conditions.

Fluid motion at depth indicates a boundary layer between the interface and a main stream. Analyses of quasi-steady magnetic, mass, and momentum transport equations indicate a thin, weak, boundary layer that neither generates much electrical current nor absorbs appreciable normal fluid flow (see, *e.g., Ball, Kahle & Vestine* [1969], *Hide & Stewartson* [1972], *Benton* [1981], *Gubbins & Roberts* [1987], *Benton* [1992]). This depends on a kinematic shear viscosity ν that is very small and a magnetic diffusivity η that is not too large, conditions met with $\nu \approx 3\times10^{-7}$ m^2/s and $\eta \equiv (\mu_0\sigma)^{-1} \approx 1.6$ m^2/s for electric conductivity $\sigma \approx 5\times10^5$ S/m [*Poirier,* 1988; *Lumb & Aldridge,* 1991; *Voorhies*, 1999; *Dobson et al.,* 2000]. A thin viscous sub-layer is also suggested by equating typical magnitudes of tangential viscous forces with Coriolis or Lorentz forces and solving for Ekman or Hartmann scale depths, $\delta_E \approx 8$ cm or $\delta_H \approx 15$ cm, respectively.

The induction equation for uniform σ, our non-conservative magnetic transport equation

$$\partial_t \mathbf{B} = \nabla\times(\mathbf{u}\times\mathbf{B}) + \eta\nabla\times\nabla\times\mathbf{B}\,, \tag{4}$$

shows core-source SV can be induced by mainly lateral motion at the base of a viscous sub-layer. A feeble boundary current implies a small jump in **B** across the sub-layer, negligibly small for the normal component. Though purely diffusive at the interface itself, the radial component of (4) has thus been used to analyze SV in terms of broad-scale fluid flow, and occasionally flux diffusion, at the top of a spherical main stream. Constrained inversions of geomagnetic secular change indicate a typical flow speed U of about 7.5 km/yr (see, *e.g., Voorhies* [1995]).

With the foregoing values for U, δ_E, ν and c_s, the main stream Reynolds number Uc_s/ν of about 3×10^9 is so much greater than the boundary number $U\delta_E/\nu$ of about 60 as to indicate some small scale motions near the top of the core, perhaps in a thicker, unsteady, second boundary layer featuring eddy mixing and enhanced diffusion – if not entrainment of the overlying viscous sub-layer. Instead of a typical width for fronts between broad regions of more uniform flow, the 4.2 km lateral length scale for fluid velocity obtained by *Benton* [1992], denoted l_0, may describe a seething mass of short-lived, rotationally polarized hydromagnetic eddies. By equation (4), such narrow scale eddies could individually induce narrow scale field variations, yet could also contribute collectively to observable, broad-scale SV.

A quantitative model of this contribution is needed to test the narrow scale flow hypothesis. A model consisting of pseudo-random walks of magnetic field line foot points with a single eddy diffusivity $Ul_0 \approx 1$ m^2/s seems too much like molecular magnetic diffusion to test; moreover, a linear diffusion term with scale-invariant diffusivity cannot describe how narrow scale flow induces broad-scale SV by mode-mixing. Deterministic inversions of (4) do not resolve narrow eddies and might misattribute eddy mixing of magnetic modes to broad-scale flow or flux diffusion. Deterministic forward models, notably numerical dynamo models that solve a system of magnetic, mass, momentum and energy transport equations closed by an equation of state (see, *e.g., Glatzmaier & Roberts* [1995a,b]), can better resolve compact eddies and effects of mode mixing.

For example, a dynamo simulation used to investigate the frozen-flux core approximation included degrees as high as 239 [*Roberts & Glatzmaier*, 2000]; yet observational field models similarly used included degrees of at most 13 and often 10 or less [*Hide & Malin*, 1981; *Voorhies & Benton*, 1982; *Voorhies*, 1984; *Benton & Voorhies*, 1987]. With $l = [8\pi c_s^2/n(n+1)]^{1/2}$, we find $l_{sim} \geq 73$ km $= 17l_0$ and $l_{obs} \geq 1300$ km $= 18l_{sim}$; therefore, finer resolution is needed to simulate eddies of scale l_0. Additional assumptions about material properties, turbulent diffusivities, boundary conditions, and initial conditions further complicate hypothesis testing via numerical simulation (see, *e.g., Glatzmaier* [2002]). Both forward and inverse deterministic models of SV induced by narrow scale flow are bypassed here via a statistical kinematic model of SV induced by compact eddies.

4. An Equivalent SV Spectrum from Narrow Scale Flow

On the narrow scale flow hypothesis, magnetic changes at different locations at the top of the core are induced by different eddies transporting different local fields in different directions at different speeds. Such changes may well appear uncorrelated on scales broader than the eddies. Mathematically, the magnetic change induced by each narrow scale eddy can be approximated in the far field, notably well outside the core, by an equivalent source of change at the base of a viscous sub-layer. Following *McLeod* [1996], quasi-static lateral magnetic transport during differential interval Δt causes differential exterior magnetic change $\Delta\mathbf{B}$ equivalent to differential dipole moment changes $\Delta\mathbf{d}_i$ scattered atop the main stream at radius c^- ($i = 1, 2, 3, \ldots, I$).

To see this, recall that a single magnetic flux vector $\mathbf{B}_0 dA$ at fixed position $\mathbf{x}_0$ on the surface A of the source region acts as the point source of a dipole field with moment proportional to $\mathbf{B}_0 dA$. The magnetostatic field at position $\mathbf{x}$ due to this equivalent source is well-known (see, *e.g., Jackson* [1975, equation 5.64]). Infinitesimal quasi-static lateral displacement $\Delta\mathbf{x}$ of this single magnetic vector, with no change in orientation and magnitude, would cause a net change in the exterior field equivalent to a differential *quadrupole* moment at $\mathbf{x}_0 + \Delta\mathbf{x}/2$. More generally, however, there is a magnetic flux vector at each position on the source surface; lateral transport replaces the vector at $\mathbf{x}_0$ with an adjacent vector of slightly different orientation and magnitude; and the change in the exterior field is equivalent to that of a differential *dipole* moment $\Delta\mathbf{d}_0$ at $\mathbf{x}_0$. Given many equivalent source changes $\Delta\mathbf{d}_i$ at $\mathbf{x}_i$ on A ($i = 1, 2, 3, \ldots, I$), the total change in the exterior field $\Delta\mathbf{B}$ at $|\mathbf{x}| > c^-$ follows by superposition. In the continuum, a differential change in surface magnetic moment density at $\mathbf{x}'$ on c^- replaces discrete $\Delta\mathbf{d}_i$ as the equivalent source for exterior secular change $\Delta\mathbf{B}(\mathbf{x},t)$, which follows by integration over $\mathbf{x}'$.

Elements of the dyad formed by two differential dipole moment changes at two well separated points, $[\Delta\mathbf{d}_1][\Delta\mathbf{d}_2]^T$, may be either positive or negative. The average over a kinematically unbiased ensemble of such dyads gives zero cross-correlations, but non-zero auto-correlations. The resulting expectation spectrum for broad-scale SV is equivalent to that from laterally uncorrelated, randomly varying dipole moments on the source shell c^-, which is given by

$$\{F_n^c(r > c^-)\} \quad = \quad C\, n\, (n + \tfrac{1}{2})(n + 1)(c^-/r)^{2n+4} \,. \tag{5}$$

Equation (5) differs slightly from *McLeod* [1996, equation (11)] because changes in horizontal as well as radial components of an equivalent source can contribute to $\{F_n^c\}$: for example, consider rotation about the vertical of a horizontal equivalent source at the equator. Positive amplitude C is proportional to $\{I(\Delta\mathbf{d}_i/\Delta t)^2\}$. Physically, C tends to increase with transport speed, field gradients and intensity; yet the mathematical derivation of spectrum (5) in Appendix C does not use equation (4). It uses steps analogous to those in *Voorhies* [1998, equations (6a)-(20b)].

Laterally uncorrelated dipole changes offer a rough model of SV rich in narrow features; yet the sum of attenuated cubic SV spectrum (5) over n converges on all spheres of radius $r \geq c > c^-$. The cubic polynomial, which modulates the exponential attenuation $(c^-/r)^{2n+4}$ of a potential field, increases with n faster than does the linear polynomial obtained from SV sources equivalent to uncorrelated Dirac delta-functions in B_r. The latter are sources for the famous "white noise" spectrum, so SV spectrum (5) is said to be "blue" or "hard" – to borrow descriptors of spectra rich in short wavelength components from optical or X-ray spectroscopy, respectively. Of course, no single $\Delta\mathbf{d}$ represents perfectly the magnetic change induced by an extended eddy, so spectrum (5) will not represent SV on scales as narrow as the eddies themselves – hence at high degrees (*e.g.*, $n \geq 4{,}200$ for l_0 above). Moreover, even transient cross-correlations may result in deviations from spectrum (5) at lower degrees. Indeed, a softer or less blue SV spectrum, such as an attenuated quadratic, may indicate contributions from partially resolved eddies.

Each set of $\Delta\mathbf{d}_i$ implies one F_n^c, so derivation of theoretical SV spectral variance $\{[F_n^c - \{F_n^c\}]^2\}$ would require additional assumptions about the ensemble of equivalent SV sources – hence additional physical hypotheses about the eddy transport they represent. Extra hypotheses tend to complicate tests of the narrow scale flow hypothesis. Low degree physical deviations $F_n^c - \{F_n^c\}$ caused by core dynamic processes can, however, be represented via cross-correlated equivalent SV sources. And cross-correlation of even a small fraction $I^{-1/2}$ of the $\Delta\mathbf{d}_i$ can cause deviations with magnitudes similar to $\{F_n^c\}$. We thus anticipate a spectral variance of magnitude similar to, or perhaps a few times larger than, $\{F_n^c\}^2$ itself. Other aspects of SV spectral covariance are noted in Appendix D. Spectrum (5) is arguably better tested against a time averaged observational spectrum, not only to help average out effects of transient cross-correlations, but to identify persistent deviations from it.

5. Tests of the SV Spectrum

Theoretical SV spectrum (5) has two parameters, amplitude C and source radius c^*, to be estimated by fitting observational SV spectra $F_n(a,t)$. For a thin sub-layer, an estimate of c^* amounts to an estimate of c, so the significance of its difference from seismologic core radius c_s provides a test of equation (5), hence the underlying narrow scale flow hypothesis. Curiously, the misfit between theoretical and observational spectra does not provide a sensitive test of the hypothesis. This is because the theoretical spectral covariance needed to fully establish the statistical significance of such misfit is not specified (see Appendix E). We therefore emphasize comparison of estimated SV source radius c^* with independently determined c_s.

To ease this estimation, and anticipating fluctuations about (5) amounting to a factor of about $e^{\pm 1}$, we minimize the sum of squared residuals to observational $ln(F_n)$ for degrees n_{min} to n_{max}. The sum of squared residuals per degree of freedom for $d = n_{max} - n_{min} + 1$ data fitted by p parameters is just

$$q^2 = (d-p)^{-1} \sum_{n=n_{min}}^{n_{max}} [ln(F_n) - ln\{F_n^c\}]^2 . \qquad (6)$$

Of course, only if the residuals were approximately log-normally distributed might one hope that the estimates approximate maximum likelihood estimates. The estimation requires computation of expected parameter covariance, with the square root of the diagonal variances indicating expected parameter uncertainties. Multiplication of these values by q yields scaled parameter uncertainty estimates. Typically $q < 1$, so the scaled uncertainties are less, and the sensitivity of the test greater, than expected.

5.1 Initial Test. The observational SV spectrum fitted first was computed from field model of *Langel, Estes & Mead* [1982]. This model GSFC 9/80 closely fits 15,206 Magsat observations, 71,000 POGO observations, measurements from 148 observatories, 300 filtered marine data, and 600 measurements from select repeat stations. It includes main field, first, second, and third time derivative coefficients through degrees 13, 13, 6, and 4, respectively, and observatory biases to account for local lithospheric magnetic anomalies. Unlike some more recent field models, it imposes no smoothness or other constraints upon a core-source field.

Table 1 lists epoch t of $F_n(a,t)$, the range of degrees n fitted (n_{min}-n_{max}), estimated core radius $c^{\cdot}$ and its scaled uncertainty for three ranges. The bottom line, labeled "Avg.", gives results from fitting theoretical spectrum (5) to $\langle F_n \rangle$ integrated over the interval 1960-1980 spanned by the model. The fit to all degrees 1-13 of the time-averaged spectrum yields $c = 3470 \pm 91$ km. Table

1 also shows results from fitting intermediate degrees 3-11 and, due to long-standing concerns about SV coefficients above degree 10 [*Voorhies,* 1984], from degrees 3-10.

The agreement between geomagnetic SV spectral estimates of core radius in Table 1 and the 3480 km seismologic value is excellent. Indeed, only one value of 18 differs from c_s by more than twice its scaled uncertainty. The tabulated results are summarized as $c^{\check{}} = 3.5 \pm 0.1$ Mm $= c_s$. The hypothesis of narrow scale flow by the top of Earth's core, as represented by the broad-scale SV spectrum (5) expected from an ensemble of compact eddies inducing laterally uncorrelated SV, thus passes our initial test.

For $n_{\min} = 1$, however, closer study reveals a systematic increase in estimates of $c^{\check{}}$ from GSFC 9/80 with $n_{\max}$. This is shown in Table 2a, which lists degree range, q^2 from equation (6), estimated core radius with scaled uncertainty estimate in km, and error relative to c_s in km. The latter are judged significant for $n_{\max} < 11$ and so indicate a failure of theoretical spectrum (5); moreover, if the larger misfits found for $n_{\max} > 10$ are due to poorly determined SV coefficients, then the smaller errors might be fortuitous. To check this, we set $n_{\min}= 3$; as shown in Table 2b, so doing tightens the fit and eliminates errors in $c^{\check{}}$ in excess of twice the scaled uncertainties. Table 2b also shows increased misfit from degrees above 10. The tabulated results imply theoretical spectrum (5) adequately describes this observational SV spectrum except for degrees one and two. Downward continuation shows these exceptional terms contribute little to SV at the CMB. They may be reconciled via degree-dependent process variances from an alternative, non-log-normal, distribution. The exceptions can be understood in terms of fast decline of a strong dipole and rapid quadrupole rebound as defined below.

Voorhies & Conrad [1996] found R_1 greater, and R_2 less, than expected based on fits of spectra (A1a) or (A1b) to observational spectra at epoch 1980 – albeit within the ranges we expected 80% of the time. We also found R_1 decreasing and R_2 increasing; moreover, for all orders m, we found $(\partial_t g_1^m)/g_1^m < 0$ and $(\partial_t g_2^m)/g_2^m > 0$. The chances of such perfect (anti-) correlations were put at 1/8 for the dipole, 1/32 for the quadrupole, and 1/256 jointly. This otherwise remarkable coincidence was viewed merely as an efficient relaxation of the field towards expectation values. The large value of F_2/R_2 and the perfect correlation, and the large value of $\partial_t R_2/R_2$ itself, are called "rapid quadrupole rebound", as distinct from the quadrupole diminution noted by *Stevenson* [1983].

5.2 A Softer SV Spectrum? To further investigate SV spectra, we use the attenuated quadratic SV spectrum expected from uncorrelated, randomly varying dipoles scattered throughout the interior of a ball of radius c, which is given by

$$\{F_n^c(a > c)\}^* = C^* n(n+1)(c/a)^{2n+4} . \qquad (7)$$

This softer SV spectrum is thought to be a proxy for contributions from some partially resolved eddies. It is not thought to describe SV from small scale eddies scattered throughout the core, or even a layer thicker than about 90 km, because equation (4) fixes the origin of core-source SV at the top of the core. Surprisingly, spectrum (7) may also be a proxy for effects of laterally heterogeneous mantle conductivity. Laterally homogeneous conductivity tends to harden a core-source SV spectrum because physical attenuation decreases with harmonic degree [*McDonald*, 1957]; however, mode coupling by lateral heterogeneity in deep mantle conductivity may in effect scatter some intense, narrow scale, core-source SV into broader scales, thereby softening a core-source SV spectrum before it emerges through Earth's surface.

Tables 3a and 3b are analogous to Tables 2a and 2b, but show results of fitting proxy spectrum (7) to the 20 year average $\langle F_n \rangle$ from model GSFC 9/80. Table 3a, with $n_{min} = 1$, shows small and insignificant errors in c for $n_{max} < 11$. Table 3b, with $n_{min} = 3$, shows large positive errors in c that exceed twice the scaled uncertainty for $n_{max} > 8$. Evidently, spectrum (7) accommodates rapid dipole decline and quadrupole rebound, but is too soft for higher degrees. Further tests against other observational spectra might better distinguish between spectra (5), (7), and promising intermediate modulation factors such as $[n(n+1)]^{5/4}$, particularly if the maximum degree of reliable $F_n(t)$ can be established.

5.3 More Tests. Model CM3 [*Sabaka, Olsen & Langel,* 2002], fitted to Magsat, POGO and observatory data from 1960 to 1985, features a more comprehensive representation of external source fields than earlier models, an internal static field through degree 65, and a temporal spline parameterization of SV. The SV model was, however, constrained to reduce the amplitude of narrow scale SV that is poorly determined by geographically sparse data before POGO and after Magsat satellite surveys. Two constraints were used. The first conflicts with spectrum (5) by forcing the mean square value of the surface Laplacian of $\partial B_r/\partial t$, averaged over the sphere of radius c_s and time, to be small – as if the SV spectral modulation factor were n^{-4} or less instead of n^{+3}. This first constraint was not as strongly imposed as the second, which forces the mean square second time derivative of B_r, also averaged over the sphere of radius over c_s and time, to be small. This reduces temporal variability in F_n from model CM3, notably at high degrees; indeed, it helps makes the F_n steady to within 18% for $n > 5$, and to within 5% for $n > 10$.

Comparison of mean SV spectra from models CM3 and GSFC 9/80, both averaged from 1960-1980, shows the two sets of $\langle F_n \rangle$ agree to within 25% for $n < 11$, with CM3 giving smaller values – typically 12% smaller. For degrees 11, 12, and 13, however, values from GSFC 9/80 exceed those from CM3 by factors of 2.6, 1.8, and 34.5, respectively. The latter exceeds the factor of e anticipated from process variance. Evidently, $\langle F_{13} \rangle$ is not reliably determined and so is not considered further.

Tables 4a and 4b show results of fitting theoretical SV spectrum (5) to the mean SV spectrum from CM3, averaged from 1960 to 1980. Low values for $c^{\cdot}$ in Table 4a might again suggest spectrum (5) is too hard; yet the increase in estimated $c^{\cdot}$ with $n_{\max}$ is evident. Table 4b confirms this to be largely due to $<F_1>$ and $<F_2>$. The fits listed in Table 4b are quite tight; indeed, for the minimum q^2 of 0.0321, deviations from $\{F_n\}$ are typically a factor of $(1.2)^{\pm 1}$ instead of $e^{\pm 1}$.

Tables 5a and 5b show fits of proxy SV spectrum (7) to CM3. They confirm that the attenuated quadratic form does fairly well considering degrees 1 through $n_{\max} < 13$, but yields errors in c that are judged significant for higher degrees 3 though $n_{\max} > 7$. Downward continuation shows the higher degrees contribute far more to SV by the CMB than do F_1 and F_2. Although extended study suggests proxy spectrum (7) might be more suitable during another, shorter epoch (see Appendix F), it is bypassed for now in favor of expectation SV spectrum (5).

6. Core Magnetic Spectrum from Narrow Scale Flow and a Weak Field

Though $(F_n/R_n)^{1/2}$ was studied empirically as a kind of summary dispersion relation for the core field, *M. G. McLeod* (1985, pers. comm.) pointed out other reasons to consider spectral ratio R_n^c/F_n^c. As it turns out, some hypotheses indirectly constrain this ratio in ways that allow a theoretical core-source spectrum R_n^c to be obtained from a theoretical SV spectrum F_n^c. One such hypothesis specifies the form of the temporal power spectrum [*McLeod,* 1996]. Another is the hypothesis of "constant aspect ratio", which asserts a single direct proportionality between horizontal wave-numbers and effective radial wave-numbers in the non-potential portion of the poloidal field near a spherical CMB. Such radial wave-numbers arise from analysis of the radial component of induction equation (4) at the top of a viscous sub-layer. A third also considers the radial component of (4), but at the top of a free-stream where narrow scale eddies mix and remix modes of a dynamically weak magnetic field. Yet another reorders arguments in Appendix B.3 to obtain a relation proportional to (B17), hence (B21b), from relations (B11b), (B18) and (4).

All these hypotheses lead from attenuated cubic SV spectrum (5) to attenuated $1/n$ core field spectra like (3b) and all seem compatible with the weak field hypothesis. For brevity, it is here argued on dimensional grounds that the available diffusivities, the weak field hypothesis, and SV spectrum (5) lead to this form for $\{R_n^c\}$.

Recall our narrow eddies at the base of a viscous sub-layer. Each eddy (i = 1, 2, 3, ... I) has a characteristic lateral speed U(i), a characteristic lateral length scale $L(\mathrm{i}) << c$, hence a lateral eddy diffusivity U(i)L(i). The ratios of lateral eddy diffusivities to molecular magnetic diffusivity η define eddy magnetic Reynolds numbers $U(\mathrm{i})L(\mathrm{i})/\eta \equiv A(\mathrm{i})$. By continuity, the normal component of eddy-induced and other SV crosses a thin, weak viscous sub-layer largely unaltered and specifies the SV signal emerging from a spherical core into a source-free exterior. At the top of the sub-layer, however, $\mathbf{u}$ vanishes and even the eddy-induced portion of the signal is transmitted via molecular diffusion. Owing to its importance in such radial magnetic transport, the molecular diffusivity may be considered a radial diffusivity. The A(i) may then be considered indices of diffusive anisotropy – ratios of lateral eddy to radial molecular diffusivities.

Neither η nor eddy diffusivities are directly measured; however, observation and analysis can reveal empirical diffusivities. In particular, for a sufficiently time-varying field, time-averaged observational spectra $\langle R_n \rangle$ and $\langle F_n \rangle$ together define regular empirical time constants

$$T_n \equiv [\langle R_n \rangle / \langle F_n \rangle]^{1/2} . \qquad (8)$$

When combined with horizontal wave-numbers, defined via the surface Laplacian operator to be $k_n^h \equiv [n(n+1)/c^2]^{1/2}$, these T_n further define lateral empirical diffusivities

$$D_n \equiv (k_n^h)^{-2} T_n^{-1} = [c^2/n(n+1)][\langle F_n \rangle / \langle R_n \rangle]^{1/2} . \qquad (9a)$$

The theoretical counterpart to definition (9a) is written

$$\zeta_n \equiv (k_n^h)^{-2} \tau_n^{-1} \equiv [c^2/n(n+1)][\{F_n^c\}/\{R_n^c\}]^{1/2} . \qquad (9b)$$

For a core-source field governed by induction equation (4), the only physical diffusivities that ζ_n can depend on are η and the eddy diffusivities, though the latter may depend on other quantities. On dimensional grounds, any dependence of ζ_n on degree n should, and arguably must, be determined from that of these diffusivities.

The magnetic field and Ohmic heating of interest are far too weak to cause either appreciable anisotropy or heterogeneity in core electric conductivity; therefore, η is effectively independent of the field, its SV and the harmonic degrees thereof. Eddy speeds and length scales, hence eddy diffusivities, do not depend directly on SV. They depend only on the fluid velocity, which can depend on the field via the Lorentz force, $\mathbf{J}\times\mathbf{B}$, in the usual momentum transport equation.

If the Lorentz force is weak compared with other forces near the core surface, as for tangentially geostrophic flow [*LeMouël*, 1984], then the fluid velocity *and* the eddy diffusivities depend but weakly on the magnetic field. If this dependence is negligible, then the eddy diffusivities must be effectively independent of the magnetic field, hence the harmonic degrees of the field. On this weak field hypothesis, neither molecular nor eddy diffusivities near the top of the core depend on the harmonic degrees n of either the core field or its SV. Yet these are the physical diffusivities from which the dependence of ζ_n on degree n must be determined. With no basis on which to construct a degree-dependent theory of ζ_n, we deduce that the ζ_n in (9b) must reduce to a single constant, independent of n,

$$\zeta_n \equiv [c^2/n(n+1)][\{F_n^c\}/\{R_n^c\}]^{1/2} = \zeta\,, \qquad (10a)$$

or

$$\{F_n^c\}/\{R_n^c\} = \zeta^2 c^{-4}[n(n+1)]^2 = \tau_n^{-2}. \qquad (10b)$$

By (10b), the core field spectrum is expected to be much softer than that of core-source SV.

To check (10b), *Voorhies & Conrad* [1996] fitted the function $\beta_0 ln[n(n+1)] - 2ln\alpha_0$ to observational values of $ln[F_n/R_n]$ from degrees 3-12 of model GSFC 9/80 at epochs 1960, 1970, and 1980. The three resulting values for β_0 average to 1.957 ± 0.156. This agrees with the expectation value $\beta_0 = 2$ in (10b). The check is independent of the radius of Earth's core. The three values for α_0 are within a factor of 1.94 of 2,640 years (not the "26,400" years misprinted in *Voorhies & Conrad* [1996]). The implications for ζ, R_m, and σ are discussed in Appendix G.

Substitution of expectation SV spectrum (5), from the narrow scale flow hypothesis, into equation (10b), from the weak field hypothesis, yields our expectation core field spectrum

$$\{R_n^c\} = C\zeta^{-2}c^4(n+½)[n(n+1)]^{-1}(c^{-}/r)^{2n+4} \quad (11a)$$

$$= K(n+½)[n(n+1)]^{-1}(c^{-}/r)^{2n+4}. \quad (11b)$$

Though obtained in different ways, quantitative distinctions between spectrum (11b) and earlier forms (A.1a), (A.1b), and (A.1c) are largely confined to degrees 1 and 2.

7. Tests of the Core Field Spectrum

Expectation spectrum (11b) has two parameters, amplitude K and source radius c^{-}, to be estimated by fitting observational spectra $R_n(a,t)$. An estimate of c^{-} again amounts to an estimate of c, so the significance of its difference from c_s provides a test of (11b), hence the underlying hypotheses of narrow scale flow and a dynamically weak field by the top of Earth's core. The hypotheses do not specify process variance $\{[R_n^c - \{R_n^c\}]^2\}$, so the misfit between theoretical and observational spectra does not provide a sensitive test of the hypotheses. This process variance might be large; therefore, we again emphasize comparison of magneto-spectral estimates of c with the seismologic value c_s.

To ease this estimation, and anticipating fluctuations about (11b) amounting to a factor of about $e^{\pm 1}$, we minimize the sum of squared residuals to observational $ln(R_n)$ for degrees n_{min} to n_{max}. The sum of squared residuals per degree of freedom for n_{max} - n_{min} + 1 data fitted by 2 parameters is just

$$s^2 = (n_{max} - n_{min} - 1)^{-1} \sum_{n=n_{min}}^{n_{max}} [ln(R_n) - ln\{R_n^c\}]^2 . \quad (12)$$

The estimation requires computation of expected parameter covariance; the square root of the variances give expected parameter uncertainties. Multiplication of these values by s yields scaled parameter uncertainty estimates. Typically, $s < 1$ and the test is more sensitive than expected.

Voorhies et al. [2002] describe a test of spectrum (11b) at Magsat epoch 1980. Here spectrum (11b) is tested against the independent observational spectrum from the Ørsted Initial Field Model (OIFM) [*Olsen, et al.*, 2000]. The epoch 2000 OIFM features a weighted least squares fit of main field coefficients through degree 19, and external field coefficients of degrees 1 and 2, to 13,859 select data acquired by the Danish Geomagnetic Research Satellite *Ørsted.* Table 6 lists the range of degrees fitted (n_{min} = 1 though n_{max}), s^2, the estimate of c with scaled uncertainty, and the error of the estimate relative to c_s = 3480 km. Selection of the minimum s^2 solution at n_{max} = 12 fixes a third parameter and yields c = 3542 ± 61 km as the core radius estimated from Ørsted. No significant errors in c are found for degree ranges 1 to n_{max} < 13, so (11b) passes this test. Estimates from these degree ranges all agree and average to 3489 ± 39 km. The increased misfit and significant errors introduced with degrees above 12 are attributed to non-core, likely crustal, source fields.

To refine the test by *Voorhies et al.* [2002], we use R_n from model CM3 of *Sabaka, Olsen and Langel* [2002] at Magsat epoch 1980. Table 7 lists the range of degrees fitted, s^2, the estimate of c with scaled uncertainty, and the error of estimate relative to c_s. Again, no significant errors in estimates of c are found for degree ranges 1 to $n_{max} < 13$. Radii from this degree range all agree and average to 3495 ± 28 km. This average more heavily weights lower degree R_n, unlike the process variance for the R_n^c distribution of *Voorhies & Conrad* [1996].

Table 8 is analogous to Table 7, but with $n_{min} = 3$. It confirms that much of the scatter, as measured by the larger values for s^2 in Table 7, comes from strong R_1 and weak R_2. Table 8 also shows the effect on estimates of c associated with R_8 being lower, and R_9 higher, than expected based on the fit of (11b). The larger residuals at degrees 1, 2, 8, and 9 may help provide some indication of geomagnetic variability about (11b), hence process variance.

A reviewer asks why analysis of magnetic spectra is any better at determining core radius than the frozen-flux method of Hide. I did not claim it is; however, estimates obtained using spectrum (11b) and the main field models are typically more accurate and precise than those obtained using the frozen-flux approximation [*Hide & Malin*, 1981; *Voorhies & Benton*, 1982; *Voorhies,* 1984]. In particular, some 44 frozen-flux core locations obtained from a few field models at various truncation levels average to 3506.2 ± 300.9 [*Voorhies,* 1984, equation (3.21)]. The first 9 estimates in Table 6 average to 3489 ± 39 km; the first 9 in Table 7 average to 3495 ± 28 km. Frozen-flux methods rely heavily upon uncertain phase information in harmonic orders m and upon uncertain secular change information from either SV models or main field models at different epochs. Analysis of F_n also relies on SV. In contrast, the main field spectral method relies on comparatively well-determined R_n alone; moreover, it does not require the frozen-flux approximation – either in section 6, in the scale analysis of Appendix B.3 and B.4 that *requires* finite σ to obtain spectrum (3b), or in the analysis of flux diffusion that returns a similar spectrum on the constant aspect ratio hypothesis. Evidently, these advantages reduce scatter.

8. Summary and Conclusions

A theoretical form for the low degree, core-source geomagnetic spectrum $\{R_n^c\}$ is deduced from the hypotheses of narrow scale flow and a dynamically weak magnetic field by the top of Earth's core. This form (11b) differs but slightly from those advanced by *Stevenson* [1983] and *McLeod* [1985, 1996]. To test these hypotheses, this theoretical spectrum is fitted to two observational spectra, one determined by analysis of data from the Ørsted satellite and the other by analysis of independent data from Magsat and POGO satellites and surface observatories. These fits yield estimates of the radius of Earth's core. These estimates differ insignificantly from the seismologically established radius, as judged by the scaled uncertainty estimates, for degree ranges 1 to $n_{max} < 13$. Because these errors of estimate are not significant, the hypotheses taken together pass these tests.

Significant errors introduced with higher degree multipole powers are attributed to non-core, likely crustal source fields. Very small errors, less than 10 km, are found by excluding the strong, rapidly declining dipole power and the weak, rapidly rebounding, quadrupole power.

To deduce the expectation spectrum (11b), we can use the theoretical low degree SV spectrum (5), which follows from the narrow scale flow hypothesis alone. There are other ways to obtain *both* spectra (11b) *and* (5), as illustrated in Appendix B; however, the geophysical foundation of all such methods known to date appears to rest on the narrow scale flow and weak field hypotheses – provided the temporal power spectrum of *McLeod* [1996] can be inferred from the approximate spatial spectra. Initial tests of SV spectrum (5) gave estimates of the radius of Earth's core that differ insignificantly from the seismologic value. This shows that the narrow scale flow hypothesis can pass a test against seismology that is independent of the surficially weak field hypothesis. Attempts to test (5) with the same rigor as (11b), however, reveal errors arising from rapid dipole decline and quadrupole rebound. Though F_1 and F_2 contribute little to SV at the CMB, such exceptional behavior of may cast doubt on (5); yet it may instead indicate a need to use degree-dependent process variances and a different distribution for residuals about SV spectrum (5).

Clearly, physical hypotheses are not proved true simply by passing a few tests, nor does falsification preclude their use in successive approximation. And there might be different physical hypotheses that yield theoretical spectra like those presented here. The hypotheses of narrow scale flow by the top of the core, as described by expectation SV spectrum (5), and of a dynamically weak core surface field, described via magnetic spectrum (11b), have nonetheless demonstrated considerable merit. By enabling magneto-locations of Earth's core, these spectra demonstrate greater utility and accuracy than far softer spectra. Evidently, far softer spectra do not provide reliable prior information on either the broad scale field or its secular variation. It is hoped these geophysical hypotheses and geomagnetic spectral forms will be further tested, and provide some basis for comparison with alternatives that may emerge, in the future.

Appendix A: Magnetic Spectral Ranges

To compare a core-source field spectrum with low degree energy, intermediate, and high degree dissipation ranges with theory, recall the magnetic energy spectrum $M(k)$ as a function of Cartesian-Fourier wave-number k (see, *e.g., Moffat* [1978], *Krause & Rädler* [1980]). The proportionality $M(k) \propto k^{-3/2}$ is indicated for the inertial sub-range of three-dimensional, homogeneous, isotropic, incompressible hydromagnetic turbulence; however, kinetic helicity injection at large k leads to an inverse cascade of magnetic helicity and $M(k) \propto k^{-1}$ at low k [*Pouquet, Frisch & Leorat*, 1976; *Stevenson,* 1983]. The latter shows similarity between an n^{-1} spectrum and a downwardly continued observational spectrum $R_n(0.55a, 1965)$ for $n \leq 8$.

Next define horizontal wave-number $k_{hn} \equiv [n(n+1)/c^2]^{1/2}$ via the surface Laplacian. Granting $M(k) \propto k^{-1}$ at low k, if $M(k) \propto \{R_n^c(c)\}$ and if $k^2 \propto k_{hn}^2$, then $\{R_n^c(c)\} \propto [n(n+1)]^{-1/2}$ at low n and

$$\{R_n^c(a)\} = K_1 [n(n+1)]^{-1/2} (c/a)^{2n+4} \quad \text{(A1a)}$$

$$\cong K_M (n+1/2)^{-1/2} (c/a)^{2n+4} \quad \text{(A1b)}$$

$$\cong K_S n^{-1} (c/a)^{2n+4} . \quad \text{(A1c)}$$

Here K_1, K_M and K_S denote constants, c core radius, and a Earth's radius. By inequality (3a) for $n \geq N_D$, if either *Stevenson's* [1983] relation (A1c), *McLeod's* [1996] rule (A1b), or equation (A1a) holds at low degrees $n \leq N_E$, then $N_E \leq N_D$, then there may indeed be an intermediate, if not inertial, sub-range between N_E and N_D.

Of course, the "ifs" strain a comparison already made difficult by possible effects of mantle heterogeneity near the CMB; anisotropy imposed by rotation at planetary angular velocity $\mathbf{\Omega}$, the CMB, and the field itself; compression; suppression of turbulence by rotation or a strong field; and sources in the mantle and crust. Yet as argued in Appendix B, the net effect might amount to a time-averaged spectrum similar to (A1a), perhaps with power distributed unevenly among the various orders within each R_n, notably R_1, due to anisotropy and lateral heterogeneity. There is evidence that spectra like (A1a) describe both modern observational spectra and time-averaged paleo-field behavior [*Voorhies & Conrad*, 1996]. Moreover, agreement between such spectra and observation [*Voorhies et al.*, 2002] offers some support for the temporal magnetic power spectrum used by *McLeod* [1996] to obtain (A1b) for $n \geq 2$.

Appendix B: Spectra from Scale Analyses

A reviewer asks if the spectrum of a magnetic field in a turbulent fluid has physical plausibility analogous to the famous Kolmogorov $k^{-5/3}$ scaling for the kinetic energy density spectrum $E(k)$ in the inertial sub-range of a turbulent flow (see, e.g., *Tennekes & Lumley* [1972]) and requests mathematical illustration. Here we show the answer is yes and offer an illustration that yields both expected spectrum (3b) and SV spectrum (5). As noted by *Pouquet et al.* [1976], however, care is needed to obtain a magnetic spectrum excited by small scale flow, instead of an Alfven-wave spectrum excited in an inertial sub-range maintained by large scale flow (see B.1 below). Additional care is needed to obtain a spectrum for a self-governing dynamo, rather than one driven by assumed rates of kinetic energy injection (see B.2). Finally, to deduce theoretical spectra comparable with observational spectra, extra care is needed to account for sphericity and the anisotropic dynamical effects of rotation, Lorentz forces, and the CMB (see B.3 and B.4).

Consider flow of a magnetized Newtonian fluid with scalar mass density ρ, kinematic shear viscosity ν, magnetic permeability μ, electric conductivity σ and magnetic diffusivity $1/\mu\sigma \equiv \eta$. Quasi-steady changes in macroscopic $\mathbf{B}$ due to fluid velocity $\mathbf{u}$ and magnetic diffusion are described by the induction equation (4). For a characteristic length scale L and flow speed U, the ratio of motional to diffusive terms scales as the magnetic Reynolds number

$$Rm \equiv UL/\eta \approx |\nabla\times(\mathbf{u}\times\mathbf{B})|\,|\eta\nabla\times\nabla\times\mathbf{B}|^{-1}\,. \qquad \text{(B1)}$$

This is also the ratio of the characteristic eddy diffusivity to the magnetic diffusivity.

Much as the kinetic energy transport equation is obtained from the inner product of $\mathbf{u}$ with the momentum equation, the magnetic energy transport equation is obtained via the inner product of $\mathbf{B}$ with the induction equation (see, e.g., *Chandrasekhar* [1981]; *Gubbins & Roberts* [1987]). The Ohmic dissipation of magnetic energy per unit volume, $-\mathbf{J}^2/\sigma$, scales as $\eta B^2/2\mu L^2$. The viscous dissipation of kinetic energy per unit volume, $-\rho\nu\mathbf{u}\bullet\nabla\mathrm{X}\nabla\mathrm{X}\mathbf{u}$ for solenoidal flow, scales as $\nu\rho U^2/2L^2$. The ratio of magnetic to viscous dissipation scales as $(\nu/\eta)^{-1}(B^2/\mu\rho U^2)$, where ν/η is the magnetic Prandtl number. In the flow of a fluid metal with $\nu/\eta << 1$, if magnetic energy density $\mathbf{B}^2/2\mu$ is at least as great as kinetic energy density $\rho\mathbf{u}^2/2$, then magnetic dissipation will be very much greater than viscous dissipation. So we focus on magnetic dissipation and dynamo action, whereby motion of the fluid conductor across the field does work against the Lorentz force, converting kinetic into magnetic energy. The latter may accumulate, dissipate into heat, or radiate away. The rate of work done against the Lorentz force per unit volume, $-\mathbf{u}\bullet(\mathbf{J}\times\mathbf{B})$, scales as $UB^2/2\mu L$.

More generally, **u** and **B** may vary on all possible length scales l and may be represented mathematically via superposition of orthogonal modes, such as Fourier transforms with wave-vector **k** and wave-number $|\mathbf{k}| = k \approx l^{-1}$. Work done by flow at one scale against the non-linear Lorentz force can energize the field over a range of scales. And magnetic energy at one scale is influenced by the flow over a range of scales. Because of this mode mixing, magnetic energy dissipated at any single scale may come from kinetic energy distributed over a range of scales.

Mode mixing is governed by the usual selection rules. We shall, however, only consider states of a hydromagnetic system with a standard spectral deviation $<[M(k) - <M(k)>]^2>^{1/2}$ that does not vastly exceed the mean $<M(k)>$. In such a state, a theoretical expectation spectrum $\{M(k)\} \approx <M(k)>$ could be of some use; moreover, there is a fair chance that a sample spectrum at a single time is within a factor of two or three of the mean. In or near such a statistically steady state, energy mixed from one mode k_α to other modes k_i at one rate is re-mixed to still other modes k_j and, on average, returns to mode k_α at about the same rate. This implies the transformed magnetic and velocity fields are so thoroughly intertwined that fields of like wave-number are physically related, as can be seen by repeated or multiple applications of the selection rules. We focus on the physical relations, with the understanding that they result from mode mixing rather than a single application of the selection rules.

Suppose **u** and **B** can be considered smooth on very small spatial scales, with scale lengths < λ_0. At length scale λ_0, let *Rm* attain unity at characteristic speed $u = \upsilon_0$:

$$\upsilon_0\lambda_0/\eta = 1 . \tag{B2}$$

At this scale and speed, eddy diffusivity $ul = \upsilon_0\lambda_0$ will equal magnetic diffusivity η and advective time-scale $\tau_a \equiv l/u$ will equal diffusive time-scale $\tau_d \equiv l^2/\eta$. At larger scales, or at greater speeds, fluid motion may curl the field faster than it diffuses away via electrical resistance to its source current of density **J** ($\nabla \times \mathbf{B} = \mu\mathbf{J}$, $\nabla \times \nabla \times \mathbf{B} = -\nabla^2\mathbf{B}$).

Let $b(l)$ denote the magnetic field of length scale l and let b_0 denote $b(\lambda_0)$. At $l = \lambda_0$ and speed υ_0, the magnetic dissipation scales as $D_0(\lambda_0) \approx \eta b_0^2/2\mu\lambda_0^2$. By equation (B2), this also scales as the rate of work against the Lorentz force: $D_0 \approx \eta b_0^2/2\mu\lambda_0^2 = \upsilon_0 b_0^2/2\mu\lambda_0$. So λ_0 is the magnetic dissipation scale, $\lambda_0 \approx (\eta b_0^2/2\mu D_0)^{1/2}$, and υ_0 is about $[2D_0/\sigma b_0^2]^{1/2}$.

B.1 Kolmogorov-Alfven Scaling

Denote by ε the total magnetic dissipation per unit volume in a hydromagnetic flow. Suppose this occurs mainly over a range of length scales less than or approximately equal to λ_0. Further suppose the magnetic energy dissipated comes from the kinetic energy $\rho U^2/2$ of a flow with speed U and large geometric length scale $L >> \lambda_0$. The energy is supplied by work done against the field B of length scale L. If this energy conversion occurs in advective time-scale L/U, then

$$\varepsilon \approx UB^2/2\mu L \approx \rho U^3/2L . \tag{B3}$$

More generally, the energy dissipated may come though a range of scales. Consider a sub-range $\lambda_0 \leq l \leq L$ and $\upsilon_0 \leq u \leq U$ through which magnetic energy cascades from the field of scale L toward the dissipation range at rate

$$\varepsilon \approx ub^2/2\mu l \approx \rho u^3/2l . \tag{B4a}$$

This *assumes* the time-scale for kinetic energy density $\rho u^2/2$ in motions $u(l)$ of scale l to be transferred to the field $b(l)$ is τ_a. Granting (B4a) for now and solving for u gives:

$$u \approx (2\mu l\varepsilon/b^2) \approx (2l\varepsilon/\rho)^{1/3}. \tag{B4b}$$

The implied kinetic energy per unit volume at wave-number $k \approx l^{-1}$ is $\rho u^2/2 \approx (\rho/2)^{1/3}(\varepsilon/k)^{2/3}$; therefore, the kinetic energy density per wave-number is

$$E_{KA}(k) = \rho u^2/2k \approx (\rho\varepsilon^2/2)^{1/3}k^{-5/3}. \tag{B5}$$

This is proportional to the Kolmogorov form for an inertial sub-range. Also by relation (B4b), the magnetic energy per unit volume at k is $b^2/2\mu \approx (\rho/2)^{1/3}(\varepsilon/k)^{2/3}$; therefore, in this case the magnetic energy density per wave-number is

$$M_{KA}(k) = b^2/2\mu k \approx (\rho\varepsilon^2/2)^{1/3}k^{-5/3}. \tag{B6}$$

As appropriate to a field of Alfven waves excited by motions in an inertial sub-range, $E_{KA}(k)$ and $M_{KA}(k)$ are approximately equal. Of course, relations (B4a) though (B6) fail when dynamical constraints render advective time-scales τ_a irrelevant to the transfer of kinetic energy.

B.2 Large Scale Flow

By presuming the rate of work done against the field equals the kinetic energy density per advective time-scale τ_a, case B.1 compels magnetic and kinetic energies to be in approximate balance, as in an Alfven wave field. For dynamo action, however, magnetic energy density $b^2(l)/2\mu$ need only be replenished over free decay time τ_d. The rate of work done against the Lorentz force still scales as $ub^2/2\mu l$, but the kinetic energy transfer only needs to occur over time l^2/η. In this case, relation (B4a) would be replaced with

$$\varepsilon \approx ub^2/2\mu l \approx \rho u^2\eta/2l^2. \tag{B7a}$$

Granting this for now and solving for u yields

$$u \approx (2l^2\varepsilon/\rho\eta)^{1/2} \approx (2\mu l\varepsilon/b^2) \approx (lb^2/\rho\eta\mu). \tag{B7b}$$

This relation implies

$$b^2/2\mu \approx (\rho\eta\varepsilon/2)^{1/2} . \tag{B8}$$

By relation (B8), the magnetic energy density at $k \approx l^{-1}$ is also about $(\rho\eta\varepsilon/2)^{1/2}$; therefore, in this second case, the magnetic energy density per wave-number is

$$M_{LSF}(k) = b^2/2\mu k \approx (\rho\eta\varepsilon/2)^{1/2} k^{-1}. \tag{B9}$$

Also from relation (B7b), the kinetic energy density at k is $\rho u^2/2 \approx (\varepsilon/\eta)k^{-2}$; therefore, in this case the kinetic energy per unit mass per wave-number is

$$E_{LSF}(k) = \rho u^2/2k \approx (\varepsilon/\eta)k^{-3}. \tag{B10}$$

This k^{-3} form describes a predominantly large scale flow. It illustrates how a strong field may suppress small scale flow. Indeed, $E(k)/M(k)$ from relations (B10) and (B9) is $(2\varepsilon/\rho\eta^3)^{1/2}k^{-2}$, which falls off as k^{-2} and increases with $\sigma^{3/2}$ and $\varepsilon^{1/2}$. The spectrum of magnetic change, obtained using relations (B9) and (B10), the motional term in equation (4), and implicit treatment of mode mixing, is proportional to that of the field itself: $F_{LSF}(k) = (\partial_t b)^2/k \approx (kub)^2/k \approx (2\varepsilon\mu)^{3/2}(\sigma/\rho)^{1/2}k^{-1}$. Evidently, this is not the case near the top of Earth's core.

Case B.2 might seem to describe a more efficient dynamo than case B.1 because, presuming $\tau_d >> \tau_a$, the kinetic energy required to maintain a large scale field would be extracted over a longer time. In fact, case B.2 still presumes: (i) Ohmic dissipation is confined to small scales $\leq \lambda_0$ on which $\tau_d \leq \tau_a$; (ii) an energy cascade from large to small scales; and (iii) a time-scale for kinetic energy transfer devoid of dynamical justification.

B.3 Small Scale Flow

Now suppose most of the kinetic energy is injected at fairly small scales near l^*, albeit with $l^* \geq \lambda_0$. This motion may set up a cascade of magnetic energy to smaller scales, but may also drive a mode-mixed reverse cascade to larger scales. Let ε indicate the overall magnetic dissipation in the flow, which realistically occurs over the full range of length scales. At larger scales $l > l^*$, and near a statistically steady state, the rate of work done against the Lorentz force approximately balances Ohmic dissipation, so

$$|\mathbf{u}\bullet(\mathbf{J}\mathrm{X}\mathbf{B})| \approx \alpha^{-1}u(l > l^*)[b(l > l^*)]^2/\mu l$$

$$\approx [b(l > l^*)]^2/\mu^2\sigma l^2 \approx |\mathbf{J}^2/\sigma|. \qquad \text{(B11a)}$$

Here α is a constant representing typical geometric factors; it is large if the field tends to be nearly Lorentz force-free, or if **u** tends to be parallel to either **J** or **B**. We solve (B11a) for

$$u(l > l^*) \approx \alpha\eta/l, \qquad \text{(B11b)}$$

and see α as a pseudo-scale invariant magnetic Reynolds number. By (B11b), we expect

$$\{E(k < k^*)\} \equiv \{\rho[u(k < k^*)]^2/2k\} \approx \alpha^2\rho\eta^2 k/2. \qquad \text{(B12)}$$

This is consistent with a predominantly small scale flow.

We cannot assume the time-scale on which work is done at the expense of kinetic energy at length scale l because it depends on dynamical constraints near the top of a self-excited, self-regulating core geodynamo. Following *Voorhies* [1991, equation (B5)] and *Benton* [1992, equation (32)], the dynamical constraints are here summarized by the magneto-geostrophic radial vorticity balance near the top of the core,

$$\rho\nabla s\bullet(2\Omega_r\mathbf{u}_s) \cong -\nabla s\bullet(B_r\mathbf{J}_s), \qquad \text{(B13)}$$

where $\nabla s\bullet$ denotes the surface divergence operator, Ω_r the radial component of planetary angular velocity ($\Omega_0\cos\theta$), B_r the radial component of **B**, and $\mathbf{u}_s$ and $\mathbf{J}_s$ the horizontal components of **u** and **J**, respectively. Anisotropies imposed by rotation and by the field are explicit in equation (B13). The published derivations explicitly account for anisotropy imposed by the CMB by omitting small terms with u_r and J_r, terms which would be zero at the spherical boundary with a rigid, insulating mantle. The condition $J_r = 0$ implies any poloidal current, hence toroidal magnetic field, is zero at the boundary; therefore, a magnetic spectrum deduced using (B13) should only be applied to the poloidal field at the top of the core and, upon upward continuation, to the potential field of core origin.

If the simple scaling of (B13) were multiplied by l, it would suggest Lorentz and Coriolis forces are of similar magnitude. Yet (B13) concerns vertical vorticity, not forces, and can hold when the Lorentz force is weak. Moreover, such scaling overlooks some geometric effects. For example, axisymmetric zonal flows contribute nothing to the left side of (B13), yet may be important in the force balance; similarly, a class of currents tangent to contours of B_r may help generate the main field, yet contribute nothing to the right side of (B13) [*Voorhies,* 1991]. To avoid any misimpression, equation (B13) is here scaled as

$$2\beta\rho\Omega u(l > l^*)/l \approx [b(l > l^*)]^2/\mu l^2, \tag{B14}$$

where β is a constant representing typical geometric factors. On the hypothesis of a very weak Lorentz force near the top of the core, $\beta << 1$.

From relation (B14), we find

$$[b(l > l^*)]^2/\mu \approx 2\beta\rho\Omega l u(l > l^*)\,. \tag{B15}$$

We use relation (B11b) to eliminate u from (B15) and obtain

$$[b(l > l^*)]^2/\mu \approx 2\alpha\beta\rho\Omega\eta\,. \tag{B16}$$

From this relation, the expected magnetic energy density per wave-number is

$$\{M(k < k^*)\} \equiv \{[b(k < k^*)]^2/2\mu k\} \approx \alpha\beta\rho\Omega\eta k^{-1}. \tag{B17}$$

If relation (B11a) between Ohmic dissipation and the rate of work done against Lorentz forces can be extended to a rate of change of kinetic energy, so that $\eta b^2/\mu l^2 \approx u b^2/\mu l \approx \rho u^2/\tau$, we can solve for the transfer time-scale $\tau \approx (\rho u^2\mu/b^2)l^2/\eta$. By relations (B11b) and (B16), this dynamical time-scale turns out to be $\tau \approx \alpha/2\Omega\beta$, another pseudo-scale invariant for $l > l^*$. Of course, for $\alpha >> 1$ and $\beta << 1$, $\tau >> \Omega^{-1}$.

With the right hand side of induction equation (4) scaled according to the motional term, and using relations (B12) and (B16), the expected SV spectrum is

$$\{F(k > k^*)\} = \{(\partial_t b)^2/k\} \approx [ku(k < k^*)b(k < k^*)]^2/k \approx 2\alpha^3\beta\eta^3\mu\rho\Omega\, k^3. \quad \text{(B18)}$$

The ratios $\{2\mu M\}/\{F\}$ from relations (B18) and (B17) define scale-variant squared time-scales $(\alpha\eta k^2)^{-2}$ which constrain α independent of β (see section 6).

If $l^* \approx \lambda_0$, an energy cascade from k^* to larger k would go directly into the dissipation range. Instead suppose $l^* > \lambda_0$ and consider the intermediate sub-range $\lambda_0 < l < l^*$. Let ε_s denote the magnetic dissipation on scales $\leq \lambda_0$, so $\varepsilon_s < \varepsilon$. This dissipated energy is re-supplied by work done against the Lorentz force which, in turn, comes from kinetic energy on scales near l^*. This energy input is now at relatively larger scale $l^* > l$, so one might re-consider cases (B.1) or (B.2). Motions in the sub-range should contain considerable kinetic energy, perhaps suggesting use of (B4a). By relation (B17), however, these motions are embedded in a relatively larger scale field, which may favor (B7a). Yet a small scale eddy cannot be everywhere parallel to a large scale field, so neither the rate of work against the Lorentz force nor the magnetic dissipation will be negligible in the intermediate range. Therefore, neither case (B.1) nor (B.2) need apply. Further analysis of an intermediate, if not inertial, sub-range is omitted for brevity.

B.4 Simple Discretization and Accounting for Sphericity

We need to relate continuous Cartesian magnetic spectrum (B17) with discrete spherical harmonic spectra (A1a), (3c) or (11c), and SV spectrum (B18) with (5), albeit only to the order of magnitude accuracy appropriate to a simple scale analysis. To do so, first integrate continuous spectrum (B17) over the small domain from $k - \Delta k/2$ to $k + \Delta k/2$,

$$\int_{k-\Delta k/2}^{k+\Delta k/2} \{M(k > k_0)\} \mathrm{d}k \approx \alpha\beta\rho\Omega\eta \int_{k-\Delta k/2}^{k+\Delta k/2} k^{-1} \mathrm{d}k \quad \text{(B19a)}$$

$$\approx \alpha\beta\rho\Omega\eta \; ln[(1 + \Delta k/2k)/(1 - \Delta k/2k)] \quad \text{(B19b)}$$

$$\approx \alpha\beta\rho\Omega\eta(\Delta k/k) . \quad \text{(B19c)}$$

where the logarithm has been expanded and approximated in terms of $\Delta k/2k << 1$.

Next, guided by the surface Laplacian operators in the transform domains, set $k^2 = n(n+1)/c^2$. This implies $\Delta k = (n + 1/2)[n(n+1)]^{-1/2} c^{-1} \Delta n$, and we set $\Delta n = 1$ to express the magnetic energy density in harmonics of integer degree n. With these identities we obtain

$$\int_{k-\Delta k/2}^{k+\Delta k/2} \{M(k < k_0)\} \mathrm{d}k \approx \alpha\beta\rho\Omega\eta \; (n + 1/2)[n(n+1)]^{-1} . \quad \text{(B20)}$$

Note k has been treated as a horizontal wave-number k_h. Distinctions between radial wave-vector component k_r and k_h are omitted for three reasons. First, we are concerned with the spectrum of the field on a thin shell by the top of the core. Second, apart from the tiny jump in horizontal components across a viscous sub-layer, this field should match a core-source potential field (solenoidal $\mathbf{B} = -\nabla V$, so $-\nabla^2 V = 0$, $k_r^2 + k_h^2 = 0$, and $k_r = \pm i k_h$). Third, the vertical vs. horizontal anisotropy implied by CMB conditions was already used to derive equation (B13), hence is implicit in relation (B17). This omission of distinctions between k_r^2 and k_h^2 is consistent with the constant aspect ratio hypothesis (see section 6).

Relation (B20) gives the magnetic energy density per harmonic degree. The expected squared magnetic field per harmonic degree at the top of the core is 2μ times this quantity, so

$$\{R_n(c^-)\} \approx (2\alpha\beta)(\rho\Omega\mu\eta) \; (n + 1/2)[n(n+1)]^{-1} \quad \text{(B21a)}$$

at low degrees. On upward continuation to radius $r > c$, this yields

$$\{R_n^c(r)\} \approx K(n + ½)[n(n+1)]^{-1} \; (c^-/r)^{2n+4} , \quad \text{(B21b)}$$

which is relation (3b). Though obtained in a very different way, it is indistinguishable from equation (11b) and differs but slightly from equations (A1a), (A1b), and (A1c). We have, however, identified $K \approx 2\alpha\beta\rho\mu\eta\Omega$. Granting $\mu = \mu_0$, $\rho = 10^4$ kg/m^{-3}, $\Omega = 7.29\times10^{-5}$ s^{-1}, and η = 1.6 m^2/s, the estimate $K = 4.5\times10^{10}$ nT^2 [*Voorhies et al.*, 2002] implies $\alpha\beta$ is roughly 1.5×10^{-2}. With $\alpha \approx R_m \approx 92$ from Appendix G, we find $\beta \approx 1.7\times10^{-4} << 1$. This is consistent with, and arguably requires, a dynamically weak field near the top of the core. These values further yield a dynamical time scale $\alpha/2\Omega\beta$ of order 120 years.

Similar operations on relation (B18), albeit with no multiplication by 2μ, yields

$$\{F_n(c^-)\} \approx 4\alpha^3\beta\eta^3\mu\rho\Omega\, c^{-4}\,(n + 1/2)[n(n + 1)] \quad \text{(B22a)}$$

for low degrees, or, on upward continuation to $r > c$,

$$\{F_n(r)\} \approx 4\alpha^3\beta\eta^3\mu\rho\Omega\, c^{-4}\, n(n + 1/2)[n + 1]\,(c^-/r)^{2n+4} \;. \quad \text{(B22b)}$$

This is indistinguishable from equation (5). The ways in which these relations are obtained could hardly be more different, but both suppose narrow scale flow and so give the same result.

Appendix C: Derivation of an Expectation SV Spectrum

The magnetic change induced by each narrow-scale eddy can be approximated in the far field by an equivalent source of change atop a mainstream of radius c^-. Though a single magnetic flux vector on c^- acts as the point source of an offset dipole field, lateral advection replaces it with an adjacent vector of slightly different orientation and magnitude in time Δt; therefore, the net change in the exterior field is equivalent to that from a differential change in magnetic dipole moment $\Delta\mathbf{d}$. Here we derive the secular variation (SV) spectrum from one such equivalent SV source, then the SV spectrum (5) expected from I random, uncorrelated equivalent SV sources.

Denote by $\Delta V_i(\mathbf{r})$ the change in scalar magnetic potential at observation point $\mathbf{r}$ due to the equivalent magnetic dipole moment change $\Delta\mathbf{d}_i$ at position $\mathbf{r}_i$ corresponding to the i^{th} eddy. Note

$|\mathbf{r}| > |\mathbf{r}_i| = c^-$ and index $i = 1, 2, 3, \ldots, I$. Further denote by $\mathbf{D}^i$ the normalized rate of change $\Delta\mathbf{d}_i/\mu_0\Delta t$. The superposition of all I such changes gives the total change in potential at $\mathbf{r}$,

$$\Delta V(\mathbf{r}) = \sum_{i=1}^{I} \Delta V_i(\mathbf{r}) = -\sum_{i=1}^{I} \frac{\mathbf{D}^i}{4\pi} \bullet \nabla \frac{1}{|\mathbf{r} - \mathbf{r}_i|} \Delta t \tag{C1a}$$

$$= \sum_{i=1}^{I} (\mathbf{D}^i/4\pi) \bullet \nabla^* |\mathbf{r} - \mathbf{r}_i|^{-1} \Delta t \tag{C1b}$$

where ∇^* denotes the gradient operator in terms of, and acting on, $\mathbf{r}_i$ coordinates (radius r_i, co-latitude θ_i, east longitude ϕ_i). With the Schmidt normalized associated Legendre function of degree n and order m denoted $P_n^m(\cos\theta)$, the spherical harmonic expansions of the ΔV_i are used to rewrite (C1b) as

$$\Delta V(\mathbf{r}) = \sum_{i=1}^{I} a\Delta t \sum_{n=1}^{\infty} (a/r)^{n+1} \sum_{m=0}^{n} [A_n^m{}_i \cos m\phi + B_n^m{}_i \sin m\phi]\, P_n^m(\cos\theta)\,. \tag{C1c}$$

To determine coefficients $(A_n^m{}_i, B_n^m{}_i)$ in expansion (C1c) from moment changes $\mathbf{D}^i$, recall

$$|\mathbf{r} - \mathbf{r}_i|^{-1} = \sum_{n=1}^{\infty} \sum_{m=0}^{n} \frac{(r_i)^n}{r^{n+1}} [C_n^m(\theta,\phi)C_n^m(\theta_i,\phi_i) + S_n^m(\theta,\phi)S_n^m(\theta_i,\phi_i)]\,, \tag{C2}$$

where $C_n^m \equiv \cos m\phi P_n^m(\cos\theta)$ and $S_n^m \equiv \sin m\phi P_n^m(\cos\theta)$ (see, *e.g.*, *Jackson* [1975, eqn. (3.70) or *Langel* [1987, eqn. (195)]). With the components of $\mathbf{D}^i$ denoted $(D_r^i, D_\theta^i, D_\varphi^i)$, we use (C2) to evaluate the gradient in (C1b), equate the result with (C1c), and obtain

$$A_n^m{}_i = (4\pi a^3)^{-1}(r_i/a)^{n-1}[D_r^i n C_n^m(\theta_i,\phi_i) + D_\theta^i \frac{\partial}{\partial\theta_i} C_n^m(\theta_i,\phi_i) - D_\varphi^i (m/\sin\theta_i) S_n^m(\theta_i,\phi_i)] \tag{C3a}$$

$$B_n^m{}_i = (4\pi a^3)^{-1}(r_i/a)^{n-1}[D_r^i n S_n^m(\theta_i,\phi_i) + D_\theta^i \frac{\partial}{\partial\theta_i} S_n^m(\theta_i,\phi_i) + D_\phi^i (m/\sin\theta_i) C_n^m(\theta_i,\phi_i)]\,. \tag{C3b}$$

In the limit as Δt approaches zero, the equivalent core-source SV coefficients are

$$\partial_t g_n^m = \sum_{i=1}^{I} A_n{}^m{}_i \tag{C4a}$$

$$\partial_t h_n^m = \sum_{i=1}^{I} B_n{}^m{}_i \,. \tag{C4b}$$

Substitution of these coefficients into equation (2) gives the equivalent core-source SV spectrum. Clearly, statistical hypotheses about moment changes $\Delta \mathbf{d}_i$, notably those implied by physical hypotheses about eddy induced SV, can be used to derive an expectation core-source SV spectra

$$\{F_n^c(r>r_i)\} = (n+1)(a/r)^{2n+4}\{ \sum_{m=0}^{n} [(\sum_{i=1}^{I} A_n{}^m{}_i)^2 + (\sum_{i=1}^{I} B_n{}^m{}_i)^2\} \tag{C5a}$$

$$\{F_n^c(r> r_i)\} = (n+1)(a/r)^{2n+4}\{ \sum_{m=0}^{n} [(\sum_{i=1}^{I} \sum_{k=1}^{I} (A_n{}^m{}_i A_n{}^m{}_k) + (B_n{}^m{}_i B_n{}^m{}_k)] \} \tag{C5b}$$

Suitably correlated, perhaps non-conserved, equivalent SV sources may also describe effects of molecular magnetic diffusion; unfortunately, the linearity of the diffusion operator might lead one to an equivalent source representation of the core field itself, which is judged ill-advised.

SV Spectrum from a Single Change in Dipole Moment

The SV spectrum from a single $\mathbf{D}^i$ is

$$F_n^c(r > r_i) = (n+1)(a/r)^{2n+4} \sum_{m=0}^{n} (A_n{}^m{}_i)^2 + (B_n{}^m{}_i)^2 \,. \tag{C6}$$

By (C3a) and (C3b), the sum over order m in (C6) is

$$\sum_{m=0}^{n} (A_n{}^m{}_i)^2 + (B_n{}^m{}_i)^2 = G_n \sum_{m=0}^{n} [(D_r^i n P_n^m)^2 + (D_\theta^i \frac{d}{d\theta_i} P_n^m)^2 + (D_\phi^i m P_n^m / \sin\theta_i)^2 + D_r^i D_\theta^i n P_n^m (\frac{d}{d\theta_i} P_n^m)^2] \tag{C7}$$

where $G_n(r_i) \equiv (4\pi a^3)^{-2}(r_i/a)^{2n-2}$ and both P_n^m and its derivative are evaluated at $\cos\theta_i$.

To cast (C7) into a more illuminating form, we use spherical harmonic identities to reduce the four terms on the right, each a sum over m. For arbitrary (θ,ϕ), the sum rule for Schmidt normalized harmonics (see, e.g., *Jackson* [1975, eqn (3.62) with $\gamma = 0$], *Langel* [1987]),

$$\sum_{m=0}^{n} [C_n^m(\theta,\phi)]^2 + [S_n^m(\theta,\phi)]^2 = \sum_{m=0}^{n} [P_n^m(\cos\theta)]^2 = 1\,, \qquad \text{(C8a)}$$

is differentiated repeatedly with respect to θ to obtain

$$\sum_{m=0}^{n} 2P_n^m \frac{d}{d\theta}P_n^m = 0\,; \qquad \text{(C8b)}$$

$$\sum_{m=0}^{n} 2P_n^m \frac{d^2}{d^2\theta}P_n^m + 2\left[\frac{d}{d\theta}P_n^m\right]^2 = 0\,. \qquad \text{(C8c)}$$

The first term on the right of (C7), the radial term, sums to $(D_r^i n)^2$ by (C8a); the fourth term, the cross term, sums to zero by (C8c).

To reduce the second, co-latitudinal term on the right of (C7), recall that r^2 times the surface Laplacian of S_n^m gives

$$\nabla_s^2 S_n^m = -n(n+1)S_n^m = (\sin\theta)^{-1}\partial_\theta\,[\sin\theta\,\partial_\theta S_n^m] + (\sin\theta)^{-2}\partial_\varphi^2 S_n^m]. \qquad \text{(C9a)}$$

By adding this to the corresponding relation for C_n^m, we find

$$\frac{d^2}{d^2\theta}P_n^m = [-n(n+1) + (m/\sin\theta)^2]P_n^m - \frac{\cos\theta}{\sin\theta}\frac{d}{d\theta}P_n^m\,; \qquad \text{(C9b)}$$

therefore,

$$\sum_{m=0}^{n} P_n^m \frac{d}{d\theta}P_n^m = \sum_{m=0}^{n} [(m/\sin\theta)^2 - n(n+1)][P_n^m]^2 - \frac{\cos\theta}{\sin\theta}P_n^m \frac{d}{d\theta}P_n^m\,. \qquad \text{(C9c)}$$

The last term on the right sums to zero by (C8b); substitution of the remainder into (C8c) yields

$$\sum_{m=0}^{n} \left[\frac{d}{d\theta} P_n^m\right]^2 = \sum_{m=0}^{n} [n(n+1) - (m/\sin\theta)^2](P_n^m)^2 . \tag{C10}$$

To reduce the third term on the right of (C7) and complete reduction of the second, we need the sum over m of $-[mP_n^m(\cos\theta_i)/\sin\theta_i]^2$. In terms of the angle γ between the **r** and $\mathbf{r}_i$, $x \equiv \cos\gamma$, and Legendre polynomials $P_n(x)$, it has been shown that this sum reduces to

$$\sum_{m=0}^{n} -[mP_n^m(\cos\theta_i)/\sin\theta_i]^2 = -\left.\frac{dP_n(x)}{dx}\right|_{x=1} . \tag{C11a}$$

Evaluation of Legendre's equation at $x = 1$ and the normalization $P_n(1) = 1$ then imply

$$\sum_{m=0}^{n} -[mP_n^m(\cos\theta_i)/\sin\theta_i]^2 = -n(n+1)/2 . \tag{C11b}$$

The proof of (C11b) by *Whaler & Gubbins* [1981, Appendix B], which they attribute to P. H. Roberts, is more elegant than that by *Voorhies* [1998].

By (C11b), (C10) is equal to $n(n+1)/2$; therefore, the second term on the right of (C7) sums to $n(n+1)(D_\theta^i)^2/2$. Similarly, the third term on the right of (C7) sums to $n(n+1)(D_\phi^i)^2/2$. Substitution of (C7), as simplified by identities (C8b,c), (C10) and (C11b), into (C6) yields

$$F_n^c(r > r_i) = (4\pi r^3)^{-2}(r_i/r)^{2n-2}\,(n^2[n+1](D_r^i)^2 + n[n+1]^2[(D_\theta^i)^2 + (D_\phi^i)^2]) . \tag{C12}$$

This is the SV spectrum from a single changing moment $\Delta\mathbf{d}_i/\Delta t = \mu_0\mathbf{D}^i$ at $r_i < r$.

Expected SV Spectrum from Uncorrelated, Randomly Oriented Changes in Moment

At an instant in geologic time, the I equivalent SV sources $\mathbf{D}^i$ representing eddy induced SV are treated as a sample population with mean square moment change $\{D^2\}$. The coefficients in (C5a,b) are given in terms of the $\mathbf{D}^i$ by (C3a,b). The source positions (θ_i,ϕ_i) on c^- are taken to be random samples from a laterally uniform spatial distribution. Such positions are expected to be uncorrelated in that $\{\mathbf{r}_i \bullet \mathbf{r}_k\} = \delta_{ik}\{(\mathbf{r}_i)^2\}$, where the Krønecker δ_{ik} is 1 if $i = k$ and is 0 if $i \neq k$.

For randomly oriented moment changes corresponding to a kinematically unbiased ensemble of eddies, any particular orientation is as likely as its opposite,

$$\{D_r^i\} = \{D_\theta^i\} = \{D_\phi^i\} = 0, \tag{C13a}$$

there is no reason to expect cross-correlated components for an individual moment change,

$$\{D_r^i D_\theta^i\} = \{D_r^i D_\phi^i\} = \{D_\theta^i D_\phi^i\} = 0, \tag{C13b}$$

there is no reason to expect cross-correlated moments

$$\{D_r^i D_r^k\} = \{D_\theta^i D_\theta^k\} = \{D_\phi^i D_\phi^k\} = 0 \text{ for } i \neq k \tag{C13c}$$

$$\{D_r^i D_\theta^k\} = \{D_r^i D_\phi^k\} = \{D_\theta^i D_\phi^k\} = 0 \text{ for } i \neq k, \tag{C13d}$$

but the auto-correlations remain perfect

$$\{D_r^i D_r^i\} = \{D_\theta^i D_\theta^i\} = \{D_\phi^i D_\phi^i\} = \{\mathbf{D}^2\}/3. \tag{C13e}$$

Equations (C13a-e), summarized by $\{D_j^i\} = 0$ and $\{D_j^i D_l^k\} = \{\mathbf{D}^2/3\}\delta_{ik}\delta_{jl}$, provide a mathematical statement of "random equivalent SV sources" associated with a kinematically unbiased ensemble of eddies. Polarizing dynamical effects, whereby eddies might tend to align with the planetary rotation axis and/or the field, are outside the focus of this appendix.

Though $\mathbf{D}^i$ depends on position (θ_i,ϕ_i), position is independent of $\mathbf{D}^i$; therefore, when evaluating $\{A_n^m{}_i\}$ and $\{B_n^m{}_i\}$ via (C3a,b), the expectation operator passes through the harmonic functions of (θ_i,ϕ_i). These expectation values are zero by (C13a), as are expected or mean SV coefficients by {(C4a,b)}, hence the expected components of SV itself. Though the expected SV vector from random equivalent sources is zero, the expected spectrum is not.

Equations (C13a-e) and (C3a,b) imply contributing coefficients are not cross-correlated

$$\{A_n^m{}_i A_n^m{}_k\} = \{B_n^m{}_i B_n^m{}_k\} = 0 \text{ for } i \neq k\,. \tag{C14}$$

Again, the auto-correlations remain. Substitution of (C14) into (C5b) yields

$$\{F_n^c(r > r_i)\} = (n+1)(a/r)^{2n+4}\{ \sum_{m=0}^{n} [\sum_{i=1}^{I} (A_n^m{}_i)^2 + (B_n^m{}_i)^2] \} \qquad \text{(C15)}$$

The order of sums in (C15) is reversible, so the expected spectrum from I uncorrelated SV sources is the sum of expected spectra from each source. With $r_i = c^-$, (C15) and (C12) imply

$$\{F_n^c(r > c^-)\} = (4\pi r^3)^{-2} I (c^-/r)^{2n-2} n(n+1)[n\{D_r^2\} + (n+1)(\{D_\theta^2\} + \{D_{\theta\phi}^2\})] \qquad \text{(C16)}$$

By (C13c), this simplifies to

$$\{F_n^c(r > c^-)\} = (4\pi r^3)^{-2}(I\{D^2\}/3)\,(c^-/r)^{2n-2}\,[n(n+1)(2n+1)] \qquad \text{(C17)}$$

With $C \equiv (4\pi c^{-3})^{-2}(2I\{D^2\}/3)$, (C17) becomes

$$\{F_n^c(r > c^-)\} = C\,n\,(n + ½)(n+1)(c^-/r)^{2n+4}, \qquad \text{(C18)}$$

which is just equation (5).

Appendix D: SV Spectral Variance

For steady c^-, equation (5) also gives the time-averaged spectrum expected from a temporal sequence of kinematically unbiased ensembles of equivalent SV sources, denoted $<\{F_n^c(t)\}>$, albeit with amplitude C replaced by $<C(t)>$. The associated sequence variance $<[\{F_n^c(t)\} - <\{F_n^c(t)\}>]^2>$ is proportional to $<[C(t) - <C(t)>]^2>$, but has the same functional dependence on n as the square of spectrum (5). The root mean square sequence deviation is therefore proportional to the expected SV spectrum itself. This is a strong restriction on the kinds of long term magnetic variations that could be adequately described by the narrow scale transport model. For example, if eddies and eddy transport change in response to a change in a driving function (*e.g.*, heat flux across a CMB, heat and buoyant component fluxes across an inner core boundary, *etc.*), then this sequence of kinematically unbiased ensembles could describe a corresponding change in the amplitude, but not the shape, of the expected low degree SV spectrum.

Different sets of Δd_i can represent broad-scale SV induced by different numbers of eddies transporting different fields at different speeds; however, a single ensemble from this temporal sequence of ensembles will give but one ensemble-mean low degree SV spectrum. Each such ensemble represents lateral magnetic transport by an ensemble of eddies with a corresponding mean number of eddies, mean speed and, arguably, a mean pattern and intensity for the field being transported. If the mean number of eddies, mean speed, and mean field properties can and do change by their own magnitude from one ensemble to the next in the sequence, say from one epoch to another, then the sequence variance will not only be proportional to, but will arguably have amplitude similar to, or perhaps somewhat greater than, $<\{F_n^c\}>^2$ itself.

Physical deviations $F_n(t)$ - $\{F_n^c\}$ may also arise from dynamic processes that cause serial, as well as lateral, correlation between SV induced by eddies which grow, vacillate and decay. Such deviations are arguably of a magnitude similar to $\{F_n^c\}$ itself and may persist for a long time, perhaps as long as dynamic $\mathbf{J}\times\mathbf{B}$ conditions associated with a polarity chron, a particular pattern of heterogeneous electrical or thermal boundary conditions, *etc.*. If so, and if spectrum (5) still described a geologically longer time average of $F_n^c(t)$, then natural fluctuations $F_n^c(t)$ - $\{F_n^c\}$ would again have variance similar to, if not somewhat greater than, $\{F_n^c\}^2$.

Appendix E: Effects of Uncertain Process Variance

The statistical significance of residuals depends on both data covariance and process covariance. Here, data covariance represents uncertainties in an observational SV spectrum $F_n(a,t)$, hence the sparse distribution of uncertain geomagnetic measurements used to determine SV coefficients. Process covariance represents fluctuations of a theoretical core-source SV spectrum $F_n^c(a,t)$ about its expectation value, plus omitted non-core contributions to the observations. For $F_n(a,t)$ determined by careful analyses of satellite and surface observatory measurements, and anticipating process variance of magnitude similar to $\{F_n^c(a)\}^2$ itself, process variance should exceed data variance at low degrees.

The narrow scale flow hypothesis specifies neither a probability distribution function for SV spectrum fluctuations nor a process variance, so one can but estimate the latter from residuals. I doubt this distribution can be determined from 10 or 12 sample residuals. Fluctuations $F_n^c(t)$ - $\{F_n^c\}$ cannot be normally distributed because $F_n^c(t)$ is positive, but many other distributions are eligible (truncated Gaussian, log-normal, chi-squared, *etc.*). Whatever the trial distribution, when process variance is estimated from but a single set of residuals, here by scaling weights so the sum of square weighted residuals per degree of freedom becomes unity, then the (in)significance of the residuals does not test the hypothesis. The test is in the comparison of estimated parameters with values determined by independent analyses of independent data. We therefore emphasize comparison of magneto-spectral estimates of source radius c with independent c_S.

Appendix F: A Softer SV Spectrum for 2000?

On orbit magnetic calibration and magneto-optical alignment checks enabled the fitting of high quality main field and SV coefficients to high precision scalar and oriented vector data acquired by Ørsted. Here we consider $F_n(2000)$ from model OSVM [*Olsen,* 2002], which fitted 24,585 two-component vector and 68,448 scalar and field aligned Ørsted data from a 2.5 year interval and data from about 100 observatories for 1998-2000. Compared with $\langle F_n \rangle$ for 1960-80, there are decreases in F_2 and perhaps F_6, increases in F_3, F_4, and F_5, but rather small changes for F_7, F_8, and F_9. The decrease in F_2 suggests less rapid quadrupole rebound.

Estimates of $c^{\cdot}$ using OSVM and spectrum (5) are typically well below c_S. Estimates using proxy spectrum (7) and $n_{min} = 1$, shown in Tables F1a, agree fairly well with c_S and have less misfit than shown in Tables 3a and 5a; moreover, for $n_{min} = 3$, the errors shown in Table F1b are rather less than found in Tables 3b and 5b. We therefore suggest spectrum (7) might be more suitable than (5) at some epochs, perhaps due to changes in lateral length scales of core surface flow, hence in the lateral correlation length scales of SV at the top of the core. Whether or not this is substantiated by other epoch 2000 models with non-steady SV, different filters of observatory data, normal weights, or different treatments of external fields remains to be seen.

A broadening in scale of core surface flow between the interval 1960-1980 and the Ørsted epoch 2000 may explain a softer SV spectrum, though other explanations are possible. If spectrum (5) were be shown to be wholly inadequate outside the 1960-1980 interval, yet (11b) remained adequate, then arguments leading to (11b) might have to be revised – possibly by including effects of Lorentz forces on eddy diffusivities. In this way it might be possible to discern deviations from tangential geostrophy by spectral methods.

Appendix G: A Bound on Core Conductivity

Until a few years ago, observational SV spectra were not thought to be well enough determined to high enough degrees to test spectrum (5) against seismology; it was only checked via (10b) in the context of (11b) [*Voorhies & Conrad,* 1996]. From the check of (10b), we infer there is but a single important lateral empirical diffusivity, independent of degree n. Moreover, from the value of α_0, this lateral empirical diffusivity is $\zeta = c^2/\alpha_0 = 147$ m^2/s with an uncertainty factor or divisor of two. If this is equal to a single ensemble mean lateral eddy diffusivity UL, then there is but one important eddy magnetic Reynolds number $R_m = \mu\sigma UL$. If one assumes $\sigma = 5\times10^5$ S/m and vacuum permeability, then this magnetic Reynolds number is $R_m = 92$ with an uncertainty factor of two. Another interpretation, which does not assume σ and allows forced dipole decay, uses α_0 to bound core magnetic diffusivity $\eta < 2(c/\pi)^2/\alpha_0 = 30$ m^2/s, and, for vacuum permeability, core electric conductivity $\sigma \geq 2.7 \times 10^4$ S/m, both with an uncertainty factor of two. This is consistent with a fluid metallic core and, in turn, provides a check on the larger theory.

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Table 1. Core radius estimated by fitting log-theoretical SV spectrum (5) to log-observational spectrum GSFC 9/80 for various epochs and degree ranges.

Epoch	n	$c^{\cdot}$ (km)	n	$c^{\cdot}$ (km)	n	$c^{\cdot}$ (km)
1980	1-13	3441 ± 96	3-11	3429 ± 135	3-10	3280 ± 127
1975	1-13	3472 ± 95	3-11	3539 ± 106	3-10	3396 ± 79
1970	1-13	3475 ± 99	3-11	3540 ± 115	3-10	3387 ± 89
1965	1-13	3490 ± 97	3-11	3547 ± 115	3-10	3397 ± 90
1960	1-13	3488 ± 87	3-11	3484 ± 99	3-10	3334 ± 55
Avg.	1-13	3470 ± 91	3-11	3511 ± 101	3-10	3360 ± 61

Table 2a: Core radius and scaled uncertainty from equation (5) and the time-averaged observational SV spectrum GSFC 9/80.

Degrees Fitted	q^2	$c^{\cdot}$ km	Error km
1-6	0.2316	3026 ± 174	- 454
1-7	0.1868	3045 ± 124	- 435
1-8	0.1982	3130 ± 108	- 350
1-9	0.1938	3184 ± 90	- 296
1-10	0.1710	3196 ± 73	- 284
1-11	0.3591	3318 ± 95	- 162
1-12	0.4138	3391 ± 91	- 89
1-13	0.5041	3470 ± 91	- 10

Table 2b: As Table 2a, but $n_{min} = 3$.

Degrees Fitted	q^2	$c^{\cdot}$ km	Error km
3-6	0.0921	3452 ± 234	- 28
3-7	0.0833	3315 ± 151	- 165
3-8	0.0709	3378 ± 108	- 102
3-9	0.0579	3398 ± 77	- 82
3-10	0.0551	3360 ± 61	- 120
3-11	0.2005	3511 ± 101	+ 31
3-12	0.2168	3579 ± 92	+ 99
3-13	0.2612	3658 ± 89	+178

Table 3a: Core radius and scaled uncertainty from equation (7) and the time-averaged observational SV spectrum GSFC 9/80.

Degrees Fitted	q^2	Core Radius km	Error km
1-6	0.2040	3488 ± 188	+ 8
1-7	0.1652	3462 ± 133	- 18
1-8	0.1526	3519 ± 106	+ 39
1-9	0.1359	3547 ± 84	+ 67
1-10	0.1210	3532 ± 68	+ 52
1-11	0.2466	3642 ± 86	+ 162
1-12	0.2701	3701 ± 80	+ 221
1-13	0.3223	3767 ± 79	+ 287

Table 3b: As Table 3a, but $n_{\min} = 3$.

Degrees Fitted	q^2	Core Radius km	Error km
3-6	0.0791	3826 ± 240	+ 346
3-7	0.0846	3644 ± 168	+ 164
3-8	0.0668	3688 ± 114	+ 208
3-9	0.0534	3687 ± 81	+ 207
3-10	0.0591	3628 ± 68	+ 148
3-11	0.1743	3774 ± 102	+ 294
3-12	0.1788	3832 ± 89	+ 352
3-13	0.2070	3902 ± 85	+ 422

Table 4a: Core radius and scaled uncertainty from equation (5) and the time-averaged constrained SV spectrum CM3.

Degrees Fitted	q^2	Core Radius km	Error km
1-6	0.2305	2994 ± 172	- 486
1-7	0.1950	3045 ± 127	- 435
1-8	0.2017	3127 ± 108	- 353
1-9	0.2064	3192 ± 94	- 288
1-10	0.1848	3211 ± 78	- 269
1-11	0.1778	3242 ± 65	- 238
1-12	0.1836	3279 ± 59	- 201
1-13	0.3096	3200 ± 66	- 280

Table 4b: As Table 4a, but $n_{\min} = 3$.

Degrees Fitted	q^2	Core Radius km	Error km
3-6	0.0743	3408 ± 208	- 72
3-7	0.0546	3342 ± 124	- 138
3-8	0.0458	3390 ± 87	- 90
3-9	0.0401	3423 ± 65	- 57
3-10	0.0373	3394 ± 51	- 86
3-11	0.0321	3400 ± 39	- 80
3-12	0.0322	3420 ± 34	- 60
3-13	0.2786	3282 ± 83	- 198

Table 5a: Core radius and scaled uncertainty from equation (7) and the time-averaged constrained SV spectrum CM3.

Degrees Fitted	q^2	Core Radius km	Error km
1-6	0.2076	3452 ± 188	- 28
1-7	0.1664	3462 ± 133	- 18
1-8	0.1517	3516 ± 106	+ 36
1-9	0.1400	3555 ± 86	+ 75
1-10	0.1228	3549 ± 69	+ 69
1-11	0.1103	3559 ± 56	+ 79
1-12	0.1045	3578 ± 48	+ 98
1-13	0.3050	3474 ± 71	- 6

Table 5b: As Table 5a, but $n_{\min} = 3$.

Degrees Fitted	q^2	Core Radius km	Error km
3-6	0.0615	3777 ± 209	+ 297
3-7	0.0514	3673 ± 132	+ 193
3-8	0.0398	3701 ± 88	+ 221
3-9	0.0323	3714 ± 63	+ 234
3-10	0.0371	3665 ± 54	+ 185
3-11	0.0324	3654 ± 42	+ 174
3-12	0.0289	3662 ± 34	+ 182
3-13	0.3198	3502 ± 94	+ 22

Table 6: Core radius and scaled uncertainty from equation (11b) and observational main field spectrum OIFM

Degrees Fitted	s^2	Core Radius km	Error km
1-4	0.5588	3441 ± 575	- 39
1-5	0.3730	3461 ± 334	- 19
1-6	0.2803	3444 ± 218	- 36
1-7	0.2326	3496 ± 159	+ 16
1-8	0.2021	3454 ± 120	- 26
1-9	0.2054	3524 ± 103	+ 44
1-10	0.1797	3522 ± 82	+ 42
1-11	0.1602	3516 ± 67	+ 36
1-12	0.1539	3542 ± 58	+ 61
1-13	0.1826	3589 ± 57	+109
1-14	0.1782	3610 ± 51	+130
1-15	0.3056	3681 ± 61	+201
1-16	0.4436	3751 ± 68	+271
1-17	0.8074	3853 ± 86	+373
1-18	1.2241	3958 ± 99	+478

Table 7: Core radius and scaled uncertainty from equation (11b) and observational main field spectrum CM3.

Degrees Fitted	s^2	Core Radius km	Error km
1-4	0.7210	3467 ± 658	- 13
1-5	0.4817	3498 ± 384	+ 18
1-6	0.3613	3503 ± 252	+ 23
1-7	0.2900	3520 ± 179	+ 40
1-8	0.2773	3432 ± 140	- 48
1-9	0.2810	3513 ± 120	+ 33
1-10	0.2466	3504 ± 96	+ 24
1-11	0.2194	3508 ± 78	+ 28
1-12	0.1975	3507 ± 65	+ 27
1-13	0.1996	3539 ± 59	+ 59
1-14	0.1956	3562 ± 52	+ 82
1-15	0.2546	3612 ± 54	+132
1-16	0.4281	3687 ± 65	+207
1-17	0.6684	3770 ± 76	+290
1-18	0.9795	3859 ± 87	+379

Table 8: As Table 7, but with $n_{min} = 3$.

Degrees Fitted	s^2	Core Radius km	Error km
3-6	0.0175	3452 ± 102	- 28
3-7	0.0126	3482 ± 62	+ 2
3-8	0.0521	3336 ± 91	-144
3-9	0.1144	3484 ± 111	+ 4
3-10	0.0958	3476 ± 83	- 4
3-11	0.0828	3486 ± 65	+ 6
3-12	0.0725	3487 ± 52	+ 7

Table F1a: Core radius and scaled uncertainty estimate from equation (7) and observational SV spectrum OSVM.

Degrees Fitted	q^2	Core Radius km	Error km
1-6	0.0731	3458 ± 112	- 22
1-7	0.0617	3426 ± 80	- 54
1-8	0.0665	3483 ± 69	+ 3
1-9	0.0765	3538 ± 63	+ 57
1-10	0.0778	3573 ± 55	+ 93
1-11	0.0763	3548 ± 47	+ 68
1-12	0.2162	3648 ± 71	+168
1-13	0.4410	3766 ± 93	+286

Table F1b: As Table F1a, but $n_{min} = 3$.

Degrees Fitted	q^2	Core Radius km	Error km
3-6	0.0561	3460 ± 183	- 20
3-7	0.0404	3408 ± 108	- 72
3-8	0.0530	3516 ± 97	+ 36
3-9	0.0620	3598 ± 87	+118
3-10	0.0586	3638 ± 68	+158
3-11	0.0664	3587 ± 60	+107
3-12	0.2116	3723 ± 94	+243
3-13	0.4230	3875 ± 33	+395