

Homogenization of a Darcy–Stokes system modeling vuggy porous media

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Abstract We derive a macroscopic model for single-phase, incompressible, viscous fluid flow in a porous medium with small cavities called vugs. We model the vuggy medium on the microscopic scale using Stokes equations within the vugular inclusions, Darcy’s law within the porous rock, and a Beavers–Joseph–Saffman boundary condition on the interface between the two regions. We assume periodicity of the medium and obtain uniform energy estimates independent of the period. Through a two-scale homogenization limit as the period tends to zero, we obtain a macroscopic Darcy’s law governing the medium on larger scales. We also develop some needed generalizations of the two-scale convergence theory needed for our bimodal medium, including a two-scale convergence result on the Darcy–Stokes interface. The macroscopic Darcy permeability is computable from the solution of a cell problem. An analytic solution to this problem in a simple geometry suggests that: (1) flow along vug channels is primarily Poiseuille with a small perturbation related

to the Beavers–Joseph slip, and (2) flow that alternates from vug to matrix behaves as if the vugs have infinite permeability.

Keywords Beavers–Joseph boundary condition · Darcy–Stokes system · Homogenization · two-scale convergence · vuggy porous media

1. Introduction

A *vug* is a cavity in a porous medium that is relatively larger than the intergranular pore space. Vugular inclusions are especially common in carbonate rocks and are endemic to many of the world’s groundwater aquifers and petroleum reservoirs. Although small, vugs can significantly increase both the effective porosity and permeability of the medium. We consider in this paper a porous medium with many small vugs scattered throughout its extent.

It is well established, both empirically and theoretically, that Darcy’s law governs fluid flow in a porous medium on scales above the pore diameter [6, 23, 27, 29]. Since the flow is expected to have a relatively low Reynolds number, the Stokes equations should adequately model fluid flow in the vugs.

In 1967, Beavers and Joseph [7] determined experimentally that a free fluid in contact with a porous medium flows faster than a fluid in contact with a completely solid surface. Although thin boundary layers arise in both cases, the latter case is generally modeled by assuming that all components of the velocity vanish at the solid contact surface. In the former case, the experiments of Beavers and Joseph demonstrate that

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the tangential velocity of the fluid cannot vanish. They proposed to account for this slippage by imposing a boundary condition of the form

$$\frac{\partial U_s}{\partial y} = \frac{\alpha}{\sqrt{K}}(U_s - U_d),$$

where $\partial/\partial y$ is the normal derivative, U_s is the tangential component of the Stokes velocity, U_d is the tangential component of the Darcy velocity, K is the permeability of the porous medium, and α is the dimensionless Beavers–Joseph slippage coefficient. Saffman [24] justified this law theoretically and showed that the term involving U_d could be dropped (see also [15, 16]). Jones [18] reinterpreted this law so that it applies to curved boundaries and nontangential flows by formulating the boundary condition in terms of the tangential component of the fluid stress tensor [see equation (1.6) below].

We model the vuggy medium on the fine scale using Stokes equations in the vugs, Darcy’s law in the porous rock, and the Beavers–Joseph–Saffman boundary condition on the interface between the two. We assume periodicity of the medium and obtain as our homogenized limit a macroscopic Darcy’s law governing the system over large scales. To illustrate the ideas, we first derive this macroscopic model formally in section 2, and then we derive it rigorously by the two-scale convergence method [1, 4, 14, 22] in section 5. In section 3, we obtain the existence, uniqueness, and energy estimate results needed in the analysis. In section 4, we develop the needed generalizations of the two-scale convergence theory needed for our bimodal medium. The convergence of the homogenization is demonstrated in section 5. The final section presents a simple analytical solution to illustrate the results.

Although our results would extend easily to \mathbb{R}^3 , for ease of presentation, we assume that the domain Ω is Lipschitz and bounded in \mathbb{R}^2 . We assume that the geometric vug and pore structure of Ω is periodic of period ϵY , where Y is a reference cell for the periodic tiling of unit volume $|Y|$. The portion of the domain consisting of the vugs is denoted Ω_s^ϵ , and that consisting of the porous rock is Ω_d^ϵ . Let Γ^ϵ be the interface between the two regions. Let η_s be the outer unit normal to $\partial\Omega_s$, and let τ be a unit tangent to Γ^ϵ .

Let D be the symmetric gradient, i.e., $D(\psi)$ is the matrix $\frac{1}{2} \left(\frac{\partial \psi_i}{\partial x_j} + \frac{\partial \psi_j}{\partial x_i} \right)$. Denote by $\mu > 0$ the fluid viscosity, $K^\epsilon = K(x/\epsilon)$ the Y -periodic, bounded, symmetric, and uniformly positive-definite permeability tensor of the porous rock matrix, and $\alpha > 0$ the Beavers–Joseph slip coefficient. The fluid velocity and pressure in the Stokes and Darcy regions are denoted u_s^ϵ , p_s^ϵ and u_d^ϵ , p_d^ϵ , respectively. These satisfy the following set of equations

[wherein $q \in L^2(\Omega)$ is an external source or sink satisfying the compatibility condition that its average over Ω vanishes, and $f \in (L^2(\Omega))^2$ is a term related to body forces such as gravitation]:

Vugular region (Stokes equations)

$$-2\mu\epsilon^2 \nabla \cdot Du_s^\epsilon + \nabla p_s^\epsilon = f \quad \text{in } \Omega_s^\epsilon, \quad (1.1)$$

$$\nabla \cdot u_s^\epsilon = q \quad \text{in } \Omega_s^\epsilon, \quad (1.2)$$

Rock matrix (Darcy equations)

$$\mu(K^\epsilon)^{-1}u_d^\epsilon + \nabla p_d^\epsilon = f \quad \text{in } \Omega_d^\epsilon, \quad (1.3)$$

$$\nabla \cdot u_d^\epsilon = q \quad \text{in } \Omega_d^\epsilon, \quad (1.4)$$

Interface

$$u_s^\epsilon \cdot \eta_s = u_d^\epsilon \cdot \eta_s \quad \text{on } \Gamma^\epsilon, \quad (1.5)$$

$$2\eta_s \cdot Du_s^\epsilon \cdot \tau = -\frac{\alpha}{\epsilon\sqrt{K^\epsilon}}u_s^\epsilon \cdot \tau \quad \text{on } \Gamma^\epsilon, \quad (1.6)$$

$$2\mu\epsilon^2 \eta_s \cdot Du_s^\epsilon \cdot \eta_s = p_s^\epsilon - p_d^\epsilon \quad \text{on } \Gamma^\epsilon, \quad (1.7)$$

Outer boundary

$$u_s^\epsilon = 0 \quad \text{on } \partial\Omega \cap \partial\Omega_s^\epsilon, \quad (1.8)$$

$$u_d^\epsilon \cdot \eta = 0 \quad \text{on } \partial\Omega \cap \partial\Omega_d^\epsilon. \quad (1.9)$$

The interface conditions represent continuity of mass flux (1.5), the Beavers–Joseph–Saffman condition on the tangential stress (1.6), and the continuity of normal stress (1.7). Examples of direct numerical simulation of these equations in a nonvuggy context (i.e., in a medium with at most a few very large vugs) can be found in, for example, [13, 19, 25]. In a vuggy context, see, for example, [3].

The homogenization problem is to determine the behavior of the system as $\epsilon \rightarrow 0$. Note that in the equations, we have scaled both the viscosity μ and the permeability K^ϵ by ϵ^2 . This is the usual scaling for deriving Darcy’s law from Stokes flow (see [27]), since as $\epsilon \rightarrow 0$, flow paths (in our case vugs) become constricted, so a corresponding decrease in viscosity is required to maintain flow rates. Moreover, when homogenizing heterogeneity (see, e.g., [8, 26]), the ratio of permeability to viscosity should be fixed, forcing a similar scaling of the permeability. These considerations then imply the stated scaling of the Beavers–Joseph boundary condition.

Below we will need to distinguish the geometry of the reference cell Y , so let Y_s denote the Stokes region, Y_d the Darcy region, and Γ the interface between the two. We assume that both Y_s and Y_d have positive measure, and thus also the one-dimensional measure of Γ is positive. As usual, x will represent a point in Ω and y a point in Y . In the sequel, let $\|\cdot\|_\omega$ and $(\cdot, \cdot)_\omega$ denote

the $L^2(\omega)$ norm and inner product, respectively, where we omit the domain ω if it is Ω .

2. Formal homogenization

We proceed to formally homogenize our system of equations in the usual manner [8, 14, 17, 26]. We make the ansatz that we can expand u_ℓ^ϵ and p_ℓ^ϵ (for $\ell = s, d$) as

$$u_\ell^\epsilon = \sum_{j=0}^\infty \epsilon^j u_{\ell,j} \left(x, \frac{x}{\epsilon} \right) \quad \text{and} \quad p_\ell^\epsilon = \sum_{j=0}^\infty \epsilon^j p_{\ell,j} \left(x, \frac{x}{\epsilon} \right),$$

where $u_{\ell,j}(x, y)$ and $p_{\ell,j}(x, y)$ are Y -periodic functions in y .

Substituting the above expressions into our system of equations (1)–(1.7), and recognizing that $\nabla = \nabla_x + \epsilon^{-1} \nabla_y$, we obtain the following equations. From the ϵ^{-1} terms of (1.1) and (1.3), and the ϵ^0 terms of (1.7), we see that

$$\nabla_y p_s^0 = 0 \quad \text{in } \Omega \times Y_s, \tag{2.1}$$

$$\nabla_y p_d^0 = 0 \quad \text{in } \Omega \times Y_d, \tag{2.2}$$

$$p_s^0 - p_d^0 = 0 \quad \text{on } \Omega \times \Gamma. \tag{2.3}$$

It follows immediately that p_s^0 and p_d^0 are independent of y and equal, so let

$$p^0(x) = p_s^0(x) = p_d^0(x) \quad \text{on } \Omega.$$

Now the ϵ^0 terms of (1.1), (1.3), and (1.5), the ϵ^{-1} terms of (1.2), (1.4), and (1.6), and the ϵ^1 terms of (1.7) imply

$$\begin{aligned} -2\mu \nabla_y \cdot D_y u_s^0 + \nabla_x p^0(x) + \nabla_y p_s^1(x, y) &= f && \text{in } \Omega \times Y_s, \\ \nabla_y \cdot u_s^0 &= 0 && \text{in } \Omega \times Y_s, \\ \mu K(y)^{-1} u_d^0 + \nabla_x p^0(x) + \nabla_y p_d^1(x, y) &= f && \text{in } \Omega \times Y_d, \\ \nabla_y \cdot u_d^0 &= 0 && \text{in } \Omega \times Y_d, \\ u_s^0 \cdot \eta_s &= u_d^0 \cdot \eta_s && \text{on } \Omega \times \Gamma, \\ 2\eta_s \cdot D_y u_s^0 \cdot \tau &= -\frac{\alpha}{\sqrt{K(y)}} u_s^0 \cdot \tau && \text{on } \Omega \times \Gamma, \\ 2\mu \eta_s \cdot D_y u_s^0 \cdot \eta_s &= p_s^1 - p_d^1 && \text{on } \Omega \times \Gamma. \end{aligned}$$

With e_j being the standard Cartesian basis vector in the j th direction, let (ω_j, Φ_j) be the periodic solution of the following auxiliary or cell problem

$$-2\nabla \cdot D\omega_j^s + \nabla \Phi_j^s = e_j \quad \text{in } Y_s, \tag{2.4}$$

$$\nabla \cdot \omega_j^s = 0 \quad \text{in } Y_s, \tag{2.5}$$

$$K^{-1} \omega_j^d + \nabla \Phi_j^d = e_j \quad \text{in } Y_d, \tag{2.6}$$

$$\nabla \cdot \omega_j^d = 0 \quad \text{in } Y_d, \tag{2.7}$$

$$\omega_j^s \cdot \eta_s = \omega_j^d \cdot \eta_s \quad \text{on } \Gamma, \tag{2.8}$$

$$2\eta_s \cdot D\omega_j^s \cdot \tau = -\frac{\alpha}{\sqrt{K}} \omega_j^s \cdot \tau \quad \text{on } \Gamma, \tag{2.9}$$

$$2\eta_s \cdot D\omega_j^s \cdot \eta_s = \Phi_j^s - \Phi_j^d \quad \text{on } \Gamma. \tag{2.10}$$

Then by linear algebra, we can express u_s^0 and u_d^0 as

$$u_\ell^0(x, y) = \frac{1}{\mu} \sum_j \left(f_j(x) - \frac{\partial p^0}{\partial x_j}(x) \right) \omega_j^\ell(y), \quad \ell = s, d. \tag{2.11}$$

Define the averaging operator \bar{v}_ℓ by averaging v_ℓ in the following sense

$$\bar{v}_\ell = \frac{1}{|Y|} \int_{Y_\ell} v_\ell(y) dy, \tag{2.12}$$

so that

$$\begin{aligned} \bar{u}_0(x) &= \bar{u}_s^0(x) + \bar{u}_d^0(x) \\ &= \frac{1}{\mu} \sum_j \left(f_j(x) - \partial_j p^0(x) \right) \left(\bar{\omega}_j^s + \bar{\omega}_j^d \right). \end{aligned}$$

Now let the matrix \tilde{K} be defined by

$$\begin{aligned} \tilde{K}_{i,j} &= \bar{\omega}_{j,i}^s + \bar{\omega}_{j,i}^d \\ &= \frac{1}{|Y|} \left(\int_{Y_d} \left(\omega_j^d \right)_i dy + \int_{Y_s} \left(\omega_j^s \right)_i dy \right). \end{aligned} \tag{2.13}$$

Then we see that

$$\mu \tilde{K}^{-1} \bar{u}_0 + \nabla p^0 = f \quad \text{in } \Omega. \tag{2.14}$$

Finally, using the ϵ^0 terms of (1.2) and (1.4), we obtain

$$\nabla_x \cdot u_\ell^0 + \nabla_y \cdot u_\ell^1 = q \quad \text{in } \Omega \times Y_s, \quad \ell = s, d,$$

so that if we again average over Y and sum, we obtain

$$\begin{aligned} \nabla \cdot \bar{u}_0 + \frac{1}{|Y|} \int_{Y_s} \nabla_y \cdot u_s^1 dy + \frac{1}{|Y|} \int_{Y_d} \nabla_y \cdot u_d^1 dy \\ = \nabla \cdot \bar{u}_0 + \frac{1}{|Y|} \int_{\partial Y_s} u_s^1 \cdot \eta_s dS + \frac{1}{|Y|} \int_{\partial Y_d} u_d^1 \cdot \eta_d dS \\ = q. \end{aligned}$$

By the periodicity of u_ℓ^1 in y , and the fact that $u_s^1 \cdot \eta_s = -u_d^1 \cdot \eta_d$ on Γ , we see that

$$\nabla \cdot \bar{u}_0 = q \quad \text{on } \Omega \tag{2.15}$$

Thus, we conclude from the formal analysis that \bar{u}_0 should satisfy a Darcy’s law on all of Ω (2.14) and (2.15), with effective permeability matrix \tilde{K} , independent of the fluid viscosity and given by (2.13).

Lemma 2.1. The tensor \tilde{K} , as defined by (2.13) and (2.4)–(2.10), is symmetric and positive definite.

Proof. The existence and uniqueness of a weak solution to (2.4)–(2.10) follow from an analysis similar to that given below for Theorem 3.1. Note that (2.4)–(2.10) are equivalent to the variational equation

$$2(D\omega_i^s, D\psi)_{Y_s} + \left(\frac{\alpha}{\sqrt{K}}\omega_i^s \cdot \tau, \psi \cdot \tau\right)_\Gamma + (K^{-1}\omega_i^d, \psi)_{Y_d} = (e_i, \psi)_Y, \tag{2.16}$$

for ψ an infinitely differentiable and periodic vector function in Y such that $\nabla \cdot \psi = 0$. With $\psi = \omega_j^s$ on Y_s and $\psi = \omega_j^d$ on Y_d (actually a sequence approaching the same), we obtain that

$$|Y| \tilde{K}_{i,j} = (e_i, \omega_j^s)_{Y_s} + (e_i, \omega_j^d)_{Y_d} = 2(D\omega_i^s, D\omega_j^s)_{Y_s} + \left(\frac{\alpha}{\sqrt{K}}\omega_i^s \cdot \tau, \omega_j^s \cdot \tau\right)_\Gamma + (K^{-1}\omega_i^d, \omega_j^d)_{Y_d},$$

and symmetry follows immediately.

To show that \tilde{K} is positive definite, take any $\lambda \in \mathbb{R}^2$ and define

$$\xi^\ell(y) = \sum_i \lambda_i \omega_i^\ell \quad \text{for } y \in Y_\ell.$$

Then, from (2.16), we conclude that

$$|Y| \lambda^T \tilde{K} \lambda = 2(D\xi^s, D\xi^s)_{Y_s} + \left(\frac{\alpha}{\sqrt{K}}\xi^s \cdot \tau, \xi^s \cdot \tau\right)_\Gamma + (K^{-1}\xi^d, \xi^d)_{Y_d},$$

and that \tilde{K} is positive semi-definite. To see definiteness, suppose that $\lambda^T \tilde{K} \lambda = 0$ and conclude that each integrand above vanishes. But now (2.16) implies that

$$0 = 2(D\xi^s, D\psi)_{Y_s} + \left(\frac{\alpha}{\sqrt{K}}\xi^s \cdot \tau, \psi \cdot \tau\right)_\Gamma + (K^{-1}\xi^d, \psi)_{Y_d} = (\lambda, \psi)_Y.$$

Since λ is constant, we can take $\psi = \lambda$ and conclude that $\lambda = 0$ and, further, that \tilde{K} is positive definite. \square

3. Existence and a priori energy estimates

In this section, we prove a theorem that gives existence of solutions u^ϵ and p^ϵ to (1.1)–(1.9) for each ϵ and energy estimates for u^ϵ and p^ϵ independent of ϵ . Let

$$V^\epsilon = V^\epsilon(\Omega) = \{v \in H(\text{div}, \Omega) \mid v_s = v|_{\Omega_s^\epsilon} \in H^1(\Omega_s^\epsilon)\},$$

$$V_0^\epsilon = V_0^\epsilon(\Omega) = \{v \in V^\epsilon(\Omega) \mid v \cdot \eta = 0 \text{ on } \partial\Omega \cap \partial\Omega_d^\epsilon \text{ and } v = 0 \text{ on } \partial\Omega \cap \partial\Omega_s^\epsilon\},$$

where η is the outward unit normal to Ω , and let $W = L^2(\Omega)/\mathbb{R}$.

We first recast the original problems (1.1)–(1.9) into a variational problem. Combine equations (1.2) and (1.4), multiply by a test function $w \in W$, and integrate over Ω . Then combine and multiply equations (1.1) and (1.3) by a test function $v \in V^\epsilon$, integrate, integrate by parts, and manipulate the boundary terms to obtain the variational form of the system for $u^\epsilon \in V_0^\epsilon$ and $p^\epsilon \in W$ satisfying

$$2\mu\epsilon^2 (Du_s^\epsilon, Dv)_{\Omega_s^\epsilon} + \left(\frac{\epsilon\mu\alpha}{\sqrt{K^\epsilon}}u_s^\epsilon \cdot \tau, v_s \cdot \tau\right)_{\Gamma^\epsilon} - (p^\epsilon, \nabla \cdot v)_\Omega + \mu((K^\epsilon)^{-1}u_d^\epsilon, v)_{\Omega_d^\epsilon} = (f, v)_\Omega, \quad v \in V_0^\epsilon, \tag{3.1}$$

$$(\nabla \cdot u^\epsilon, w)_\Omega = (q, w)_\Omega, \quad w \in W, \tag{3.2}$$

where $u_d^\epsilon = u|_{\Omega_d^\epsilon}$ and $u_s^\epsilon \cdot \eta_s = u_d^\epsilon \cdot \eta_s$ on Γ^ϵ is implicit from $u^\epsilon \in V_0^\epsilon$.

Theorem 3.1. For each ϵ , there exists $(u^\epsilon, p^\epsilon) \in V_0^\epsilon \times W$ satisfying (1.1)–(1.9) weakly, i.e., (3.1) and (3.2), such that

$$\epsilon \|\nabla u_s^\epsilon\|_{\Omega_s^\epsilon} + \sqrt{\epsilon} \|u_s^\epsilon \cdot \tau\|_{\Gamma^\epsilon} + \|u^\epsilon\| + \|\nabla \cdot u^\epsilon\| + \|p^\epsilon\| \leq C(\|f\| + \|q\|) \tag{3.3}$$

with C independent of ϵ .

In order to prove this result, we first prove a lemma related to Korn’s inequality.

Lemma 3.2. There exists C independent of ϵ such that for all $v \in V_0^\epsilon(\Omega)$,

$$\|v_s\|_{\Omega_s^\epsilon} + \epsilon \|\nabla v_s\|_{\Omega_s^\epsilon} \leq C(\epsilon \|Dv_s\|_{\Omega_s^\epsilon} + \sqrt{\epsilon} \|v_s \cdot \tau\|_{\Gamma^\epsilon} + \|v_d\|_{\Omega_d^\epsilon} + \epsilon \|\nabla \cdot v\|).$$

If Ω_d is not empty, the above inequality holds for $v \in V^\epsilon(\Omega)$.

Proof. First we show that a similar result holds for $\hat{v} \in V(Y) = \{v \in H(\text{div}, Y) \mid v_s = v|_{Y_s} \in H^1(Y_s)\}$, and then we use a translation and scaling argument to pass to all of Ω . Suppose it is not true that there exists \hat{C} such that

$$\|\hat{v}_s\|_{Y_s} + \|\nabla \hat{v}_s\|_{Y_s} \leq \hat{C}(\|D\hat{v}_s\|_{Y_s} + \|\hat{v}_s \cdot \tau\|_\Gamma + \|\hat{v}_d\|_{Y_d} + \|\nabla \cdot \hat{v}\|_Y). \tag{3.4}$$

Then there exists a sequence $\{\hat{v}_n\}_{n=1}^\infty \in V(Y)$ such that

$$\|\hat{v}_{n,s}\|_{Y_s} + \|\nabla \hat{v}_{n,s}\|_{Y_s} = 1 \tag{3.5}$$

and

$$\|D\hat{v}_{n,s}\|_{Y_s} + \|\hat{v}_{n,s} \cdot \tau\|_\Gamma + \|\hat{v}_{n,d}\|_{Y_d} + \|\nabla \cdot \hat{v}_n\|_Y \leq \frac{1}{n}. \tag{3.6}$$

The last two terms on the left-hand side above tell us that $\hat{v}_{n,d} \rightarrow 0$ in $H(\text{div}, Y_d)$, and (3.5) implies that $\hat{v}_{n,s} \rightharpoonup \hat{v}_s$ weakly in $H^1(Y_s)$ for some \hat{v}_s . Let \hat{v} be the extension by zero of \hat{v}_s to Y . Since \hat{v}_n is bounded in $H(\text{div}, Y)$, it converges weakly, and we conclude that in fact

$$\hat{v}_n \rightharpoonup \hat{v} \quad \text{in } H(\text{div}, Y).$$

Since

$$\|\hat{v}_{n,d} \cdot \eta\|_{(H_0^{1/2}(\Gamma))^*} \leq C\|\hat{v}_{n,d}\|_{H(\text{div}, Y_d)} \rightarrow 0,$$

where the norm on the left-hand side is the norm of the dual space of $H_0^{1/2}(\Gamma)$ (see [20]), we conclude that $\hat{v}_d \cdot \eta = \hat{v}_s \cdot \eta = 0$ on Γ . On the other hand, $\|\hat{v}_{n,s} \cdot \tau\|_\Gamma \rightarrow 0$, so that $\hat{v}_s \cdot \tau = 0$ and further that $\hat{v}_s = 0$ on Γ . Now Korn’s inequality can be applied on Y_s to show that

$$\|\hat{v}_{n,s}\|_{Y_s} + \|\nabla \hat{v}_{n,s}\|_{Y_s} \leq C\|D\hat{v}_{n,s}\|_{Y_s}.$$

But the left-hand side is 1 by (3.5), and the right-hand side tends to 0 by (3.6), contradicting the assumption that the inequality (3.4) fails to hold. Now we have inequality (3.4) for any $\hat{v} \in V(Y)$, wherein \hat{C} does not depend on ϵ .

We remark that we also have the inequality for functions defined only on an open subset $Y_p \subset Y$, as long as $Y_p \cap Y_d$ is not empty. When $Y_p \subset Y_s$, we can obtain the same inequality provided that we have the boundary condition $\hat{v} = 0$ on some positive measure subset of ∂Y_p .

By the structure of Ω , we can write it as $\Omega = \bigcup_{i \in \mathcal{I}} \epsilon Y^i$,

where $\epsilon Y^i = (\epsilon(Y + \bar{n}_i)) \cap \Omega$, \bar{n}_i is some vector whose components are integers, and \mathcal{I} is some appropriate index set. Let $v \in V_0^\epsilon(\Omega)$ and define for $y \in Y$, $\hat{v}^i(y) =$

$v(\epsilon(y + \bar{n}_i)) \in V(Y)$. By our remark above, if $\partial\Omega$ intersects the interior of a cell (i.e., Ω is not tiled exactly by scaled translates of Y), the inequality (3.4) still holds on that truncated cell since $v \in V_0^\epsilon$. Thus, inequality (3.4) holds on each Y^i , so

$$\|\hat{v}_s^i\|_{Y_s^i} + \|\nabla \hat{v}_s^i\|_{Y_s^i} \leq \hat{C}(\|D\hat{v}_s^i\|_{Y_s^i} + \|\hat{v}_s^i \cdot \tau\|_{\Gamma^i} + \|\hat{v}_d^i\|_{Y_d^i} + \|\nabla \cdot \hat{v}^i\|_{Y^i}),$$

and if we sum over all i and make the change of variables $x = \epsilon(y + \bar{n}_i)$ on each Y^i , we obtain

$$\epsilon^{-1}\|v_s\|_{\Omega_s^\epsilon} + \|\nabla v_s\|_{\Omega_s^\epsilon} \leq C(\|Dv_s\|_{\Omega_s^\epsilon} + \epsilon^{-1/2}\|v_s \cdot \tau\|_{\Gamma^\epsilon} + \epsilon^{-1}\|v_d\|_{\Omega_d^\epsilon} + \|\nabla \cdot v\|)$$

which gives the desired result $v \in V_0^\epsilon$.

For $v \in V^\epsilon$, as long as Ω_d is not empty, we can adjoin any truncated cell to an entire cell lying beside it (at least for ϵ small). Since there will be a portion of Γ within this new composite cell, we have the inequality (3.4) on the composite cell regardless of the outer boundary conditions. Hence, if Ω_d is not empty, the theorem holds also for all $v \in V^\epsilon$. \square

Proof. Proof of Theorem 3.1. The theorem follows from the inf–sup theory of saddle point problems [5, 10–12]. For $u, v \in V_0^\epsilon(\Omega)$ and $w \in W$, we have the bilinear forms

$$a^\epsilon(u, v) = 2\mu\epsilon^2(Du_s, Dv)_{\Omega_s^\epsilon} + \left(\frac{\epsilon\mu\alpha}{\sqrt{K^\epsilon}}u_s \cdot \tau, v_s \cdot \tau\right)_{\Gamma^\epsilon} + (\mu(K^\epsilon)^{-1}u_d, v)_{\Omega_d^\epsilon},$$

$$b(v, w) = (w, \nabla \cdot v)_\Omega.$$

Then (3.1) and (3.2) can be rewritten as follows: find $(u^\epsilon, p^\epsilon) \in V_0^\epsilon \times W$ such that

$$a^\epsilon(u^\epsilon, v) - b(v, p^\epsilon) = (f, v), \quad v \in V_0^\epsilon, \tag{3.7}$$

$$b(u^\epsilon, w) = (q, w), \quad w \in W. \tag{3.8}$$

We endow $V_0^\epsilon(\Omega)$ with the norm

$$\| \|u\| \|_\epsilon = (\|u\|^2 + \|\nabla \cdot u\|^2 + \epsilon^2\|\nabla u_s\|_{\Omega_s^\epsilon}^2)^{1/2},$$

for which it is complete. We claim that

$$\left(\frac{\epsilon\mu\alpha}{\sqrt{K^\epsilon}}u_s \cdot \tau, v_s \cdot \tau\right)_{\Gamma^\epsilon} \leq C\|u_s \cdot \tau\|_{\Gamma^\epsilon}\|v_s \cdot \tau\|_{\Gamma^\epsilon} \leq C\| \|u\| \|_\epsilon \| \|v\| \|_\epsilon, \tag{3.9}$$

so that both a^ϵ and b are bounded (i.e., continuous) with constants independent of ϵ . To see the claim, compute

$$\begin{aligned}
 \epsilon \|u_s \cdot \tau\|_{\Gamma^\epsilon}^2 &= \epsilon \sum_{i \in \mathcal{I}} \|u_s \cdot \tau\|_{\epsilon \Gamma^i}^2 \\
 &= \epsilon \sum_{i \in \mathcal{I}} \epsilon \|\hat{u}_s \cdot \hat{\tau}\|_{\Gamma^i}^2 \\
 &\leq \epsilon^2 \sum_{i \in \mathcal{I}} \hat{C} \|\hat{u}\|_{Y_s^i} \|\hat{u}\|_{H^1(Y_s^i)} \\
 &\leq \hat{C} \epsilon \sum_{i \in \mathcal{I}} \|u\|_{\epsilon Y_s^i} \|u\|_{H^1(\epsilon Y_s^i)} \\
 &\leq \frac{\hat{C}}{2} \sum_{i \in \mathcal{I}} [\|u\|_{\epsilon Y_s^i}^2 + \epsilon^2 \|u\|_{H^1(\epsilon Y_s^i)}^2] \\
 &= \frac{\hat{C}}{2} [\|u\|_{\Omega_s^\epsilon}^2 + \epsilon^2 \|u\|_{H^1(\Omega_s^\epsilon)}^2] \\
 &\leq C \|u\|_\epsilon^2. \tag{3.10}
 \end{aligned}$$

Moreover, a^ϵ is coercive on $V_0^\epsilon \cap \{v \in V_0^\epsilon : \nabla \cdot v = 0\}$ by Lemma 3.2, with bound independent of ϵ .

It remains to show the inf-sup condition, but this follows from the corresponding condition known for the Stokes system on Ω ; that is,

$$\begin{aligned}
 \inf_{w \in W} \sup_{v \in V_0^\epsilon} \frac{(\nabla \cdot v, w)}{\|v\|_\epsilon \|w\|} &\geq \inf_{w \in W} \sup_{v \in (H_0^1)^2} \frac{(\nabla \cdot v, w)}{\|v\|_\epsilon \|w\|} \\
 &\geq \inf_{w \in W} \sup_{v \in (H_0^1)^2} \frac{(\nabla \cdot v, w)}{2 \|v\|_{(H^1(\Omega))^2} \|w\|} \\
 &\geq \gamma > 0,
 \end{aligned}$$

for some γ independent of ϵ , since $(H_0^1)^2 \subset V_0^\epsilon$ and $2 \|v\|_{(H^1(\Omega))^2} \geq \|v\|_\epsilon$.

Now the inf-sup theory provides the existence and uniqueness of a solution to our systems (3.7) and (3.8) [12]. Moreover,

$$\|u^\epsilon\|_\epsilon + \|p^\epsilon\| \leq C(\|f\| + \|q\|), \tag{3.11}$$

where C depends on γ , the coercivity bound for a^ϵ , and the continuity bounds for a^ϵ and b , each of which is independent of ϵ . Finally, (3.10) and (3.11) imply (3.3). \square

4. Two-scale convergence results for bimodal media

In this section, we make note of some slight extensions of the two-scale convergence results of Allaire [1, 14]. Lemmas 4.1 and 4.4 can be deduced easily from the proof of Theorem 2.7 in [1]. We include the following statements and proofs for clarity and completeness. We first recall that $\mathcal{D}(\Omega; C_\#^\infty(Y))$ is the set of infinitely

differentiable functions in $\Omega \times Y$ that have compact support in Ω and are periodic in Y , and we recall the following definition.

Definition 4.1. If $\{u^\epsilon\}_\epsilon \subset L^2(\Omega)$ and $u_0(x, y) \in L^2(\Omega \times Y)$ are such that

$$\begin{aligned}
 \lim_{\epsilon \rightarrow 0} \int_\Omega u^\epsilon(x) \phi(x, x/\epsilon) dx \\
 = \frac{1}{|Y|} \int_\Omega \int_Y u_0(x, y) \phi(x, y) dy dx
 \end{aligned}$$

for any function $\phi \in \mathcal{D}(\Omega; C_\#^\infty(Y))$, then $\{u^\epsilon\}_\epsilon$ is said to two-scale converge in $\Omega \times Y$ to $u_0(x, y)$, and we write this as

$$u^\epsilon \rightharpoonup u_0 \quad \text{in } \Omega \times Y \text{ as } \epsilon \rightarrow 0.$$

Lemma 4.1. Let $\ell = s$ or d and χ_ℓ^ϵ be the characteristic function on Ω_ℓ^ϵ . If u^ϵ is such that $\|u^\epsilon\| \leq C$ for some constant C independent of ϵ , then a subsequence of $\chi_\ell^\epsilon u^\epsilon$ two-scale converges to some $\psi_0^\ell \in L^2(\Omega \times Y)$ such that $\text{supp}(\psi_0^\ell) \subset \Omega \times \bar{Y}_\ell$. Moreover, if u^ϵ two-scale converges to $u_0 \in L^2(\Omega \times Y)$, then $\chi_\ell^\epsilon u^\epsilon \rightharpoonup u_0|_{\Omega \times Y_\ell}$ in $\Omega \times Y_\ell$.

Proof. Because $\|\chi_\ell^\epsilon u^\epsilon\| \leq \|u^\epsilon\| \leq C$, a subsequence of $\chi_\ell^\epsilon u^\epsilon$ two-scale converges to some $\psi_0^\ell \in L^2(\Omega \times Y)$ [1]. Take a test function $\phi \in \mathcal{D}(\Omega; C_\#^\infty(Y))$ supported in $\Omega \times Y_k$ where $k \neq \ell$. Then $\phi^\epsilon(x) = \phi(x, x/\epsilon)$ is supported in Ω_k^ϵ and

$$\begin{aligned}
 0 &= \lim_{\epsilon \rightarrow 0} \int_\Omega \chi_\ell^\epsilon u^\epsilon \phi^\epsilon dx \\
 &= \frac{1}{|Y|} \int_\Omega \int_Y \psi_0^\ell \phi dy dx \\
 &= \frac{1}{|Y|} \int_\Omega \int_{Y_k} \psi_0^\ell \phi dy dx.
 \end{aligned}$$

This holds for all such ϕ , so $\psi_0^\ell = 0$ on $\Omega \times Y_k$.

Now, take a test function $\phi \in \mathcal{D}(\Omega; C_\#^\infty(Y))$ with support in $\Omega \times Y_\ell$. Then

$$\int_\Omega \chi_\ell^\epsilon u^\epsilon \phi^\epsilon dx = \int_\Omega u^\epsilon \phi^\epsilon dx,$$

so, taking the limit as $\epsilon \rightarrow 0$,

$$\frac{1}{|Y|} \int_\Omega \int_{Y_\ell} \psi_0^\ell \phi dy dx = \frac{1}{|Y|} \int_\Omega \int_Y u_0 \phi dy dx.$$

An application of Lusin’s theorem completes the lemma. \square

Lemma 4.1 allows us to make the following definition and gives the following corollary. Note that a function

in $\mathcal{D}(\Omega; C^\infty_\#(Y_\ell))$ is considered to be only Y -periodic in y , with no condition imposed on Γ .

Definition 4.2. If $\ell = s$ or d and $\{u_\ell^\epsilon\}_\epsilon \subset L^2(\Omega_\ell^\epsilon)$ is such that, for any function $\phi(x, y)$ in $\mathcal{D}(\Omega; C^\infty_\#(Y_\ell))$,

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_{\Omega_\ell^\epsilon} u_\ell^\epsilon(x) \phi(x, x/\epsilon) dx \\ = \frac{1}{|Y|} \int_\Omega \int_{Y_\ell} u_0(x, y) \phi(x, y) dy dx \end{aligned}$$

for some $u_0(x, y)$ in $L^2(\Omega \times Y_\ell)$, then $\{u_\ell^\epsilon\}_\epsilon$ is said to *two-scale converge* in $\Omega \times Y_\ell$ to $u_0(x, y)$ as $\epsilon \rightarrow 0$.

Corollary 4.2. If $u_\ell^\epsilon \in L^2(\Omega_\ell^\epsilon)$ and there exists $C > 0$ such that $\|u_\ell^\epsilon\|_{\Omega_\ell^\epsilon} \leq C$ for all $\epsilon > 0$, then there exists a subsequence that two-scale converges in $\Omega \times Y_\ell$ to $u_0 \in L^2(\Omega \times Y_\ell)$.

The following lemma is immediate and illuminates the connection between weak and two-scale convergence.

Lemma 4.3. If $\{u_\ell^\epsilon\}_\epsilon$ two-scale converges to $u_0(x, y)$ in $\Omega \times Y_\ell$, and \hat{u}_ℓ^ϵ denotes the extension of u_ℓ^ϵ by zero to Ω , then \hat{u}_ℓ^ϵ converges weakly to $\frac{1}{|Y|} \int_{Y_\ell} u_0(x, y) dy$ in $L^2(\Omega)$.

The next results will be needed to prove our homogenization result. As usual, $H^\#_1(Y)$ denotes the Y -periodic functions in $H^1(Y)$.

Lemma 4.4. Fix $\ell = s$ or d .

- (a) If $\|u_\ell^\epsilon\|_{\Omega_\ell^\epsilon} \leq C$ and $\|\epsilon \nabla u_\ell^\epsilon\|_{\Omega_\ell^\epsilon} \leq C$ for some constant C , then there exists $u_{0,\ell} \in (L^2(\Omega; H^\#_1(Y_\ell)))^2$ such that some subsequence of $\{u_\ell^\epsilon\}_\epsilon$ two-scale converges in $\Omega \times Y_\ell$ to $u_{0,\ell}$ and $\{\epsilon \nabla u_\ell^\epsilon\}_\epsilon$ two-scale converges in $\Omega \times Y_\ell$ to $\nabla_y u_{0,\ell}$.
- (b) If $\|u_\ell^\epsilon\|_{\Omega_\ell^\epsilon} \leq C$ and $\|\nabla \cdot u_\ell^\epsilon\|_{\Omega_\ell^\epsilon} \leq C$ for some constant C , then there exists $u_{0,\ell} \in (L^2(\Omega; H(\text{div}, Y_\ell)))^2$ such that some subsequence of $\{u_\ell^\epsilon\}_\epsilon$ two-scale converges in $\Omega \times Y_\ell$ to $u_{0,\ell}$ and $\{\nabla \cdot u_\ell^\epsilon\}_\epsilon$, extended to Ω by zero, converges weakly in $L^2(\Omega)$ to $\frac{1}{|Y|} \int_{Y_\ell} \nabla_x \cdot u_{0,\ell} dy$. Moreover, $\nabla_y \cdot u_{0,\ell} = 0$.

Proof. For result (a), by Corollary 4.2, we have for some subsequence both

$$\begin{aligned} u_\ell^\epsilon &\rightharpoonup u_{0,\ell} \quad \text{in } \Omega \times Y_\ell, \\ \epsilon \nabla u_\ell^\epsilon &\rightharpoonup \psi_{0,\ell} \quad \text{in } \Omega \times Y_\ell. \end{aligned}$$

Let $\phi \in (\mathcal{D}(\Omega; C^\infty_\#(Y_\ell)))^2$ be such that $\phi|_{\Omega \times \Gamma} = 0$, and let $\phi^\epsilon(x) = \phi(x, x/\epsilon)$. Compute

$$(\epsilon \nabla u_\ell^\epsilon, \phi^\epsilon)_{\Omega_\ell^\epsilon} = -(u_\ell^\epsilon, \epsilon \nabla_x \cdot \phi^\epsilon + \nabla_y \cdot \phi^\epsilon)_{\Omega_\ell^\epsilon},$$

so that as $\epsilon \rightarrow 0$,

$$\begin{aligned} (\psi_{0,\ell}, \phi)_{\Omega \times Y_\ell} &= -(u_{0,\ell}, \nabla_y \cdot \phi)_{\Omega \times Y_\ell} \\ &= (\nabla_y u_{0,\ell}, \phi)_{\Omega \times Y_\ell} - (u_{0,\ell} \cdot \eta, \phi)_{\Omega \times (\partial Y_\ell \setminus \Gamma)}. \end{aligned}$$

With $\phi|_{\Omega \times \partial Y_\ell} = 0$, we conclude that $\psi_{0,\ell} = \nabla_y u_{0,\ell}$. Then we further conclude that $u_{0,\ell}$ is periodic in $y \in Y_\ell$, i.e., that $u_{0,\ell} \in (L^2(\Omega; H^\#_1(Y_\ell)))^2$.

For (b), Corollary 4.2 gives us two-scale convergence of u_ℓ^ϵ to $u_{0,\ell}$, and then weak convergence of $\nabla \cdot u_\ell^\epsilon$ to $\frac{1}{|Y|} \int_{Y_\ell} \nabla_x \cdot u_{0,\ell} dy$ follows easily. To obtain $\nabla_y \cdot u_{0,\ell} = 0$, note that for $\phi \in \mathcal{D}(\Omega; C^\infty_0(Y))$,

$$\int_{\Omega_\ell^\epsilon} \nabla \cdot u_\ell^\epsilon \phi^\epsilon dx = - \int_{\Omega_\ell^\epsilon} u_\ell^\epsilon \cdot (\nabla_x \phi^\epsilon + \epsilon^{-1} \nabla_y \phi^\epsilon) dx,$$

By Corollary 4.2, the left-hand side and the first term on the right-hand side both converge as $\epsilon \rightarrow 0$. Thus, we obtain

$$\lim_{\epsilon \rightarrow 0} \int_{\Omega_\ell^\epsilon} u_\ell^\epsilon \cdot \nabla_y \phi dx = 0,$$

which implies that $\nabla_y \cdot u_{0,\ell} = 0$. □

Lemma 4.5. If u_s^ϵ is such that $\|u_s^\epsilon\|_{\Omega_s^\epsilon}$ and $\|\epsilon \nabla u_s^\epsilon\|_{\Omega_s^\epsilon}$ are bounded independent of ϵ and Γ^ϵ is a smooth submanifold of Ω^ϵ , then for $\phi \in (\mathcal{D}(\Omega; C^\infty_\#(Y_s)))^2$,

$$\lim_{\epsilon \rightarrow 0} \epsilon (u_s^\epsilon \cdot \tau, \phi \cdot \tau)_{\Gamma^\epsilon} = |Y|^{-1} (u_{0,s} \cdot \tau, \phi \cdot \tau)_{\Omega \times \Gamma},$$

where $u_{0,s}$ is the two-scale limit of u_s^ϵ in $\Omega \times Y_s$. Moreover,

$$\lim_{\epsilon \rightarrow 0} \epsilon (u_s^\epsilon \cdot \eta_s, \phi \cdot \eta_s)_{\Gamma^\epsilon} = |Y|^{-1} (u_{0,s} \cdot \eta_s, \phi \cdot \eta_s)_{\Omega \times \Gamma}.$$

Proof. Let $\Phi \in (\mathcal{D}(\Omega; C^\infty_\#(Y_s)))^{2 \times 2}$ and $\Phi^\epsilon(x) = \Phi(x, x/\epsilon)$. Then

$$\begin{aligned} (\epsilon \nabla u_s^\epsilon, \Phi^\epsilon)_{\Omega_s^\epsilon} &= -\epsilon (u_s^\epsilon, \nabla_x \cdot \Phi^\epsilon)_{\Omega_s^\epsilon} \\ &\quad - (u_s^\epsilon, \nabla_y \cdot \Phi^\epsilon)_{\Omega_s^\epsilon} + \epsilon (u_s^\epsilon, \Phi^\epsilon \cdot \eta_s)_{\Gamma^\epsilon}. \end{aligned}$$

Taking the limit of both sides as $\epsilon \rightarrow 0$, we obtain from Lemma 4.4

$$\begin{aligned} |Y|^{-1} (\nabla_y u_{0,s}, \Phi)_{\Omega \times Y_s} &= -|Y|^{-1} (u_{0,s}, \nabla_y \cdot \Phi)_{\Omega \times Y_s} \\ &\quad + \lim_{\epsilon \rightarrow 0} \epsilon (u_s^\epsilon, \Phi^\epsilon \cdot \eta_s)_{\Gamma^\epsilon}. \end{aligned}$$

This implies that

$$|Y|^{-1} (u_{0,s}, \Phi \cdot \eta_s)_{\Omega \times \Gamma} = \lim_{\epsilon \rightarrow 0} \epsilon (u_s^\epsilon, \Phi^\epsilon \cdot \eta_s)_{\Gamma^\epsilon}. \tag{4.1}$$

Since Γ is smooth, the tubular neighborhood theorem from topology allows us to extend the normal vector field η_s on Γ to a smooth vector field $\hat{N}(y)$ on Y . If we do this locally, and patch the results together using a partition of unity argument, we can obtain smooth vector fields after periodic extension. We can then define N^ϵ and T^ϵ on all of Ω by setting $N^\epsilon(x) = N(x/\epsilon) = \hat{N}(y)$ and $T^\epsilon(x) = T(x/\epsilon) = \hat{T}(y)$.

Now take $\Phi = T^\epsilon(\phi \cdot T^\epsilon)(N^\epsilon)^T$ in (4.1), where $\phi \in (\mathcal{D}(\Omega; C^\infty_\#(Y_s)))^2$. This yields the first result. Replacing T^ϵ with N^ϵ gives the second result. \square

In fact, the following more general definition makes sense and was previously stated in [2, 9, 21].

Definition 4.3. If $\ell = s$ or d and $\{u_\ell^\epsilon\}_\epsilon \subset L^2(\Gamma^\epsilon)$ is such that, for any function $\phi(x, y)$ in $\mathcal{D}(\Omega; C^\infty_\#(\Gamma))$ which is Y -periodic in y ,

$$\begin{aligned} &\lim_{\epsilon \rightarrow 0} \int_{\Gamma^\epsilon} u_\ell^\epsilon(x) \phi(x, x/\epsilon) dx \\ &= \frac{1}{|Y|} \int_\Omega \int_\Gamma u_0(x, y) \phi(x, y) dS dx \end{aligned}$$

for some $u_0(x, y)$ in $L^2(\Omega \times \Gamma)$, then $\{u_\ell^\epsilon\}_\epsilon$ is said to *two-scale converge* on $\Omega \times \Gamma$ to $u_0(x, y)$ as $\epsilon \rightarrow 0$.

5. Proof of the homogenization result

We now prove rigorously the homogenization results obtained formally in section 2. In the first theorem, we obtain only weak convergence in $H(\text{div}, \Omega) \times W$ to the solution of the homogenized problem. In the second theorem, we show that, in fact, we have strong convergence of (u^ϵ, p^ϵ) in $L^2(\Omega) \times W$.

Theorem 5.1. There exists $(u, p) \in H(\text{div}, \Omega) \times W$ such that the velocity u^ϵ converges weakly to u in $H(\text{div}, \Omega)$, p^ϵ converges weakly to p in W , and (u, p) is the unique solution to the homogenized Darcy problem

$$\mu \tilde{K}^{-1}u + \nabla p = f \quad \text{in } \Omega, \tag{5.1}$$

$$\nabla \cdot u = q \quad \text{in } \Omega, \tag{5.2}$$

$$u \cdot \eta = 0 \quad \text{on } \partial\Omega, \tag{5.3}$$

where the tensor \tilde{K} is defined by (2.4)–(2.10) and (2.13).

Proof. By our energy estimates in Theorem 3.1 and the two-scale convergence results of [1, 14], Corollary 4.2, and Lemma 4.4, for $\ell = s, d$, there exists $p_0(x, y) \in$

$L^2(\Omega \times Y)$ and $u_0(x, y) \in L^2(\Omega \times Y)$ such that the following two-scale convergences hold:

$$p^\epsilon \rightharpoonup p_0 \quad \text{in } \Omega \times Y, \tag{5.4}$$

$$u^\epsilon \rightharpoonup u_0 \quad \text{in } \Omega \times Y, \tag{5.5}$$

$$u_\ell^\epsilon \rightharpoonup u_{0,\ell} \quad \text{in } \Omega \times Y_\ell, \tag{5.6}$$

$$\epsilon \nabla u_s^\epsilon \rightharpoonup \nabla_y u_{0,s} \quad \text{in } \Omega \times Y_s. \tag{5.7}$$

Moreover, $u_{0,s} \in (L^2(\Omega; H^1_\#(Y_s)))^2$,

$$\nabla \cdot u^\epsilon = q, \tag{5.8}$$

and

$$\nabla_y \cdot u_0(x, y) = 0. \tag{5.9}$$

Let $\Psi(x, y) \in C^\infty_0(\Omega \times Y)$ be Y -periodic. Take $v(x) = \epsilon \Psi(x, x/\epsilon)$ in the variational problem (3.1), so that

$$(p^\epsilon, \nabla_y \cdot \Psi) + O(\epsilon) = 0.$$

As $\epsilon \rightarrow 0$, we obtain

$$(p_0, \nabla_y \cdot \Psi)_{\Omega \times Y} = 0.$$

This implies that $\nabla_y p_0 = 0$, so that $p_0(x, y) = p_0(x)$ only.

Next, take $\Psi \in (\mathcal{D}(\Omega; C^\infty_\#(Y)))^2$ with $\nabla_y \cdot \Psi = 0$, and let $v(x) = \Psi(x, x/\epsilon)$ in the variational equation (3.1), so that

$$\begin{aligned} &2\mu \epsilon^2 (Du_s^\epsilon, D\Psi^\epsilon)_{\Omega_s^\epsilon} + \left(\frac{\epsilon \mu \alpha}{\sqrt{K^\epsilon}} u_s^\epsilon \cdot \tau, \Psi^\epsilon \cdot \tau \right)_{\Gamma^\epsilon} \\ &+ \left(\mu (K^\epsilon)^{-1} u_d^\epsilon, \Psi^\epsilon \right)_{\Omega_d^\epsilon} - (p^\epsilon, \nabla_x \cdot \Psi^\epsilon) = (f, \Psi^\epsilon). \end{aligned}$$

Using Lemma 4.5, passing to the two-scale limit gives

$$\begin{aligned} &2\mu (D_y u_{0,s}, D_y \Psi)_{\Omega \times Y_s} + \left(\frac{\mu \alpha}{\sqrt{K}} u_{0,s} \cdot \tau, \Psi \cdot \tau \right)_{\Omega \times \Gamma} \\ &+ (\mu K^{-1} u_{0,d}, \Psi)_{\Omega \times Y_d} \\ &- (p_0, \nabla_x \cdot \Psi)_{\Omega \times Y} = (f, \Psi)_{\Omega \times Y}. \end{aligned}$$

Integrating by parts and collecting terms, we obtain

$$\begin{aligned} &(-2\mu \nabla_y \cdot D_y u_{0,s} + \nabla_x p_0 - f, \Psi)_{\Omega \times Y_s} \\ &+ (\mu K^{-1} u_{0,d} + \nabla_x p_0 - f, \Psi)_{\Omega \times Y_d} \\ &+ \left(\frac{\mu \alpha}{\sqrt{K}} u_{0,s} \cdot \tau + 2\mu \tau \cdot D_y u_{0,s} \cdot \eta_s, \Psi \cdot \tau \right)_{\Omega \times \Gamma} \\ &+ (2\mu \eta_s \cdot D_y u_{0,s} \cdot \eta_s, \Psi \cdot \eta_s)_{\Omega \times \Gamma} = 0. \tag{5.10} \end{aligned}$$

It is a well-known result that if $(\Phi, \Psi)_{\Omega \times Y} = 0$, with $\nabla_y \cdot \Psi = 0$ and $\Psi \in \mathcal{D}(\Omega; C^\infty_0(Y))^2$, then $\Phi = \nabla_y \psi$ for some $\psi \in L^2(\Omega; H^1(Y))$ [28]. Restrict to $\Psi \in (\mathcal{D}(\Omega; C^\infty_0(Y_s)))^2$ to obtain

$$(-2\mu \nabla_y \cdot D_y u_{0,s} + \nabla_x p_0 - f, \Psi)_{\Omega \times Y_s} = 0.$$

Thus, there exists $p_{1,s}(x, y) \in L^2(\Omega; H^1(Y_s))$ such that in $\Omega \times Y_s$,

$$-2\mu \nabla_y \cdot D_y u_{0,s} + \nabla_x p_0 - f = -\nabla_y p_{1,s}.$$

Likewise, we obtain $p_{1,d}(x, y) \in L^2(\Omega; H^1(Y_d))$ such that in $\Omega \times Y_d$,

$$K^{-1}u_{0,d} + \nabla_x p_0 - f = -\nabla_y p_{1,d}.$$

Thus, for all $\Psi \in \mathcal{D}(\Omega; C_{\#}^{\infty}(Y))^2$ satisfying the divergence constraint,

$$\begin{aligned} &(-\nabla_y p_{1,s}, \Psi)_{\Omega \times Y_s} + (-\nabla_y p_{1,d}, \Psi)_{\Omega \times Y_d} \\ &+ \left(\frac{\mu\alpha}{\sqrt{K}} u_{0,s} \cdot \tau + 2\mu\tau \cdot D_y u_{0,s} \cdot \eta_s, \Psi \cdot \tau \right)_{\Omega \times \Gamma} \\ &+ (2\mu\eta_s \cdot D_y u_{0,s} \cdot \eta_s, \Psi \cdot \eta_s)_{\Omega \times \Gamma} = 0, \end{aligned}$$

so integrating by parts yields

$$\begin{aligned} &\left(\frac{\mu\alpha}{\sqrt{K}} u_{0,s} \cdot \tau + 2\mu\tau \cdot D_y u_{0,s} \cdot \eta_s, \Psi \cdot \tau \right)_{\Omega \times \Gamma} \\ &+ (2\mu\eta_s \cdot D_y u_{0,s} \cdot \eta_s - p_{1,s} \\ &+ p_{1,d}, \Psi \cdot \eta_s)_{\Omega \times \Gamma} = 0. \end{aligned} \tag{5.11}$$

By noting, for example, that there exists a weak solution (see [28] or the proof of Theorem 3.1) $\Psi \in L^2(\Omega; (H_{\#}^1(Y))^2)$, $w \in L^2(\Omega; L^2(Y)/\mathbb{R})$ to

$$\begin{aligned} -\Delta_y \Psi + \nabla_y w &= 0 && \text{on } \Omega \times Y_s, \\ \nabla_y \cdot \Psi &= 0 && \text{on } \Omega \times Y_s, \\ \Psi \cdot \eta_s &= 0 && \text{on } \Omega \times \Gamma, \\ \Psi \cdot \tau &= \frac{\mu\alpha}{\sqrt{K}} u_{0,s} \cdot \tau + 2\mu\tau \cdot D_y u_{0,s} \cdot \eta_s && \text{on } \Omega \times \Gamma, \\ \Psi &= 0 && \text{on } \Omega \times (\partial Y_s \setminus \Gamma), \end{aligned}$$

and by the fact that $\mathcal{D}(\Omega; C_{\#}^{\infty}(Y))$ is dense in $L^2(\Omega; H_{\#}^1(Y))$, we obtain that each individual term in equation (5.11) vanishes. We then finally obtain that

the two-scale variational equations (5.9) and (5.10) are equivalent to

$$-2\mu \nabla_y \cdot D_y u_{0,s} + \nabla_x p_0(x) + \nabla_y p_{1,s}(x, y) = f \quad \text{in } \Omega \times Y_s, \tag{5.12}$$

$$\nabla_y \cdot u_{0,s} = 0 \quad \text{in } \Omega \times Y_s, \tag{5.13}$$

$$\mu K^{-1}u_{0,d} + \nabla_x p_0(x) + \nabla_y p_{1,d}(x, y) = f \quad \text{in } \Omega \times Y_d, \tag{5.14}$$

$$\nabla_y \cdot u_{0,d} = 0 \quad \text{in } \Omega \times Y_d, \tag{5.15}$$

$$2\mu \eta_s \cdot D_y u_{0,s} \cdot \eta_s = p_{1,s} - p_{1,d} \quad \text{on } \Omega \times \Gamma, \tag{5.16}$$

$$2\tau \cdot D_y u_{0,s} \cdot \eta_s = -\frac{\alpha}{\sqrt{K}} u_{0,s} \cdot \tau \quad \text{on } \Omega \times \Gamma. \tag{5.17}$$

Let (ω_j^s, Φ_j^s) and (ω_j^d, Φ_j^d) be Y -periodic solutions to the auxiliary problem on Y_s and Y_d given in (2.4)–(2.10). Then, because the above problem has a unique solution, it is clear that we can express $u_{0,s}$ and $u_{0,d}$ as in (2.11):

$$u_{0,\ell}(x, y) = \frac{1}{\mu} \sum_{j=1}^N \left(f_j(x) - \frac{\partial p_0}{\partial x_j}(x) \right) \omega_j^{\ell}(y),$$

where $\ell = s$ or d . Averaging over Y_s and Y_d as in (2.12), we get

$$\bar{u}_0 = \bar{u}_{0,s} + \bar{u}_{0,d},$$

$$= \frac{1}{\mu|Y|} \sum_{j=1}^N \left(f_j(x) - \partial_{x_j} p_0(x) \right)$$

$$\left(\int_{Y_s} \omega_j^s(y) + \int_{Y_d} \omega_j^d(y) \right)$$

$$= \frac{\tilde{K}}{\mu} (f - \nabla p_0), \tag{5.18}$$

where \tilde{K} is as in equation (2.13).

By Lemma 4.3 and equation (5.8), we obtain the weak convergence results in $L^2(\Omega)$

$$p^\epsilon \rightharpoonup \frac{1}{|Y|} \int_Y p_0(x, y) dy = p_0(x), \tag{5.19}$$

$$u^\epsilon \rightharpoonup \frac{1}{|Y|} \int_Y u_0(x, y) dy = \bar{u}_0, \tag{5.20}$$

$$q = \nabla \cdot u^\epsilon \rightharpoonup \nabla \cdot \bar{u}_0 = q. \tag{5.21}$$

Setting $p = p_0$ and $u = \bar{u}_0$, equations (5.18)–(5.21) give the theorem, since convergence on the boundary of the domain is trivial. \square

Theorem 5.2. Let $(u_0, p_0) \in L^2(\Omega \times Y) \times W$ be the two-scale limit of (u^ϵ, p^ϵ) as before. Then $u^\epsilon - u_0^\epsilon$ converges to 0 strongly in $L^2(\Omega)$, where $u_0^\epsilon = u(x, x/\epsilon)$, and p^ϵ converges strongly to $p_0(x)$ in W .

Proof. Since (\bar{u}_0, p_0) satisfies (5.1) and (5.2) and μ and \tilde{K} are smooth, if $f \in H^1(\Omega)$ and $q \in L^2$, then $p_0 \in H^2(\Omega)$. It follows that $\bar{u}_0 \in H^1(\Omega)$ and $\nabla_x u_0(x, y) \in L^2(\Omega)$. We also know that $\nabla_y u_0(x, y) = 0 \in L^2(\Omega \times Y_s)$, so $u_0(x, x/\epsilon) \in V^\epsilon$. Since $\mathcal{D}(\Omega)$ is dense in V^ϵ , we can use u_0^ϵ as a two-scale test function. By two-scale convergence

$$\lim_{\epsilon \rightarrow 0} a^\epsilon(u^\epsilon, u_0^\epsilon) = \bar{a}(u_0, u_0),$$

where

$$\begin{aligned} \bar{a}(v, z) &= 2\mu (D_y v, D_y z)_{\Omega \times Y_s} \\ &+ \left(\frac{\mu \alpha}{\sqrt{K}} v_s \cdot \tau, z_s \cdot \tau \right)_{\Omega \times \Gamma} + \left(\mu K^{-1} v, z \right)_{\Omega \times Y_d}. \end{aligned}$$

Similarly,

$$\lim_{\epsilon \rightarrow 0} a^\epsilon(u_0^\epsilon, u_0^\epsilon) = \bar{a}(u_0, u_0)$$

by two-scale convergence. Subtracting, we obtain $\lim_{\epsilon \rightarrow 0} a^\epsilon(u^\epsilon - u_0^\epsilon, u_0^\epsilon) = 0$.

We can also show that $\lim_{\epsilon \rightarrow 0} a^\epsilon(u^\epsilon, u^\epsilon) = \bar{a}(u_0, u_0)$. Since

$$a^\epsilon(u^\epsilon, u^\epsilon) - b(u^\epsilon, p^\epsilon) = (f, u^\epsilon), \tag{5.22}$$

$$b(u^\epsilon, p^\epsilon) = (q, p^\epsilon), \tag{5.23}$$

we know that $a^\epsilon(u^\epsilon, u^\epsilon) = (f, u^\epsilon)_{\Omega^\epsilon} + (q, p^\epsilon)_{\Omega^\epsilon}$. Taking the limit of both sides as $\epsilon \rightarrow 0$, by our weak convergence results in Theorem 5.1, the right-hand side converges to $(f, \bar{u}_0)_\Omega + (q, p_0)_\Omega$. By (5.2), $q = \nabla \cdot \bar{u}_0$, so integrating by parts gives that

$$\lim_{\epsilon \rightarrow 0} a^\epsilon(u^\epsilon, u^\epsilon) = \bar{a}(u_0, u_0),$$

since \bar{u}_0 satisfies the system of equations (5.12)–(5.17). Finally, by the symmetry of $a^\epsilon(\cdot, \cdot)$, we have

$$\lim_{\epsilon \rightarrow 0} a^\epsilon(u^\epsilon - u_0^\epsilon, u^\epsilon - u_0^\epsilon) = 0,$$

and by Lemma 3.2, we see that $\lim_{\epsilon \rightarrow 0} \{ \|u^\epsilon - u_0^\epsilon\|_{L^2(\Omega^\epsilon)} + \epsilon \|\nabla(u_s^\epsilon - u_{0,s}^\epsilon)\|_{L^2(\Omega_s)} \} = 0$.

Now we show the strong convergence of the pressure p^ϵ to p_0 in W . By the inf-sup condition for Stokes,

$$\sup_{v \in H_0^1} \frac{(w, \nabla \cdot v)}{\|v\|_1} > \gamma \|w\|_0,$$

there exists a sequence v^ϵ such that $\|v^\epsilon\|_1 = 1$ and

$$(p^\epsilon - p_0, \nabla \cdot v^\epsilon)_{\Omega^\epsilon} > \frac{\gamma}{2} \|p^\epsilon - p_0\|_0. \tag{5.24}$$

By Theorem 3.5 in Appendix A of [14], there exists $v_0(x) \in H^1(\Omega)$ and $v_1(x, y) \in L^2(\Omega; H_\#^1(Y)/\mathbb{R})$ such that (possibly after passing to a subsequence)

$$v^\epsilon \rightharpoonup v_0 \quad \text{and} \quad \nabla v^\epsilon \rightharpoonup \nabla_x v_0(x) + \nabla_y v_1(x, y).$$

Note that we can assume v_1 is such that its average over Y is zero. It follows that

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} (p_0, \nabla \cdot v^\epsilon)_{\Omega^\epsilon} &= \left(p_0, \nabla \cdot v_0 + \int_Y \nabla v_1 dy \right)_\Omega \\ &= (p_0, \nabla \cdot v_0)_\Omega, \end{aligned}$$

by the periodicity of v_1 . Now,

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} (p^\epsilon, \nabla \cdot v^\epsilon)_{\Omega^\epsilon} &= \lim_{\epsilon \rightarrow 0} (p^\epsilon, \nabla \cdot (v^\epsilon - v_0))_{\Omega^\epsilon} \\ &+ (p_0, \nabla \cdot v_0)_\Omega \\ &= \lim_{\epsilon \rightarrow 0} a^\epsilon(u^\epsilon, v^\epsilon - v_0) \\ &+ (p_0, \nabla \cdot v_0)_\Omega. \end{aligned}$$

Finally, we can show

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} a^\epsilon(u^\epsilon, v^\epsilon - v_0) \\ &= \lim_{\epsilon \rightarrow 0} \left\{ 2\mu(\epsilon Du^\epsilon, \epsilon D(v^\epsilon - v_0))_{\Omega_s^\epsilon} \right. \\ & \quad \left. + \left(\mu\alpha\sqrt{\frac{\epsilon}{K}}u_s^\epsilon \cdot \tau, \sqrt{\epsilon}(v^\epsilon - v_0)_s \cdot \tau \right)_{\Gamma^\epsilon} \right. \\ & \quad \left. + (\mu K^{-1}u^\epsilon, v^\epsilon - v_0)_{\Omega_d^\epsilon} \right\} \\ &= \lim_{\epsilon \rightarrow 0} \left\{ 2\mu(\epsilon D(u^\epsilon - u_0^\epsilon), \epsilon D(v^\epsilon - v_0))_{\Omega_s^\epsilon} \right. \\ & \quad \left. + \left(\mu\alpha\sqrt{\frac{\epsilon}{K}}(u^\epsilon - u_0^\epsilon)_s \cdot \tau, \sqrt{\epsilon}(v^\epsilon - v_0)_s \cdot \tau \right)_{\Gamma^\epsilon} \right. \\ & \quad \left. + (\mu K^{-1}(u^\epsilon - u_0), v^\epsilon - v_0)_{\Omega_d^\epsilon} \right. \\ & \quad \left. + 2\mu(\epsilon Du_0^\epsilon, \epsilon D(v^\epsilon - v_0))_{\Omega_s^\epsilon} \right. \\ & \quad \left. + \left(\mu\alpha\sqrt{\frac{\epsilon}{K}}u_{0,s}^\epsilon \cdot \tau, \sqrt{\epsilon}(v^\epsilon - v_0)_s \cdot \tau \right)_{\Gamma^\epsilon} \right. \\ & \quad \left. + (\mu K^{-1}u_0^\epsilon, v^\epsilon - v_0)_{\Omega_d^\epsilon} \right\} \\ &= 0, \end{aligned}$$

where the first three terms above converge to zero because $\lim_{\epsilon \rightarrow 0} \|u^\epsilon - u_0\|_0 = 0$, $\lim_{\epsilon \rightarrow 0} \epsilon \|\nabla(u^\epsilon - u_0^\epsilon)\|_0 = 0$, and because $v^\epsilon - v_0$ is bounded in H^1 . The final three terms above converge to zero by the two-scale convergence results for v^ϵ (and because one term has an extra factor of ϵ). Putting everything together, we have that

$$\lim_{\epsilon \rightarrow 0} (p^\epsilon - p_0, \nabla \cdot v^\epsilon) = 0$$

as $\epsilon \rightarrow 0$. By (5.24), $p^\epsilon \rightarrow p_0$ as $\epsilon \rightarrow 0$ strongly in W . □

6. A simple analytical solution of the auxiliary problem

It is not so easy to construct analytical solutions to the auxiliary problems (2.4)–(2.10), except at least in the following case. Let $Y = (0, \ell) \times (0, \ell)$ be a square of side length $\ell > 0$. With $Y_s = (0, \ell) \times (0, h)$ and $Y_d = (0, \ell) \times (h, \ell)$ repeated periodically, we have a horizontally layered medium. Note that Γ consists of two segments, $y_2 = h$ and, by periodicity, $y_2 = 0 \iff y_2 = \ell$.

When $j = 1$, it is easy to verify that the solution is

$$\omega_1^s(y) = \frac{1}{2} \left(-y_2^2 + hy_2 + \frac{\sqrt{K}}{\alpha}h \right) e_1, \tag{6.1}$$

$$\omega_1^d(y) = Ke_1, \tag{6.2}$$

$$\Phi_1(y) = 0, \tag{6.3}$$

which has flow in the y_1 -direction only. It follows from (2.13) that $\tilde{K}_{21} = 0$ (so \tilde{K} is diagonal), and

$$\tilde{K}_{11} = \frac{1}{\ell} \left(\frac{1}{12}h^3 + \frac{\sqrt{K}}{2\alpha}h^2 + K(\ell - h) \right). \tag{6.4}$$

This should be contrasted to the situation in which the porous matrix is replaced by an impermeable medium. Then $\omega_j^s = 0$ on Γ , and we have the well-known problem of Poiseuille flow in a pipe. The solution is

$$\check{\omega}_1^s(y) = -\frac{1}{2}y_2(h - y_2)e_1, \tag{6.5}$$

$$\check{\Phi}_1(y) = 0. \tag{6.6}$$

In this case, we would compute the effective permeability for a unit pressure drop in the y_1 -direction (which corresponds to the forcing function e_1) as

$$\tilde{K}_{11, \text{Poiseuille}} = \frac{h^3}{12\ell}, \tag{6.7}$$

which is the first term on the right side of (6.4).

If instead we assume the vugular region is impermeable, the bulk flow would be reduced from the porous medium case by the geometric factor $(\ell - h)/\ell$:

$$\tilde{K}_{11, \text{Darcy}} = \frac{\ell - h}{\ell}K. \tag{6.8}$$

This is the last term on the right side of (6.4). Thus, when considering flow in the direction of the vugular channel, we have the representation

$$\tilde{K}_{11} = \tilde{K}_{11, \text{Poiseuille}} + \tilde{K}_{11, \text{Beavers-Joseph}} + \tilde{K}_{11, \text{Darcy}}, \tag{6.9}$$

where

$$\tilde{K}_{11, \text{Beavers-Joseph}} = \frac{\sqrt{K}}{2\alpha\ell}h^2 \tag{6.10}$$

represents the Beavers–Joseph interface effect of fluid slippage. This term increases the Darcy–Stokes flow from the arithmetic average of the pure Stokes “pipe flow” and the pure Darcy flow. Note that we generally have $K \ll h^2$ and $\alpha = O(1)$. Thus, $\tilde{K}_{11, \text{Poiseuille}}$ is the leading term, and $\tilde{K}_{11, \text{Beavers-Joseph}}$ is the next order term in the expansion.

When $j = 2$ in the auxiliary problems (2.4)–(2.10), it is easy to verify that the solution is

$$\omega_2(y) = \frac{\ell}{\ell - h}Ke_2, \tag{6.11}$$

$$\Phi_2^s(y) = y_2, \tag{6.12}$$

$$\Phi_2^d(y) = \frac{h}{\ell - h}(\ell - y_2). \tag{6.13}$$

Again, $\tilde{K}_{12} = \tilde{K}_{21} = 0$ and

$$\tilde{K}_{22} = \frac{\ell}{\ell - h} K. \quad (6.14)$$

In this case, the flow is entirely in the y_2 -direction.

It is well known and easily verified that one-dimensional flow across a porous medium of permeability k_1 for distance h and k_2 for distance $\ell - h$ results in a flow rate that is the same as that in a uniform medium of permeability $k = \frac{\ell k_1 k_2}{h k_2 + (\ell - h) k_1}$, which is the harmonic average permeability. If we apply this result to our case, assuming that the vugular channel has infinite permeability, we obtain exactly (6.14).

In conclusion, this example suggests that the effective permeability represents an average of two extremes. When the vugs are interconnected in some direction, we have primarily Poiseuille flow behavior, with a low order correction term for the effective permeability related to the Beavers–Joseph slip (and an even lower order correction related to flow entirely in the porous matrix). When the flow is along paths that alternate from vug to matrix, the fluid behaves as if it were flowing in a porous medium with the vugs having infinite permeability.

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