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Сферическая симметрия гальки: скорость изменения отношения объема к поверхности

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Стремление горных пород и камней к обретению сферической симметрии в процессе эрозии предложено описывать с помощью скорости изменения отношения объема к поверхности. Проведено математическое моделирование с использованием в качестве эллипсоидов мраморной гальки различных форм и размеров. Скорость изменения отношения объема к поверхности $\mathrm{d}V/\mathrm{d}A$ рассчитана для плоской гальки, толщина которой принимается равной единице. Расчеты проводились в рамках гипотезы о том, что изменение формы и размеров гальки можно описывать с помощью обобщенной кривой, полученной из соотношения $\mathrm{d}A/\mathrm{d}L$, которая является двумерным вариантом соотношения $\mathrm{d}V/\mathrm{d}A$. Предположено, что сферическая симметрия возникает при округлой, не удлиненной форме независимо от степени шероховатости поверхности и минералогического состава. Эрозионность и абразивность поверхности, а также временные эффекты связаны с симметричностью, однако рассмотрение данного вопроса выходит за рамки настоящей статьи.

Keywords: симметрия, сферичность, округлость, удлиненность, эллипсоид, сфероид, галька, объем, поверхность, эрозия, абразивный износ

Tendency towards sphericity symmetry of pebbles: The rate change of volume with surface

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The primary effect of rocks and stones tending to erode towards sphericity symmetry can be identified with the rate change of volume with surface. Marble pebbles of various shapes and sizes are recorded and modeled mathematically as ellipsoids. The rate change of volume with surface $\mathrm{d}V/\mathrm{d}A$ is computed for flat-shaped pebbles where the thickness dimension is assumed to be unity. Invoked is the hypothesis that the change of shape and size of pebbles can be characterized by a master curve derived from $\mathrm{d}A/\mathrm{d}L$, the 2D version of $\mathrm{d}V/\mathrm{d}A$. The sphericity symmetry is assumed to be concerned with the preference of roundness to slenderness regardless of the surface roughness, smoothness and the mineralogical composition. The erosiveness and abrasiveness of the surface and time effects can interact with sphericity, which is an issue beyond the scope of this work.

Keywords: symmetry, sphericity, roundness, slenderness, ellipsoid, spheroid, pebble, volume, surface, erosion, abrasion

1. Introduction

Stones and rocks tend to be rounded when transported long distance by winds and rains. Similar phenomena can be observed for objects as large as the stars and planets and as small as the biological cells. For whatever causes of erosion and/or abrasion, there appears to be a natural law governing the change of shape and size of objects. That is the tendency towards the highest geometrical symmetry. To this end, the rate change of volume with surface $\mathrm{d}V/\mathrm{d}A$ will be depicted as the candidate to explain this natural law. It took more than 30 years since the 1980s to realize that the neglect of interaction between the surface and volume have left many physical phenomena unexplained. Classical continuum physics and mechanics disconnected surface and volume by assuming that the body forces vanished in the limit faster than the surface tractions [1]. The so called "size effect" was left out of continuum mechanics and thermodynamics was introduced as a separate discipline. This may be one of the fundamental reasons why the tendency for objects to strive for higher geometrical symmetry was never considered. The assumption, $\mathrm{d}V/\mathrm{d}A \to 0$, has in effect disregarded the natural law of objects tending towards higher symmetry. It has also affected the thinking of researchers for more than a century.

Granted surface texture affects the size shape and roundness of objects engulfed by outer space, ocean and desert. Poorly sorted samples can be unmanageable when seeking for the fundamentals of cause and effect. A well sorted group of samples would made note of the size scale and shape or the sphericity which refers to the roundness and slenderness of the objects in general, in contrast to the restricted definition of the ratio of the surface area of the sphere to that of the object (referred to as particle) in [2]. An arbitrary condition, however, was invoked to set the volume of the sphere equal to that of the particle without physical nor mathematical explanation except the fact that “volume” should somehow interact with “surface”. This appears to be the only way the sphericity parameter [2] can be made to depend on volume of the particle (object) and area of the particle. The fortuitous result gave the impression that the volume to surface effects were interactive. The intuition deserves credit even though the derivation lacks logic. The mathematical equivalence of volume of the object and sphere is irrevocable and cannot be justified by hand waving statements.

Sphericity, in general, is associated with the rates of exchange of volume with surface that determine the roundness and slenderness of objects. It does not have the same meaning as that considered in [2]. It involves a time dependent event with the change of geometry. These effects are revealed by a master curve that exhibit a directional preference and hence time effect, they agreed well with the data of pebbles.

2. Specificity of location and weathering

Roundness and roughness appearance can provide transport information on the type of objects. The characteristic abrasion patterns of space dusts and meteorites on the dwarf planets can reveal their time history. The same applies to a grain of sand with the same shape with different abrasive surface when observed under high magnification. Although the texture of their surface differs, they have a striking common feature, “the shape” resembling that of an ellipsoid pebble. Moreover, there is a fundamental invariant propriety of the tendency towards a higher degree of geometrical symmetry, regardless of location and weathering. This characteristic of nature will be explored mathematically and experimentally for the pebbles.

Figure 1 gives a close-up view of sand from the Gobi Desert in Mongolia. Under the electron microscope, the sand particles from 10^{-2} to 1 mm in size can be enlarged to pebbles of different sizes and shapes. Desert sands, transported long distances by winds, are typically rounded. Comparable sizes and shapes of sand when enlarged also resemble those collected from beaches of Thassos, facing the northern side of the Aegean sea which is far away from the Gobi Desert. The sand pebbles and those at the beaches of Thassos share similar geometrical configurations. Figure 2 shows an unsorted collection of mixture of large, medium and small pebbles while Fig. 3 gives an orderly display of ellipsoids and spheroids of marble pebbles.

The Thassos rivers transport the rock sediments to the ocean. They are eroded by the ocean waves over the years and washed onto the beaches of Thassos. Their variations in sizes and shapes are manifestation of the difference in ocean current velocity, pressure, weathering and rock properties giving rise to different rates of erosion and abrasion. These incidental details need not enter the analysis for establishing the tendency for pebbles to evolve to a higher state of symmetry from carefully selected samples at the beaches of Thassos.

3. Sphericity symmetry: Mathematical model

The term sphericity was introduced by Wadell [2] to measure the roundness (spherical) of an object. An index Ψ, defined as the ratio of the surface area A_s of the sphere to the surface area A_p of the object, was referred to as the

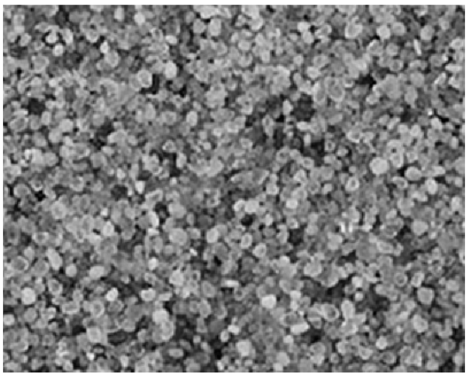

Fig. 1. Close-up of sand from the Gobi Desert, Mongolia

Fig. 2. Irregular shapes and size

Fig. 3. Ellipsoids and spheroids

"sphericity", such that $\Psi = A_s / A_p$. As such, the definition involved only the surface areas of the sphere and object (or particle). The stipulation was made for A_s to have the same volume V_p of the object (or particle). This is mathematically equivalent to setting $V_s = V_p$, an irrevocable manipulation that is not immune to dispute. Interestingly, the stipulation recognized the interaction between the surface and volume [1] that has been overlooked in the development of theories in physics and mechanics. It was not until the 1980s that the body forces were recognized to be interactive with the surface tractions via the rate change of volume with surface of the element. In this way, the so called "size effect" were brought into continuum mechanics. Up to that point, surface and volume effects are known to be related physically, but joined mathematically by invoking additional hypotheses. This is basically due to setting the limit $\mathrm{d}V/\mathrm{d}A \to 0$ that leads to the separation of surface and volume effects [1]. Surface is part of the volume and vice versa. They increase and decrease simultaneously. Their separation would results in inconsistencies.

Sphericity, as defined presently, is concerned with the roundness and slenderness of an object. It does not have the same meaning as that considered in [2]. More specifically, roundness and slenderness pertain to shape while roughness and smoothness to surface texture. They are related to size of the object. Small particles have a large surface area in contrast to large objects that are less likely to be affected by abrasion. In this light, the surface of a grain of sand is expected to be roughened more than a large pebble. The size of the object comes into play via the exchange of the surface and volume. The individual effects are interwoven. Figures 4 and 5 exhibit two seemingly similar objects with different surface textures. One of them is an artistic image of a Jacobi-ellipsoid dwarf planet resembling an ellipsoid pebble. Dwarf planets are about 600–1500 km in diameter. Their natural tendency is to become spheroids. Figure 5 is a grain of sand from the Gobi Desert in Mongolia. The actual size is about 100 μm magnified ×200. Indeed, the size and surface tell the time history of abrasion by space dusts in contrast to abrasion by sand sediments. Figures 4 and 5 lack the resolution to illustrate the details of the effects of roundness and roughness.

3.1. Ellipsoidal symmetry

Consider an ellipsoid with distinct semiaxes $a \geq b \geq c$. The volume of the internal part of the ellipsoid is given by

$$V = \frac{4}{3}\pi abc. \quad (1)$$

The surface area enclosing the ellipsoid is

$$A = 2\pi c^2 + \frac{2\pi ab}{\sin\varphi}\left[E(\varphi, k)\sin^2\varphi + F(\varphi, k)\cos^2\varphi\right], \quad (2)$$

where parameters

$$\cos\varphi = c/a, \quad k^2 = \frac{a^2(b^2 - c^2)}{b^2(a^2 - c^2)}, \quad (3)$$

$F(\varphi, k)$ and $E(\varphi, k)$ are incomplete elliptic integrals of the first and second kind respectively.

3.2. Elliptical approximation

The two-dimensional version of the ellipsoid is an ellipse with distinct semiaxes $a \geq b$. The volume of the internal part of the ellipse is given by

$$A = \pi ab. \quad (4)$$

The perimeter of the ellipse is

$$L = 4aE(k). \quad (5)$$

Fig. 4. Artist's conception of Haumea, a Jacobi-ellipsoid dwarf planet

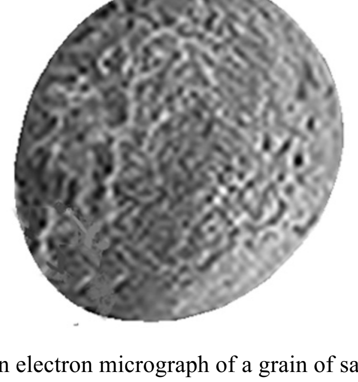

Fig. 5. An electron micrograph of a grain of sand, ×200

The argument k is given by

$$k = \frac{1}{2} - \frac{1}{2}\left(\frac{b}{a}\right)^2. \quad (6)$$

Substituting Eqs. (4) and (5) into $\mathrm{d}A/\mathrm{d}L$, there results

$$\frac{\mathrm{d}A}{\mathrm{d}L} = \pi\left(b + a\frac{\partial b}{\partial a}\right)\frac{a^2 k}{4E(k)a^2 k + 4b^2[E(k) - K(k)]}, \quad (7)$$

where $K(k)$ and $E(k)$ are complete elliptic integrals of the first and second kind respectively.

4. Tendency towards higher symmetry

The symmetry properties in physics and mechanics apply mainly to existing geometrical symmetry by physical or mathematical representation. The one-way preference towards higher symmetry for objects in three- or two-dimensional space has not received the attention that it deserves. The tendency for objects to gain stability by changing shape and size is implicitly a dynamic process. This is the non-equilibrium character of nature where space and time interact continuously. Events that seem to be in equilibrium at the macroscopic scale can be highly chaotic and unpredictable at the atomic or subatomic scale. Too often, premature conclusions are made on the behavior of the end states. This is particularly true in the field of particle physics when the observations are limited to small space and short time. Obviously, the results will change for long time, especially when the event may be unstable. "Uncertainty" is a common reference made in particle physics. These remarks are most relevant to studying the directional preference of symmetry with reference to the changing of shape and size of pebbles. Hypotheses will be necessary to develop mathematical models that can be checked by data collected from the Thassos beaches.

4.1. Hypotheses

Erosion and abrasion are space and time dependent processes that affect geometrical symmetry. Evolution obeys a hierarchy of changes that are space and time scale dependent. Extrapolation of results to the very small and very large is not obvious. Hypotheses will be made to guide the extrapolation.

Hypothesis I: Nature tends to evolve from a state of lower to higher geometrical symmetry to achieve kinematic stability.

Hypothesis II: The interim states of preferred geometrical symmetry are space and time dependent.

Hypothesis III: The highest states of geometrical symmetry correspond to complete isotropy in any dimensions.

The following three corollaries are stated to indicate what can be proven and what cannot. Even though the subject of geometrical symmetry is of fundamental interest, the progress will be slow since the subject will not get the attention of the main stream scientists and mathematicians. Even the project for computing the rate change of volume with surface $\mathrm{d}V/\mathrm{d}A$ for an ellipsoid initiated in 1982 [3] was left incomplete up to this date.

Corollary I: The lower order symmetry "ellipse" tends toward the higher order symmetry "circle", which can be stated as

$$b + a\frac{\partial b}{\partial a} \to 2b. \quad (8)$$

Corollary II: The more slender object tend to become less slender in time occupying less space.

Corollary III: The existence of perfect isotropy cannot be known since 96 % of the Universe remains as unknown.

A mathematical version of the model has been developed by assuming that the lower order symmetry "ellipse" tends toward the higher order symmetry "circle". The three-dimensional counterpart would be "ellipsoid" to "spheroid".

4.2. Validation: The "master curve"

Application of Eq. (8) to Eq. (7) renders the rate change of volume with surface in two dimensions given by $\mathrm{d}A/\mathrm{d}L$. The result can be further normalized with respect to the semi-major axis a to yield the expression for the master curve. It shows the tendency of b/a rising from zero to one as shown in Fig. 6. It can thus be stated that objects, in general, have the tendency to evolve from a low to high slenderness ratio with a maximum of $b/a = 1$.

Pebbles of different shapes and sizes that belong to the conical family were collected and measured. Corollary I is validated as all data points of Fig. 7 lie on the master curve even though they were gathered at the different locations. Group I of pebbles corresponds to b/a of 0.8 to 1.0 which indicate the roundness of the pebbles. Data points between 0.65 and 0.75 were also found for Group II. These samples were more slender. Group III pertains to 0.7 and 0.8. Clearly, a tendency of directional preference has been established: Group III $\to$ II $\to$ I with increasing b/a. Note that no data points were found for b/a less than 0.4. This lower limit represents very narrow pebbles whose geometrical symmetry are difficult to sustain. The shape and size effects are shown to be the major factors while the surface textures are secondary.

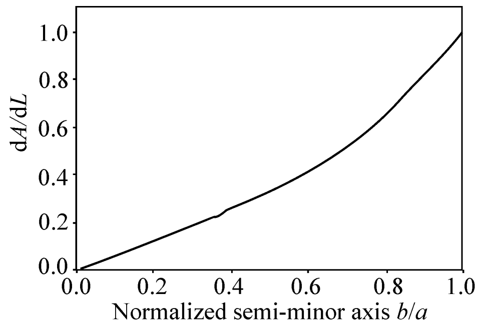

Fig. 6. Master curve

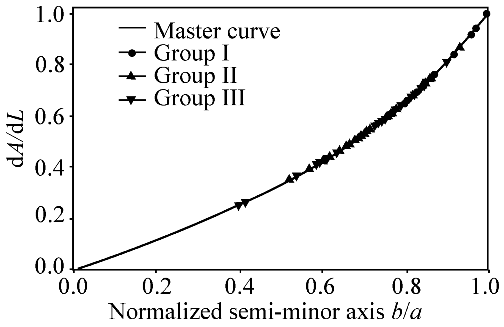

Fig. 7. Data for Group I, II, and III

5. The nature's way

Symmetries suggest explanations for the order of nature. The causes and effects, however, are not always apparent because similar patterns can occur at the different space and time scale. Mathematical models are required to understand the evolution of the visible patterns developed gradually over time by the naked eye or otherwise. When the explanations cover the behavior of a wide range of living things, laws can be stated to facilitate application. Incidentally, all things are living if their constituents alter with space and time. Strictly speaking, geometrical symmetry is constantly changing which makes all static predictions obsolete in time. The "incomplete theorem" of Gödel and the uncertain principle of Heisenberg are cases in point. Computer models is of no help to what the human mind cannot comprehend even when it can exceed the intelligence of the mind.

There is no use to do the impossible. That is to outdo nature. Mathematical models are for the logical mind. The following law of higher symmetry is proposed as it seems to apply to known space and time and can serve a useful purpose: *Higher symmetry law*: *Objects tends to higher geometrical symmetry regardless of size, shape and their internal constituents.*

There stands a good chance that the rate change of volume with surface dV/dA can be applied to explain other problems of symmetry in physics and mechanics. Non zero dV/dA admits the interaction of symmetry with energy dissipation in the "isoenergy density space" [1]. That is in this space there are no energy types. The only distinction is the available (before) and dissipated (after) energy. Objects, overloaded with energy, will change symmetry to dissipate the additional energy. A tree will branch. A crack will bifurcate. Thermal and deformation energy are one of the same physical process. They had to be separated by introducing thermodynamics because of the limit $dV/dA \to 0$. The title "thermomechanics" was used in [1] to emphasize this fact.

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