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Estimation of the sustainable yields of boreholes in fractured rock formations

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Abstract

The simplest way to derive an estimate for the sustainable yield of a borehole is to study the behaviour of drawdowns observed during a hydraulic (also known as a pumping test) of the borehole, through an appropriate conceptual model. The choice of this model is probably the most difficult choice that the analyst of such a hydraulic test has to make, since a wrong model can only lead to the wrong conclusions and failure of the borehole.

This paper discusses a semi-analytical and two numerical methods that can be used to simplify the analyses of hydraulic tests in fractured rock formations. The first method, called the Method of Derivative Fitting (MDF), uses a new approach to identify the conceptual model needed in such analyses. This is achieved by characterizing the various flow periods in fractured rock aquifers with numerical approximations of the first logarithmic derivative of the observed drawdown (the derivative of the drawdown with respect to the logarithm of the time). Semi-analytical expressions are used to estimate the influence that boundaries may have on the observed drawdown and the sustainable yield of a borehole — the rate at which a borehole can be pumped without lowering the water level below a prescribed limit. An effort has also been made to quantify errors in the estimates introduced by uncertainties in the parameters, such as the transmissivity and storativity, through a Gaussian error propagation analysis. These approximations and the MDF, called the Flow Characteristics Method (FCM) have been implemented in a user-friendly EXCEL notebook, and used to estimate the sustainable yield of a borehole on the Campus Test Site at the University of the Orange Free State.

The first numerical method, a two-dimensional radial flow model, is included here because it allows the user more freedom than the FCM, although it requires more information. One particular advantage of the method is that it allows one to obtain realistic estimates of the storativity and transmissivity of Karoo aquifers in particular, which is required in the estimation of the sustainable yield of a borehole.

There is no doubt that a three-dimensional numerical model, the second numerical method discussed here, is the best method with which to analyse a hydraulic test in a fractured aquifer. The method was consequently used to evaluate the accuracy of the implementation of the MDF in the Excel notebook and its application to the borehole on the Campus Test Site. The good agreement between the sustainable yield estimated with the three-dimensional numerical model and the FCM indicates that the FCM can be used with confidence to estimate the sustainable yields of boreholes in fractured media.

Keywords: Aquifers; Fractured; Sustainable; Yield; Borehole

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Nomenclature

Latin symbols

a, b, c	Parameters used to denote generic equation coefficients and distances
C	Parameter in the Cooper–Jacob approximation of the Theis solution [l]
d	Thickness of the aquifer [L]
$\mathrm{D}_z f$	Partial derivative of the function f with respect to the variable z
$f(\mathbf{x}, t)$	Strength of sources or sinks in the aquifer [T^{-1}]
$\mathbf{K}(\mathbf{x}, t)$	Hydraulic conductivity tensor [$\mathrm{L\,T}^{-1}$]
Q	The discharge rate [$\mathrm{L}^3\,\mathrm{T}^{-1}$]
Q_{obs}	Discharge rate of a constant rate test [$\mathrm{L}^3\,\mathrm{T}^{-1}$]
Q_{p}	Prescribed discharge rate [$\mathrm{L}^3\,\mathrm{T}^{-1}$]
Q_{sus}	The sustainable yield [$\mathrm{L}^3\,\mathrm{T}^{-1}$]
r	Radial co-ordinate [L]
r_{b}	Radius of a borehole [L]

1. Introduction

South Africa can no longer satisfy the demand for potable water from surface sources alone. This has led to the drilling of numerous boreholes in the past few years, principally to supply in the needs of rural communities. Since the communities are spread all over the country it is simply not economical to study each aquifer in detail to ensure that the boreholes will not fail. However, the country also cannot afford the costs associated with the failure of a large number of the boreholes, which is often the case. One reason for this is that more than 80% of the aquifers in South Africa occur in hard rock formations, while the methods used to determine sustainable yields for the boreholes have all been developed for porous aquifers.

The motion of groundwater has been studied extensively since World War II. However, the procedures conventionally used in the analysis of hydraulic test date mainly apply to homogeneous porous aquifers (Gringarten, 1982), and are therefore inadequate for the analysis of data from fractured-rock formations. This applies in particular to the analysis of data from hydraulic tests, commonly (but incorrectly) also known as pumping tests (Domenico and Schwartz, 1990). Although the literature abounds with 'recipes' with which hydraulic tests can be analysed, e.g. (Duffield and Rumbaugh, 1991; Kruseman and De Ridder, 1991), very little attention is usually paid to the basic physical principles involved in applying the 'recipes'. Since these tests play such an important role in groundwater investigations, it may be worthwhile to briefly first review the basic physical principles on which the tests are based before continuing with this discussion.

Groundwater is a physical phenomenon and therefore subject to the laws of physics, particularly the laws of mechanics and hydrodynamics that governs the flow of fluids. If the geometry of the geological formations (in which an aquifer occurs) is known, the application of these laws allows one to describe the flow of groundwater in the aquifer with a mathematical equation. In groundwater flow this so-called governing equation is usually a partial differential equation. The flow of groundwater can therefore, in principle, be evaluated by studying the mathematical solution of the governing equation. However, there are at least two constraints that must be satisfied to do this:

(a) One must get rid of the indeterminate constants (or functions in two or three-dimensions) introduced by the integration of the differential equation.

(b) Differential equations that describes the motion of one substance through another usually contain one or more what Botha (1994) calls 'relational parameters', that must be known explicitly before the differential equation can be solved.

In the case of groundwater flow the relational parameters depend implicitly on how the geometry of the formations affects their interaction with the fluid (Black, 1993; Botha et al., 1998). They will consequently be referred to as aquifer or hydraulic parameters in the discussion that follows.

The previous constraints are usually satisfied in the classical physical and engineering sciences by prescribing suitable initial and boundary conditions for the domain of interest, while the relational parameters are determined from laboratory measurements. This approach is, unfortunately, not applicable in geohydrology, because it is difficult to determine the exact extent of an aquifer, while laboratory measurements of the aquifer parameters are often inconsistent with field data.

Another difficulty with the previous approach is that there is not a 'universal' equation that can be used to describe the flow of groundwater. Most groundwater investigations are consequently based on the classical differential equation for flow of a constant density fluid in a saturated porous medium, which assumes in the case of groundwater flow the form (Bear, 1979; Botha, 1996).

$$S_0(\mathbf{x},t)\mathrm{D}_t\varphi(\mathbf{x},t) = \boldsymbol{\nabla}\cdot[\mathbf{K}(\mathbf{x},t)\boldsymbol{\nabla}\varphi(\mathbf{x},t)] + f(\mathbf{x},t) \qquad (1)$$

where $\varphi(\mathbf{x}, t)$ is the piezometric head and $f(\mathbf{x}, t)$ the strength of sources or sinks in the aquifer at the position $\mathbf{x} = \{x, y, z\}$ and the time t. The two aquifer parameters, $S_0(\mathbf{x}, t)$ and $\mathbf{K}(\mathbf{x}, t)$ are, respectively, known as the specific storativity and hydraulic conductivity of the aquifer. Since the rocks on earth are mainly anisotropic, the hydraulic conductivity is a tensor (as indicated by the bold font used to denote it). Eq. (1) is consequently a complex three-dimensional partial differential equation. One or more of the following simplifying assumptions are therefore often made when applying Eq. (1) to estimate aquifer parameters in practice:

(a) Both the aquifer parameters are scalars that do not depend on space and time.
(b) Flow in the aquifer is essentially horizontal, in other words satisfies the hydraulic approach (Bear, 1972).
(c) Flow in the aquifer is three-dimensional, but symmetric in the radial direction, see for example (Botha et al., 1998; Neuman, 1975).

The advantage of these assumptions is that they reduce the complexity of Eq. (1) considerably. For example, the first two assumptions allow one to reduce the dimensions of Eq. (1) by integrating it over the vertical thickness (d) of the aquifer, to obtain (Bear, 1979)

$$S\mathrm{D}_t s(x,y,t) = T\boldsymbol{\nabla}^2 s(x,y,t) + F(x,y,t) \qquad (2)$$

where $s(x, y, t)$ is the drawdown of the water level in a borehole with reference to its value at a time t_0, usually taken as zero. The parameters $S(= S_0 d)$ and $T(= Kd)$, d the thickness of the aquifer, are conventionally known as the storativity or storage coefficient and transmissivity of the aquifer, respectively, while

$$F(x,y,t) = \int_0^d f(\mathbf{x},t)\,\mathrm{d}z$$

Although Eq. (2) is mathematically considerably simpler than Eq. (1), one still needs to prescribe suitable boundary and initial conditions and know the aquifer parameters and the strength of sources or sinks explicitly before it can be solved. As mentioned above this is not always achievable in practice. One approach to circumvent this problem is to study the behaviour of an aquifer inversely. In this approach the water level in one borehole is disturbed and the subsequent response of the water levels in this and or adjacent boreholes measured. The measured water levels are then fitted numerically (or graphically) to what is known as a *conceptual model* of the aquifer. Such a conceptual model essentially consists of an analytical solution of Eqs. (1) or (2), also known as a *type curve*, or a numerical solution based on an assumed set of hydraulic parameters, boundary and initial conditions. This procedure is known as a *hydraulic test* in geohydrology, and the *inverse problem* in mathematics.

The inverse problem for Eqs. (1) or (2) suffers, unfortunately, from at least two inherent difficulties. The first is that the method is not well posed in the sense of Hadamard (1932). In the normal formulation of complexity theory, this means that the solution of the inverse problem cannot be computed with a specified error. The possibility therefore exists that the observed response of an aquifer can be fitted with two or more sets of aquifer parameters, boundary and initial conditions that differ completely from one another.

The second difficulty is: how does one choose the conceptual model? The correct approach to answer this question, at least from a physical point of view, would be to study the physical dimensions and geometry of the aquifer in detail. However, this is a very extensive and costly exercise. A method, commonly used to circumvent this difficulty, is to compare the observed water levels with the analytical solution of a known conceptual model, also known as a *type curve*, see, e.g. Kruseman and De Ridder (1991). Such an approach may be acceptable, when the aquifer is single-layered, but can lead to completely wrong results in the case of multi-layered aquifers, especially when the water levels are measured in boreholes that penetrate all the aquifers. Botha and Verwey (1992) obtained a near perfect fit between the water levels in a two-layered confined aquifer system and the type curve of Neuman (1975) for a phreatic aquifer.

2. Water levels and the geometry of an aquifer

A physical phenomenon, such as the flow of groundwater, is always related to the three basic constituents of the Universe — space, time and matter. The governing equations of such phenomenon are therefore nothing more than an abstract description of how these three constituents interact with one another to produce a specific phenomenon. The possibility consequently exists that one may not be aware of some vital interactions in a physical phenomenon, by concentrating on the governing equation alone and neglect them in applications of the equation. For example, Eqs. (1) and (2) are nothing more than a combination of the general law of mass conservation from physics and Darcy's law from fluid mechanics. The law of mass conservation is a universal law. However, Darcy derived his law from observations on water percolating through a column of sand with its *characteristic geometry* of voids and grains — commonly known as a *porous medium*. The law may therefore not be applicable to media with different geometric properties. Since the geometry of a porous medium is so simple, it is often neglected in the analysis of hydraulic tests (Black, 1993).

The geometries of aquifers that occur in hard-rock formations can be conveniently divided into two classes. The first class, called *fractured aquifers*, are commonly associated with igneous and metamorphic rocks. These very dense rocks can only store significant quantities of water if permeated by large numbers of arbitrarily orientated intersecting fractures, unless they are weathered. Since the behaviour of these aquifers can only be studied with special methods, such as percolation theory (Berkowitz and Balberg, 1993; Stauffer and Aharony, 1992), they will not be discussed further here.

The second class of aquifers occur mainly in layered sedimentary rocks, such as the Karoo formations in South Africa. These formations, which still display the geometry of a porous medium, but with large variations in porosity, are sometimes intersected by bedding-plane fractures. Botha et al. (1998) consequently refer to them as *multi-porous fractured aquifers*. The bedding-plane fractures often have significant apertures ($\sim$1–10 mm) and underlay large areas. The flow field in these aquifers and the drawdown in boreholes, therefore, differ considerable from that observed in a porous aquifer. An example of this difference is illustrated in Fig. 1. This figure compares the drawdown observed in a typical Karoo borehole near the town of Philippolis south-west of Bloemfontein, with the drawdowns computed from the type curve of Gringarten and Ramey (1974) for a horizontal fracture and the Theis curve for a conventional porous aquifer.

3. Model identification

3.1. General

It follows from the preceding discussion that it is very important to know the geometry of an aquifer before selecting a type curve (or the numerical model) with which to analyse the data from a hydraulic test. However, this can be a daunting task and not very practical in those cases where one is only interested in supplying small communities with water from a few boreholes. This led the authors to introduce the Method of Derivative Fitting (MDF) for the analysis of constant rate tests, which is the conventional type of hydraulic test used in developing water supply schemes for rural communities in South Africa. This method will now be described in more detail.

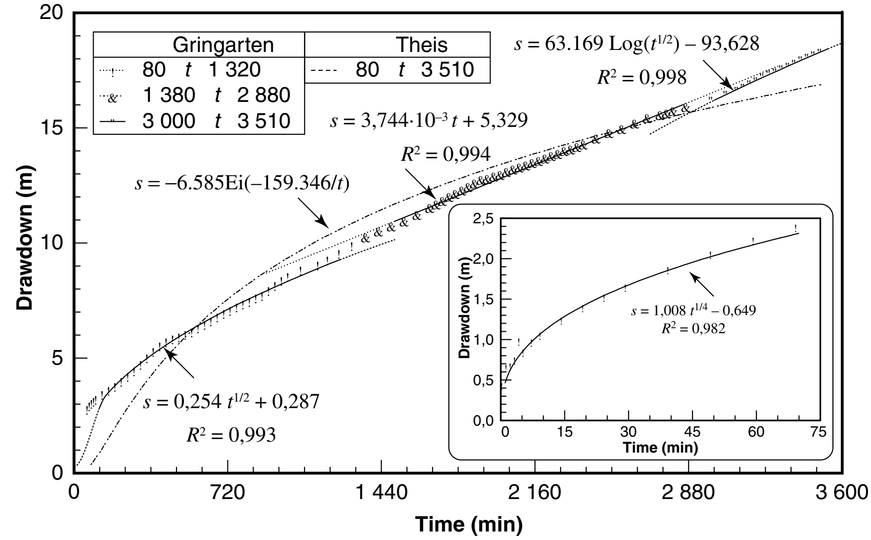

Fig. 1. The drawdown observed in borehole A05 at Philippolis by (Botha et al., 1996) fitted to the type curve of Gringarten and Ramey for an aquifer intersected by a horizontal fracture and the Theis curve for a horizontal porous aquifer.

3.2. The method of derivative fitting

The time derivative of the pressure (p) has been used for years in the oil industry to analyse the data from well tests (Bourdet et al., 1984; Horne, 1997). The advantage of this approach is that small variations in the pressure can be more easily recognized in a graph of the time derivative of the pressure, than in a graph of the pressure. This approach can of course also be applied to the piezometric head, $\phi(t)$, which is related to the pressure by the equation

$$\phi(t) = \int_{p_0}^{p} \frac{dp}{\rho g} + z$$

where t is the time and z a suitable reference elevation, or to the drawdown defined as

$$s(t) = \phi(t) - \phi(t_0)$$

with t_0 as a suitable time. This is the Method of Derivative Fitting (MDF) referred to above.

One way in which to apply the MDF is to simply fit the derivative of a conventional type curve directly to the first divided differences of the observed drawdown, s, defined by the equation

$$\frac{\Delta s_i}{\Delta t_i} = \frac{s(t_{i+1}) - s(t_i)}{t_{i+1} - t_i}$$

However, this means that one has to select a suitable type curve a priori, which may not be obvious, as pointed out above. To circumvent this difficulty the flow characteristics of a number of aquifers in South Africa were analysed. This analysis indicated that there are essentially three types of flow that occur in the hard-rock formations of South Africa: radial flow, flow in vertical and horizontal fractures and flow in a dual porosity medium.

Radial flow can be characterized by the Theis equation

$$s(r,t) = \frac{Q}{4\pi T}\int_u^\infty \frac{\exp(-x)}{x}\,dx = \frac{Q}{4\pi T}W(u) \quad \left(u = \frac{Sr^2}{4Tt}\right) \tag{3}$$

where Q is the discharge rate, T the transmissivity and S the storativity of the aquifer. The first derivative of this equation with respect to the logarithm of time, henceforth referred to as the log derivative,

$$s'(r,t) = \frac{\partial s}{\partial \ln(t)} = \frac{Q}{4\pi T}\exp\left[-\frac{Sr^2}{4Tt}\right]$$

will therefore approach a straight line parallel to the

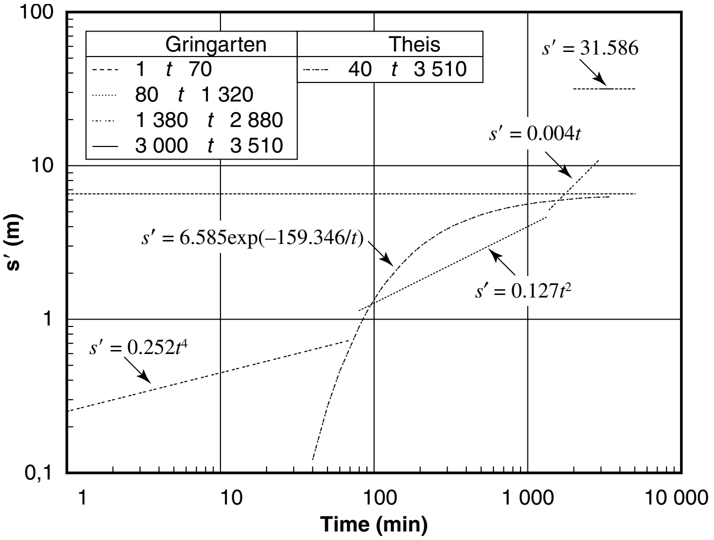

Fig. 2. The log derivatives of the Gringarten and Theis fits to the water levels of borehole A05 in Fig. 1.

time axis on a log–log graph of $s'(r, t)$ as $t \to \infty$, as illustrated in Fig. 2.

Flow in fractures and dual porosity media can be characterized similarly. For example, the drawdown in an aquifer intersected by a horizontal fracture will show three but sometimes four characteristic periods (Gringarten and Ramey, 1974) in the absence of storage flow, as illustrated in Fig. 1. At a very early time the drawdown often resembles storage flow in which case it can be described by a linear function of the time. This period is followed first by one in which the drawdown can be described by a function of $t^{1/4}$ (only if the horizontal fracture has significant storage), and then one in which the drawdown can be described as a function of $\sqrt{t}$. During the last period the dominant flow, conventionally referred to as linear flow, is from the aquifer matrix to the fracture. This period is followed by one in which the drawdown can again be described by a linear function of t. However the period will be present only if the areal extent (radius) of the fracture is larger then the thickness of the aquifer. This period is followed by one in which drawdown behaves as a function of $\ln(\sqrt{t})$. The type of flow during this period will be called pseudo-radial flow in the discussion that follows. An infinite conductive, vertical fracture behaves in exactly the same way, except that it never displays the third period where the drawdown behaves as a function of t. The log derivatives of the four periods in Gringarten and Ramey's model for a horizontal fracture, $s'(r, t)$ will hence all display straight lines on a log–log graph of $s'(r, t)$ as a function of time, with slopes 1/4, 1/2, 1 and 0, respectively, as illustrated in Fig. 2.

The previously described flow periods are slightly modified in the case of a finite conductive fracture in that a period in which water flows from both the fracture and the matrix to the borehole appears between the storage flow and linear flow periods. During this bi-linear flow period the drawdown should behave as a function of $t^{1/4}$ (Cinco et al., 1978), as illustrated in the inset of Fig. 1.

One assumption in the model of Gringarten and Ramey (1974) for a horizontal fracture is that the flow from the matrix to the fracture is constant. This assumption will only be satisfied in situations where the piezometric level in the rock matrix does not change much with time. The flow from the matrix to the fracture, however, will in practice be determined by the piezometric level in the aquifer, which tend to increase steadily from the borehole towards the boundaries of the aquifer (Botha et al., 1998). There is thus a possibility that an aquifer intersected by a horizontal fracture may be able to supply water to a borehole at a steady rate for a prolonged period of time. The water level in such a

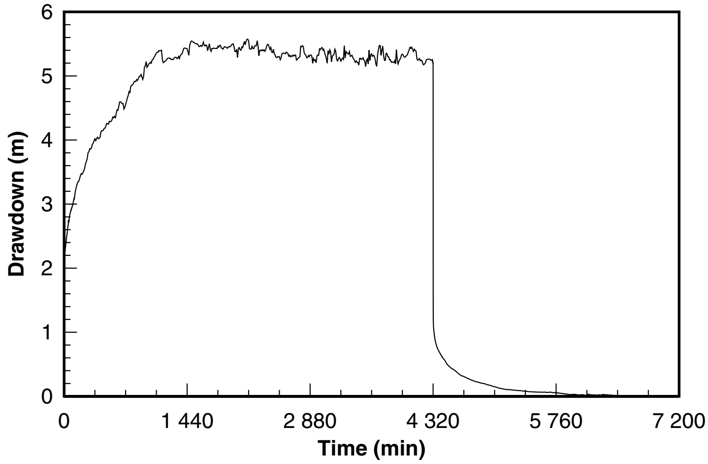

Fig. 3. The drawdown observed in borehole D20 at Philippolis during a constant rate test with a discharge rate of 6.84 $m^3 h^{-1}$ (Botha et al., 1996).

borehole will display a constant value if the borehole is pumped at a rate less than or equal to the rate at which the aquifer can supply water to the fracture, as illustrated in Fig. 3.

A borehole in a dual porosity aquifer will withdraw its water first from the highly permeable fractures and then the less permeable matrix. The water level in such a borehole therefore at first tend to decline normally, but then tend to stabilize, before it continues to decline again. Both the first and last legs of the drawdown curve can be described by a conventional Theis type curve-the first with hydraulic parameters representative of the fractured medium and the second with hydraulic parameters representative of the rock matrix (Moench, 1984). The first derivative of the drawdown in dual porosity flow should therefore display a clear minimum.

The advantage of the MDF method-its ability to delineate small changes in the observed water levels-unfortunately, also means that it is very sensitive to noise in the observed water levels. The method should therefore only be applied to observed drawdowns free of noise in theory.

There are a number of approaches that can be used to reduce the influence of noise in the observed data. One is to compute the derivative from a least squares approximation of the first divided difference, which assumes in the case of the log derivative the form

$$\frac{\partial s}{\partial \log(t)} = \frac{n\sum_{i=1}^{n} \log(t_i)s(t_i) - \sum_{i=1}^{n} \log(t_i)\sum_{i=1}^{n} s(t_i)}{n\sum_{i=1}^{n} [\log(t_i)]^2 - \left[\sum_{i=1}^{n} \log(t_i)\right]^2}$$

where n is a suitable number of data points (usually 3, 5 or 7) that have to be determined experimentally to achieve the best smoothing. Another approach, suggested by Horne (1997), is to use data points that are separated by at least 0.2 of a log cycle, rather than adjacent data points.

3.3. Boundary conditions

As mentioned above a differential equation can only be solved in general if one prescribes suitable boundary conditions — conditions that exist at the point, line or surface that bounds the domain of the differential equation, but does not form part of the domain. However, there is a tendency in the groundwater literature to follow the petroleum literature and regard discontinuities in the coefficients of the differential equation also as ‘internal boundaries’. One reason for this is probably because the discontinuities that arise in grondwater flow are always related to an abrupt change in the lithology of the rocks that host an aquifer, or the presence of a fracture or a fault.

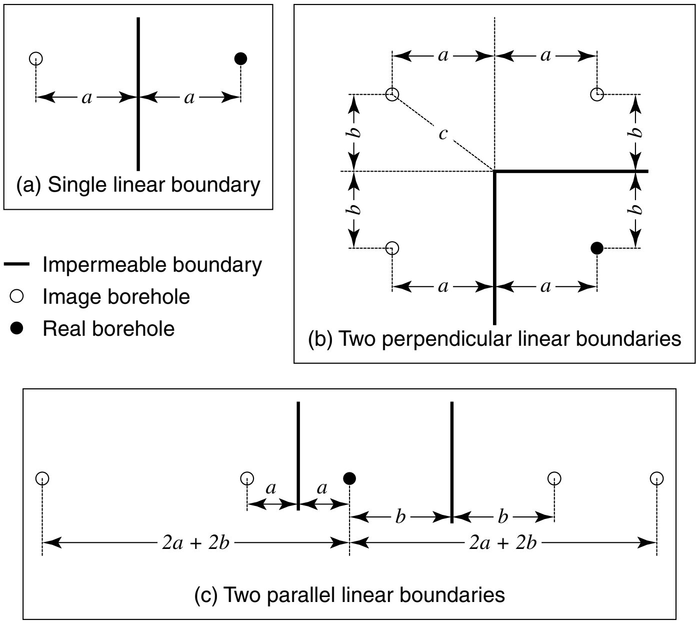

Fig. 4. Schematic examples of a few linear impermeable boundaries and the positions of image boreholes required to simulate their responses on the drawdown in the real borehole.

It must be kept in mind though that these discontinuities always occur within the domain of the differential equation and therefore do not form boundaries. The theory of differential equations consequently only allows one to prescribe continuity conditions and not boundary conditions, for discontinuities. It may therefore be more appropriate to describe them with the term *geometric boundaries*, if one wants to introduce another term.

There are essentially two types of boundaries that may affect the behaviour of the drawdown in an aquifer during a constant rate test — Dirichlet or prescribed head boundaries and Neumann or prescribed flux boundaries. However, the exact shape of aquifer boundaries can vary enormously in practice. The following discussion will therefore be restricted to the few that have been observed in South African aquifers.

(a) A constant or variable head boundary. This type of boundary, also referred to as a recharge boundary, occurs where an aquifer is adjoint to or intersected by a body of surface water, such as a river or dam. The water level of a borehole in such an aquifer will in general approach a limit during a constant rate test, while its first and higher derivatives will tend to zero.

(b) A closed impermeable boundary. This type of boundary occurs where the aquifer is completely bounded by impermeable formations. The characteristic dolerite ring dykes of the Karoo formations are prime examples of an impermeable boundary. The water level of a borehole in such an aquifer will ultimately behave as a linear function of the time with slope 1 during a constant rate test, as shown by the analytical solution of Muskat (1937), or a water balance of the aquifer (Horne, 1997).

(c) Linear and intersecting linear impermeable boundaries. The linear dolerite dykes that occur so frequently in the Karoo Sequence are prime examples of this type of boundary. Since the intersecting dykes can form complex patterns it is not possible to describe their influence on the drawdown in general terms. However, there are a few common situations where the method of images (Bear, 1979) can be used to derive expressions for the drawdown in a borehole.

To illustrate the method of images consider first the case of a borehole situated at a distance a from a single impermeable boundary, as shown in Fig. 4(a). Let q_a be the flux of water at the position of the boundary towards the borehole in the absence of the boundary. It is not

Table 1
Analytical expressions for the drawdown in a borehole surrounded by the various impermeable boundaries. The subscripts of u give the distances needed to compute the values of u

Single linear boundary	$s(t) = \frac{Q}{4\pi T}[W(u_{\mathrm{r}}) + W(u_{2a})]$
Two perpendicular boundaries	$s(t) = \frac{Q}{4\pi T}[W(u_{\mathrm{r}}) + W(u_{2a}) + W(u_{2b}) + W(u_{2c})]$
Two parallel boundaries	$s(t) \simeq \frac{Q}{4\pi T}[W(u_{\mathrm{r}}) + W(u_{2a}) + W(u_{2b}) + 2W(u_{2a+2b}) + W(u_{2a+4b}) + W(u_{4a+2b}) + 2W(u_{4a+4b})]$
Impermeable square boundary with sides 2a and borehole at its centre (Streltsova, 1988)	$s(t) \simeq \frac{Q}{4\pi T}[W(u_{2a}) + W(2u_{2a})]$ $\left(\frac{Tt}{Sa^2} < \frac{1}{\pi}\right)$; $s(t) \simeq \frac{Q}{4\pi T}\left[-2.608 - \frac{6}{\pi}\exp\left(-\frac{\pi^2 Tt}{Sa^2}\right) + \frac{\pi Tt}{Sa^2} + 2\ln\left(\frac{2a}{r}\right)\right]$ $\left(\frac{1}{\pi} \le \frac{Tt}{Sa^2} < 1\right)$; $s(t) \simeq \frac{Q}{4\pi T}\left[-2.608 + \frac{\pi Tt}{Sa^2} + 2\ln\left(\frac{2a}{r}\right)\right]$ $\left(\frac{Tt}{Sa^2} \ge 1\right)$
Impermeable circular boundary with radius R and borehole at its centre (Muskat, 1937)	$s(r,t) \simeq \frac{Q}{2\pi T}\left[\frac{2r^2-3}{4} + \ln\frac{1}{r} + \frac{2T}{SR^2}t \quad \left(\frac{2T}{SR^2}t \gtrsim 2.5\right)\right]$

difficult to see that an imaginary image of the borehole — a borehole with the same discharge rate but situated at a distance, a, on the opposite side of the boundary will cause a similar, but oppositely directed, flux of water towards itself. The sum of the two fluxes will therefore be zero at the boundary, thereby simulating the impermeable boundary. The true drawdown in the real borehole can therefore be expressed as the sum of the unrestricted drawdowns in the two boreholes, that is

$$s(t) = s_{\mathrm{r}}(t) + s_{\mathrm{i}}(t)$$

where the subscripts r and i refer to the real and image borehole, respectively. One can therefore expect that the drawdown and the slope of its log derivative will tend to double near the boundary.

A similar line of reasoning can also be used to derive expressions for more complex impermeable (or prescribed head) boundaries. The only difficulty is that the response of some of these boundaries, such as the two parallel impermeable boundaries in Fig. 4(c), can in theory only be simulated with an infinite number of image boreholes. However, the contribution of the individual boreholes will decrease with their distance, so that one need only to include those closest to the boundary in computing the response of the boundary. A few analytical expressions for the true drawdowns in aquifers with impermeable boundaries are listed in Table 1. These expressions are all based on the assumption that the unrestricted drawdown in the boreholes can be described by the Theis solution in Eq. (3) and therefore only apply to radial symmetric flow. However, this does not mean that the method of images applies only to radial symmetric flow. Other types of flows can also be handled by merely replacing the expression for the unrestricted drawdown with one that describes the specific type flow.

The behaviour of the first logarithmic derivatives of the drawdown for some of the boundary conditions and drawdown curves described above are illustrated graphically in Fig. 5.

4. The analysis of constant rate tests

4.1. General

As will be shown below, the MDF can be used directly to determine the sustainable yield of a

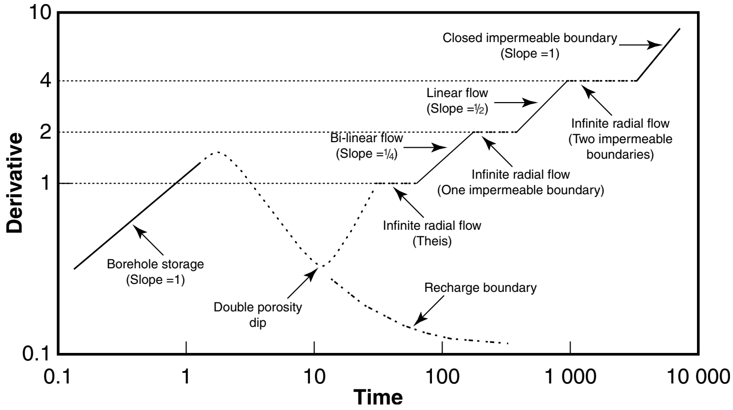

Fig. 5. Graphs of the first logarithmic derivative of the drawdown in a borehole for a few types of geometries and boundaries.

borehole. However, there are situations where more information is needed on the hydraulic parameters of an aquifer, than what can be obtained from the MDF. There is little doubt that a three-dimensional numerical model will be the best choice for the analysis of the multi-porous fractured aquifers in the Karoo Sequence. Unfortunately, it is not always possible to justify the use of such a model in practice, because of the large amounts of data required by the models. Two methods that proved to be very useful in such situations will therefore be briefly described in this section.

The first method, referred to as the basic model below, is based on the Cooper–Jacob approximation of the Theis solution. The second method — a numerical method — was developed specifically with the geometry of Karoo aquifers in mind (Botha et al., 1996a,b, 1998). The last method has been implemented in a user-friendly computer package RPTsolv, copies of which are available on the website of the Institute for Groundwater Studies.

4.2. The basic model

It is not difficult to show that the Theis solution for an infinite homogeneous aquifer in Eq. (3) is a particular solution of Eq. (2). Type curves for an aquifer with an infinite domain must consequently reduce to this equation at large times. Meier et al. (1998) have shown that in many of these cases the Cooper–Jacob approximation of Eq. (3) can be used to compute an effective T-value for the aquifer from the drawdown data at a sufficiently late time, t_0 say, if expressed in the form

$$s = 2.3\frac{Q}{4\pi T}\log C \qquad (4)$$

where C is a parameter that depends on the aquifer type. Values of C for a few of these aquifers are listed in Table 2. However, this T is not representative of the fracture or the rock matrix, but the system of fracture and rock matrix, except for the homogeneous porous

Table 2
Values of the parameter C in Eq. (4) for a few typical types of aquifers, with X_f the half width of the vertical fracture, W_d the width of the dyke or fault and β a constant = 1/3 for an orthogonal fracture system and 1 for a linear system (Kruseman and De Ridder, 1991)

C[a]	c	Aquifer
$(2.25Tt_0)/(r^2S)$	$2.25/r^2$	Homogeneous porous (Theis-model)
$(2.25T_ft_0)/(r^2S_f)$	$2.25/r^2$	Dual porosity (early time)
$(2.25T_ft_0)/[r^2(S_f+\beta S_m)]$	$2.25/r^2$	Dual porosity (late time)
$(16.59Tt_0)/[S(X_f)^2]$	$16.59/[(X_f)^2]$	Single vertical fracture
$(40T^3t_0)/[S(W_dT_d)^2]$	$(40T^2)/[(W_dT_d)^2]$	Conductive dyke or fault zone

[a] The subscripts f, m and d refer to the fracture, rock matrix and dyke, respectively.

aquifer of course. This value of T can be easily calculated from the log derivative of the drawdown in Eq. (4), given by

$$\frac{\partial s}{\partial \log t_0} = 2.3\frac{Q}{4\pi T} \tag{5}$$

An estimate of the storativity can likewise be obtained form the log derivative of log(s) in Eq. (4)

$$\frac{\partial \log(s)}{\partial[\log(t)]} = \frac{1}{\ln(C)} \equiv \frac{1}{\ln(\alpha c t_0)} = \frac{1}{\beta \ln(10)}$$

so that

$$\alpha = \frac{T}{S} = \frac{10^{\beta}}{c t_0} \tag{6}$$

4.3. RPTsolv

The first attempt to use a numerical model for the analysis of hydraulic test data is probably that of Rushton and Booth (1976). This work was later expanded and improved by Rathod and Rushton (1984, 1991). However, their numerical models are cumbersome and do not address vertical flow, which, as the preceding discussion shows, seems to be a characteristic property of many aquifers in South Africa.

There are two approaches that one can use to study vertical flow. The first is to use a full three-dimensional model and the second to use a vertical two-dimensional model. Since three-dimensional models require vast quantities of data and sophisticated computer equipment they are often not very suitable to use in practice. This leaves one only with the two-dimensional vertical model. However, it must be remembered that the flow of groundwater is essentially a three-dimensional phenomenon. This means that one must get rid of one of the horizontal directions to apply a two-dimensional vertical flow model.

There are essentially two methods that can be used to reduce the dimensions of a three-dimensional model. The first method, which Botha (1996) calls the *physical approach*, is to simply discard the direction one is not interested in. However, this approach should only be applied if the phenomenon is naturally, or artificially, restricted to two dimensions. The second alternative is what Botha calls the *mathematical reduction of dimensions*. What is done in this case, is to integrate the mathematical model for the phenomenon over the dimension to be neglected (Bear, 1979, 1977). This approach allows one to express the vertical flow model in the form (Botha et al., 1998; Verwey et al., 1995)

$$rS_0(r,z,t)\mathrm{D}_t\varphi(r,z,t) = r\boldsymbol{\nabla}\cdot[\mathbf{K}(r,z,t)\boldsymbol{\nabla}\varphi(r,z,t)] + \frac{Q(t)}{2\pi d}\delta(r-r_0) \tag{7}$$

where $\varphi(r, z, t)$ is the piezometric pressure head, $\mathbf{K}(r, z, t)$ the hydraulic conductivity tensor and $S_0(r, z, t)$ the specific storativity at the point (r, z) and time t in the aquifer, with d the constant thickness of the aquifer, $Q(t)$ the discharge rate of a borehole situated at $r = r_0$ and $\delta(\rho)$ the Dirac delta-function.

Eq. (7) differs considerably at first sight from the equation conventionally used to describe the flow of groundwater in a horizontal plane

$$S(x,y,t)\mathrm{D}_t h(x,y,t) = \boldsymbol{\nabla}\cdot[\mathbf{T}(x,y,t)\boldsymbol{\nabla}h(x,y,t)] + Q(t)\delta(x-x_0)\delta(y-y_0) \tag{8}$$

where $h(x, y, t)$ is the observed water level, $\mathbf{T}(x, y, t)$ the transmissivity tensor and $S(x, y, t)$ the storativity of the aquifer. However, a closer examination shows that the only differences are that Eq. (7) is based on the piezometric head, hydraulic conductivity and specific storativity and not the water level, transmissivity and storativity as in Eq. (8). The two equations are consequently equivalent from the mathematical point of view. The computer program Gcon, developed by (Botha et al., 1990) for horizontal flow was consequently adapted to solve Eq. (7).

5. Estimation of the sustainable yield of a borehole

5.1. General

As mentioned above the major objective of constant rate tests is often to determine the *sustainable yield* of a borehole. This quantity is here defined as the discharge rate that will not cause the water level in the borehole to drop below a prescribed limit (e.g. the position of a major water strike).

It is not difficult to derive an expression for the sustainable yield in those cases where the drawdown in the borehole is radial symmetric, in other words satisfies Eq. (2), with a forcing function that assumes in radial co-ordinates the form

$$F(r,t) = Q\delta(r)$$

since one can then express the ratio of the drawdown s at a certain time (t_0 say) and the corresponding pumping rate, Q, in the form

$$\frac{s(t_0)}{Q} = cW(T,S) \tag{9}$$

where c is a characteristic parameter of the type curve that represents $s(t_0)$.

Let t_l now be the operation time in which the drawdown of a borehole shall not exceed the prescribed limit, s_p say, when pumped at a discharge rate of Q_p, the sustainable yield. Also let $s_{obs}(t_l)$ be the drawdown observed in the borehole during a constant rate test with a discharge rate, Q_{obs} say, at the time t_l. It then follows from Eq. (9) that Q_p is related to Q_{obs} through the equation

$$Q_p = Q_{obs}\frac{s_p}{s_{obs}(t_l)} \tag{10}$$

provided that $s_{obs}(t_l)$ is known.

A major difficulty experienced with the practical application of Eq. (10), is that constant rate tests are usually restricted to a few days, while one would prefer to compute the sustainable yield over a much longer period, say two to five years. This would not be a problem if one had an explicit expression for the drawdown. However, this is rarely the case in practice, with the result that one usually has to extrapolate $s_{obs}(t_l)$ to the required time. Extrapolation is generally not a very stable numerical procedure. However, as will be shown below the method is able to yield reliable estimates of the sustainable yield.

5.2. Extrapolation of the observed drawdown

The simplest approach to calculate the required value of $s_{obs}(t_l)$ is to compute values for S and T from Eqs. (5) and (6) above and then use Eq. (4) to predict the value of $s_{obs}(t_l)$ at the appropriate time. This approach will be ideal for the aquifers listed in Table 2, provided one is certain that the drawdown will not be influenced by boundaries or the geometry of the aquifer, in other words that the flow at these late times will remain radial and infinite. However, the discussion in Section 3 shows that this will probably be rarely the case in South Africa.

A more general extrapolation procedure is to compute $s_{obs}(t_l)$ from the Taylor series expansion of the observed drawdown curve and its first few derivatives

$$s_{obs}(\tau_l) = s_{obs}(\tau_0) + (\tau_l - \tau_0)D_\tau s_{obs}(\tau_0) + \frac{(\tau_l-\tau_0)^2}{2}D_{\tau\tau}s_{obs}(\tau_0) + \cdots \tag{11}$$

where τ is used to denote either t or the logarithm of t, with $D_\tau s(\tau)$ and $D_{\tau\tau}s(\tau)$ the first and second derivatives of $s_{obs}(t_0)$ at a time t_0 near the end of the constant rate test. One source of critique against this approach is that the Taylor series is an infinite series and that the derivatives are not bounded. However, as shown by the list of the first two derivatives of drawdowns in Table 3, it is only in the case of an aquifer that intersects a closed or one or more impermeable boundaries that the derivatives may be unbounded and the method cannot be applied directly. The derivatives in all the other cases are bounded and, more importantly, alternate in sign. This means that in these cases one can

Table 3
The first two ordinary and logarithmic derivatives, denoted, respectively, by ($D_t s$, $D_{tt}s$) and ($D_\tau s$, $D_{\tau\tau} > s$), for the most common types of the drawdowns observed during constant rate tests in South Africa. [note that ln(z) = log(10) log(z)]

	Type of flow			
	Radial	Closed boundary	Horizontal fracture	Other
$s(t)$	$a + b\ln(at)$	$a + bt$	$a + b\ln(\sqrt{\alpha t})$	$a + bt^n$
$D_t s$	b/t	b	$b/(2t)$	$bnt^{(n-1)}$
$D_{tt}s$	$(-b/t^2)$	0	$-b/(2t^2)$	$bn^{(n-1)}t^{(n-2)}$
$D_\tau s$	b	bt	$b/2$	bnt^n
$D_{\tau\tau}s$	0	bt	0	bn^2t^n

actually compute the error in the approximation from the first derivative neglected in Eq. (11).

There are especially two difficulties associated with the application of the Taylor series in estimating the sustainable yield of a borehole. The first is that it is difficult to calculate reliable derivatives from noisy drawdown data and the second that the test may not have been performed long enough to display the influence of boundaries on the drawdown. There is not much that one can do to correct for noise in the drawdown data, except to fit a curve to the drawdown data as in Fig. 1, or use suitable averages. However, there are a few methods that can be used to estimate the effect of boundaries, one of which will now be described in more detail.

5.3. Boundary effects

Although the effect of boundaries on the long-time drawdown of a borehole has been known for many years, boundary effects are not usually included in the estimation of the sustainable yield of boreholes in Southern Africa, except for the recent release of DWA (1997). The most common practice is to assume that the aquifer has an infinite areal extent and that a constant rate test can be analysed with the Theis solution in Eq. (3), even in situations where the existence of impermeable boundaries have been confirmed by field investigations. It was therefore thought worthwhile to develop a new method for the estimation of the sustainable yield that takes the effect of impermeable boundaries into account.

The extrapolation procedure described by Eq. (11) will take care of those situations where the observed drawdown displays the effect of an impermeable boundary. The following discussion is therefore mainly aimed at situations where it is known that an impermeable boundary exist, but the boundary is situated too far from the pumped borehole to influence the drawdown observed during a constant rate test.

The most accurate estimation of the boundary's influence on the drawdown will be obtained from a numerical model of the aquifer. However, the information available from an ordinary constant rate test will rarely be sufficient to develop such a model. A very simple approach to use in these situations is to express the observed drawdown in the form

$$s_{\mathrm{obs}}(t_{\mathrm{l}}) = s_{\mathrm{a}}(t_{\mathrm{l}}) + s_{\mathrm{b}}(t_{\mathrm{l}}) \tag{12}$$

where $s_{\mathrm{a}}(t_{\mathrm{l}})$ refers to the contribution of the aquifer and $s_{\mathrm{b}}(t_{\mathrm{l}})$ to the contribution of the boundary. The value of $s_{\mathrm{a}}(t_{\mathrm{l}})$ is once again computed from Eq. (11) and $s_{\mathrm{b}}(t_{\mathrm{l}})$ from the drawdowns given in Table 1.

5.4. The propagation of parameter uncertainties

One difficulty with the application of the solutions in Table 1 is that one need to know the type of the impermeable boundary, the distance(s) from the boundary to the borehole and the hydraulic parameters of the aquifer. Since these quantities are usually not known (with the possible exception of the distances to the boundaries) they have to be estimated in one way or another. It may therefore be worthwhile to study the effect that uncertainties in the parameters may have on the computed solution.

Kunstmann and Kinzelbach (1998) discuss a number of computational methods to study the propagation of parameter uncertainties in groundwater modelling. One of the methods, Gaussian Error Propagation, is particularly suitable for the analysis of analytical equations. The application of this method will now be illustrated by applying it to the solutions in Table 1.

It follows from the preceding discussion that the drawdown in the pumped borehole can be described by an equation of the form

$$s = s(t, Q, T, S, a, b) \tag{13}$$

provided that the behaviour of the aquifer satisfies Eq. (2). Since the time, t, and discharge rate, Q, can usually be measured accurately, the rest of the discussion will be restricted to the transmissivity, T, storativity, S, and the boundary distances, a and b. Assume now that these parameters are normally distributed with mean values and standard deviations given, respectively, by

$$(\bar{T}, \bar{S}, \bar{a}, \bar{b}) \text{ and } (\sigma_T, \sigma_S, \sigma_a, \sigma_b)$$

The uncertainty in the drawdown, s in Eq. (13), can then be expressed in terms of its standard deviation through the chain rule of differentiation as

$$\sigma_s = [(\mathrm{D}_T s|_{T=\bar{T}})^2 \sigma_T^2 + (\mathrm{D}_S s|_{S=\bar{S}})^2 \sigma_S^2 + (\mathrm{D}_a s|_{a=\bar{a}})^2 \sigma_a^2 + (\mathrm{D}_b s|_{b=\bar{b}})^2 \sigma_b^2]^{1/2} \tag{14}$$

where the symbol $|_{\mathrm{p}}$ indicates the value of the variable for which the derivative to its left, also known as the sensitivity of the parameter, must be evaluated.

The sensitivities associated with Eq. (12) will in general consist of two components-one associated with the contribution of the aquifer and the other with the contribution of the boundary. If P is used to denote any one of the parameters in Eq. (14), the sensitivity of the parameter can be expressed as

$$\mathrm{D}_P s(t_l) = \mathrm{D}_P s_a(t_l) + \mathrm{D}_P s_b(t_l) \tag{15}$$

or in a divided difference form

$$\mathrm{D}_P s(t_l) \simeq \frac{s_a(P+\Delta P) - s_a(P)}{\Delta P} + \frac{s_b(P+\Delta P) - s_b(P)}{\Delta P}$$

The advantage of the sensitivity analysis is that it allows one to establish a more risk-based estimate of the drawdown at large times, given by

$$\hat{s}_{obs}(t_l) = s_{obs}(t_l) \pm n\sigma_s$$

which yields on substitution into Eq. (10) the risk-based sustainable discharge rate

$$Q_{sus} = Q_{obs}\frac{s_p}{s_{obs}(t_l) \pm n\sigma_s}$$

where n is a statistical parameter that indicates the significance of the result. For example, $n = 1$ indicates that there is a 68.3% chance that a new estimate of $\hat{s}_{obs}(t_l)$ will fall in the range $s_{obs}(t_l) \pm \sigma_s$, while there is a 95.5% chance that the new estimate will fall in the range $s_{obs}(t_l) \pm 2\sigma_s$, if $n = 2$.

The approximations described above to estimate the sustainable yield of a borehole, henceforth referred to as the Flow Characteristics Method (FCM), has been implemented in a user friendly Excel workbook, discussed in more detail below. Since one cannot compute the sensitivities of an $s_a(t_l)$ derived from Eq. (11) for the parameters in Eq. (13), these sensitivities are approximated with the derivatives of the Theis solution, in Eq. (3), in the present version of the workbook.

The sustainable yield of a borehole does not only

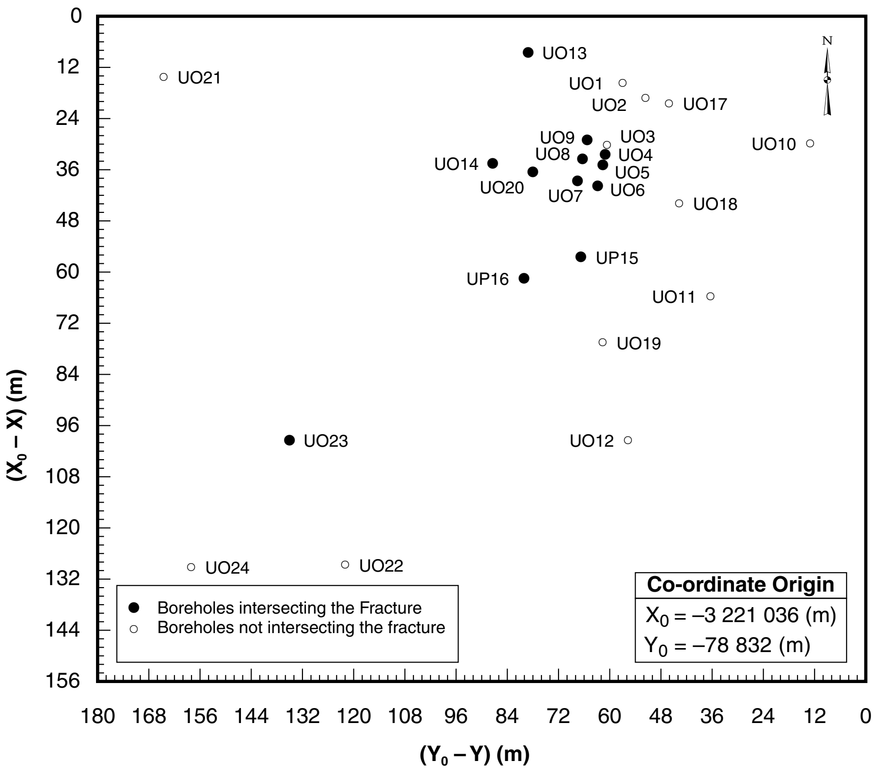

Fig. 6. Map of the boreholes on the Campus Test Site.

depend on the flow characteristics and hydraulic parameters of an aquifer, but also the rate at which the aquifer can be recharged by precipitation. This quantity must be estimated independently of the workbook and supplied as a parameter to the workbook.

The workbook allows the user to obtain two estimates. The first estimate, known as the basic solution, is based on the extrapolation of the observed drawdowns in Eq. (11), subjectively adjusted for the presence of impermeable boundaries. The second estimate, the advanced solution, uses the principle of error propagation discussed above to provide the user with a more qualified and risk-based estimate. However, the advanced solution can only be applied if prior information is available for the values of T and S, and the distances to impermeable boundaries.

6. A case study: borehole UO5 at the campus test site

6.1. Geology of the site

The Campus Test Site. Located at the University of the Orange Free State, Bloemfontein, South Africa, is underlain by a series of mudstones and sandstones from the Adelaide Subgroup of the Beaufort Group of formations in the Karoo Sequence. There are three aquifers present on the Site. The top, a phreatic aquifer, occurs within the upper mudstone layers on the Site. This aquifer is separated from the middle and main aquifer, which occurs in a sandstone layer between 8 and 10 m thick, by a layer of carbonaceous shale with a thickness of 0.5–4 m. The bottom aquifer occurs in the mudstone layers (more than 100 m thick that underlies the sandstone unit (Botha et al., 1998).

A major characteristic of the main aquifer is the presence of a horizontal fracture that coincides approximately with the centre plane of the middle aquifer and which intersects borehole UO5 and the other 11 boreholes with significant yields on the site, see Fig. 6. The remaining 12 boreholes all have insignificant yields. The fracture itself has an aperture of approximately 10 mm but the adjacent 200 mm of the sandstone is also highly permeable, see Fig. 7. This observation and experience gained in drilling boreholes at two other sites underlain by Karoo formations led Botha et al. (1998) to the conclusion that horizontal fractures serve as the conduits of water towards boreholes in Karoo aquifers, but that the water is stored in the Karoo formations themselves. Since this geometry of the formations differs significantly from that usually associated with the type curves available in the literature, it stands to reason that one cannot analyse the data from these aquifers with the conventional type curves.

Fig. 7. Photograph of the horizontal fracture on the Campus Test Site as it appears in a core sample.

Table 4
Drawdowns measured in a few boreholes during a constant rate test on borehole UO5 at the campus site and their distances from UO5 (the distance given for UO5 is the borehole radius)

Time (min)	Boreholes and distances			
	UO5, 0.08 m	UO6, 5.0 m	UP15, 22 m	UP16, 32 m
1.5	0.171	0.068	0.033	0.030
10.5	0.779	0.695	0.601	0.587
20.5	1.065	0.986	0.907	0.899
30.5	1.254	1.186	1.118	1.111
40.5	1.391	1.332	1.272	1.264
50.5	1.485	1.435	1.382	1.372
60.5	1.557	1.512	1.466	1.454
70.5	1.624	1.583	1.541	1.527
80.5	1.682	1.643	1.606	1.590
90.5	1.736	1.698	1.666	1.650
100.5	1.789	1.752	1.724	1.706
110.5	1.831	1.792	1.766	1.747
120.5	1.887	1.850	1.829	1.808
130.5	1.935	1.897	1.880	1.857
140.5	1.980	1.941	1.930	1.906
150.5	2.019	1.982	1.975	1.951
160.5	2.057	2.018	2.014	1.990
170.5	2.095	2.054	2.051	2.028
180.5	2.128	2.085	2.085	2.062
190.5	2.158	2.117	2.117	2.095
200.5	2.189	2.146	2.146	2.124
210.5	2.211	2.170	2.171	2.151
220.5	2.235	2.198	2.197	2.179
230.5	2.271	2.229	2.229	2.209
240.5	2.305	2.262	2.261	2.241
250.5	2.330	2.295	2.288	2.272
260.5	2.354	2.318	2.313	2.297
270.5	2.368	2.341	2.338	2.322
280.5	2.394	2.365	2.365	2.349
290.5	2.419	2.391	2.392	2.374
300.5	2.439	2.412	2.416	2.397
310.5	2.460	2.435	2.440	2.420
320.5	2.483	2.461	2.467	2.446
330.5	2.510	2.487	2.495	2.473
340.5	2.532	2.510	2.518	2.496
350.5	2.557	2.536	2.546	2.522
360.5	2.577	2.556	2.569	2.543
370.5	2.599	2.578	2.592	2.566
380.5	2.617	2.598	2.614	2.587
390.5	2.641	2.621	2.620	2.610

6.2. Analysis of a constant rate test on borehole UO5 with the Theis curve

It is often tempting to ascribe differences in the form of the drawdown observed during a constant rate test on a borehole in Karoo formations and that described by the Theis equation to measurement errors, especially if the drawdowns are measured by hand. These tests are consequently mainly analysed with either the Theis type curve, or the Cooper–Jacob approximation of the curve. The same method was consequently also applied to the drawdowns observed with pressure transducers during a constant rate test in which borehole UO5 was pumped at a constant rate of $1.25\ \mathrm{l\ s^{-1}}$, given in Table 4. This analysis yielded a value of $19\ \mathrm{m^2\ d^{-1}}$ for T and 8.6 for S. The value of T is not unrealistic and in fact agrees excellently with the value derived from the data of the other boreholes in Table 4. However, the value of S is completely unrealistic. Indeed, it can theoretically only be true if the aquifer is more than 10,000 m thick.

The unrealistic S-value was investigated further by computing the S-values for the other boreholes in Table 4 also with the Cooper–Jacob approximation. An interesting feature of these results, given in Table 5 is that the S-value decreases rapidly with the distance between the pumped and the observation boreholes. This behaviour can be briefly explained as follows. An analysis of the results of a three-dimensional numerical model of the Campus Test Site (Botha et al., 1998) has shown that there is a continuous vertical flux of water from the rock matrix towards the fracture, or its plane in areas outside the fracture, on both sides of the fracture. This flux, which assumes its maximum value at the pumped borehole decreases steadily with distance from the borehole towards the boundary of the aquifer, but never vanishes, a type of flow not considered in the derivation of the Theis equation. The result is that the Theis equation tends to overestimate the drawdown during early times and underestimate it during late times, as illustrated in Fig. 8. The only way that the Theis curve can 'interpret' the additional water is to 'view' it as water released from matrix storage, thereby yielding a larger S-value closer to the borehole than farther away. The Gringarten and Ramey model, on the other hand, fits the observed drawdown almost perfectly.

The previous results cast some doubt on the proposed use of the derivatives of the Theis curve to represent the sensitivities of the drawdown $s_a(t_l)$ in Eq. (15). However, it must be remembered that the high value of S is the direct result of using the borehole radius to compute the value of S from the Theis

Table 5
Cooper–Jacob estimates of the T and S values for the boreholes in Table 4

Borehole	Distance from UO5	T (m^2 d^{-1})	S
UO5	0.08	19	8.6
UO6	5	18	2.7×10^{-3}
UO15	22	17	1.7×10^{-4}
UO16	32	17	8.5×10^{-5}

solution. A heuristic approach to circumvent the unrealistic S-value is to view the influence that the fracture has on the drawdown as a 'skin-effect' (Horne, 1997), and work with the 'effective radius' and 'effective storativity' of the borehole defined by the equation

$$r_e^2 S_e = r_b^2 S \tag{16}$$

instead of the true storativity, S, and borehole radius, r_b.

6.3. Estimation of the sustainable yield

The application of the FCM to compute the sustainable yield of a borehole is illustrated in Fig. 9 with the drawdown data of borehole UO5 in Table 4. The first step in using the FCM is to supply the workbook with the observed drawdowns to compute the first derivatives of the drawdown as well as the second time derivatives of the drawdown, required by Eq. (11). The sheet in the workbook displayed in Fig. 9 is then used to supply the workbook with a number of parameters — the extrapolation time, specified maximum drawdown (that must not be exceeded at the end of the extrapolation time) and average annual recharge. The workbook then uses Eqs. (10) and (11), supplemented with subjective information on boundaries, to compute an average sustainable yield and sustainable yield.

The advanced solution can be used in those cases where distances to no-flow boundaries are known. In this solution uncertainties in late-time storativity and transmissivity and the distances to no-flow boundaries are used to estimate a risk-based sustainable yield by applying Gaussian Error Propagation, as defined in Eq. 16. In the case of borehole UO5 the advanced solution estimated the sustainable yield of the borehole as 0.36 l s^{-1} with a 95% confidence level.

As mentioned above, the main flow direction of water in a Karoo aquifer is vertical from the rock matrix to the fracture and then from the fracture to the borehole. This suggests that a hydraulic test in such an aquifer is best analysed with a three-dimensional numerical model. Since the geometry of the Campus Test Site is known (Botha et al., 1998), it may be useful to compare the FC-estimate of the sustainable yield for borehole UO5 above with that obtained from a three-dimensional numerical model.

The numerical model used for this purpose was based on the somewhat simplified geometry of the

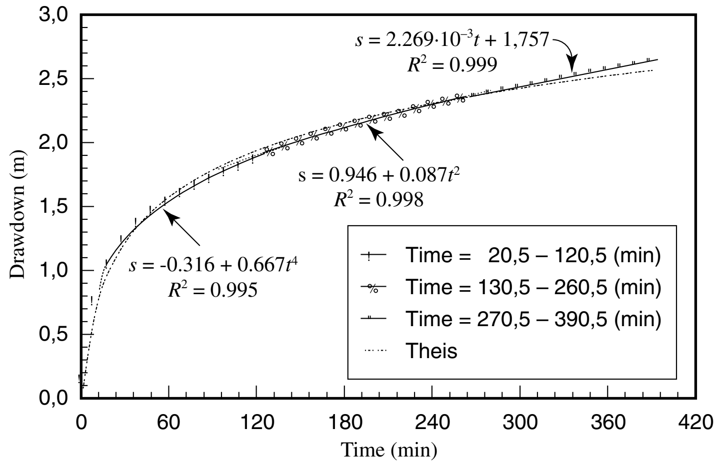

Fig. 8. Comparison of the drawdown in UO5 given in Table 4 and its least squares fit to the first three periods of Gringarten's model for a horizontal fracture and the classical Theis solution.

FC-METHOD : Estimation of the sustainable yield of a borehole

Borehole: UO5 at 1.25 L s^{-1}

Extrapolation time in years = (enter)	2	1051200	Extrapol.time in minutes	
Effective borehole radius (r) = (enter)	22 ←	22.45 ←	Est. r_e	From r(e) sheet
Q (L s^{-1}) from pumping test =	1.25	30.70 ←	Est. r_e	Qualified guess
s_a (available drawdown), sigma_s = (enter)	7	1 ←	Sigma_s from risk analysis	
Annual effective recharge (mm) =		6.00	s_available working drawdown(m)	
t (end) and s (end) of pumping test =	390.5	2.64	End time and drawdown of test	
Average maximum derivative = (enter)	1.8 ←	1.8	Estimate of average of max deriv	
Average second derivative = (enter)	0 ←	0.1	Estimate of average second deriv	
Derivative at radial flow period = (enter)	1 ←		Read from derivative graph	
	T-early (m^2 d^{-1}) =	19.76		
T and S estimates from derivatives	T-late (m^2 d^{-1}) =	10.98	Est. min S-late=1.10E-03	
(To obtain correct S-value, use program RPTSOLV)	S-late =	1.10E-03	S-estimate could be wrong	

BASIC SOLUTION

(Using derivatives + subjective information about boundaries)

(No values of T and S are necessary)

		Maximum influence of boundaries at long time		
	No boundaries	1 no-flow	2 no-flow	Closed no-flow
sWell (Extrapol.time) =	8.81	14.99	21.16	39.68
Q_sust (L s^{-1}) =	0.85	0.50	0.35	0.19
	Best case	→		**Worst case**
Average Q_sust (L s^{-1}) =	**0.41**			
with standard deviation =	0.28			

(If no information exists about boundaries skip advanced solution and go to final recommendation)

ADVANCED SOLUTION

(Using derivatives+ knowledge on boundaries and other boreholes)

(Late T-and S-values a priori + distance to boundary)

T-late (m^2 d^{-1}) = (enter) →	11.00			
S-late = (enter) →	1.00E-03			
1. BOUNDARY INFORMATION (choose a or b)	(Code =9999 = dummy value if not applicable)			
(a) Barrier (no-flow) boundaries →	Closed Square	Single	Intersect. 90	2 Parallel
Bound. distance a(m) : (enter)	9999	9999	9999	400
Bound. distance b[meter] : (enter)			9999	800
s_Bound (t = Extrapol.time) (m) =	0.00	0.00	0.00	7.94
(b) Fix head boundary + no-flow →	Closed Fix	Single Fix	90 Fix+no-flow	// Fix+no-flow
Bound. distance to fix head a (m) : (enter)	9999	9999	9999	9999
Bound. distance to no-flow b (m) : (enter)			9999	9999
s_Bound (t = Extrapol.time) (m) =	0.00	0.00	0.00	0.00
2. INFLUENCE OF OTHER BOREHOLES →	Q (L s^{-1})	r (m)	u_r	W(u,r)
BH1			0.00E+00	
BH2			0.00E+00	
s_(influence of BH1,BH2) =	0.00	0.00		

SOLUTION INCLUDING BOUNDS AND BH's

Fix head + No-flow : Q_sust (L s^{-1}) =	9999	9999	9999	9999
No-flow : Q_sust (L s^{-1}) =	9999	9999	9999	0.45
Enter selected Q for risk analysis = (enter) →	0.52			

(Go to Risk sheet and perform risk analysis from which sigma_s will be estimated : only for barrier boundaries)

FINAL RECOMMENDED ABSTRACTION RATE

Abstraction rate (L s^{-1}) for 24 h	**0.36**
Total volume of water that can be abstracted per month (m^3) =	933

COMMENTS

Q_sust with 68% safety = 0.48

Q_sust with 95% safety = 0.36

Fig. 9. Estimation of the sustainable yield of borehole UO5 with the FC-method as obtained from the EXCEL Workbook.

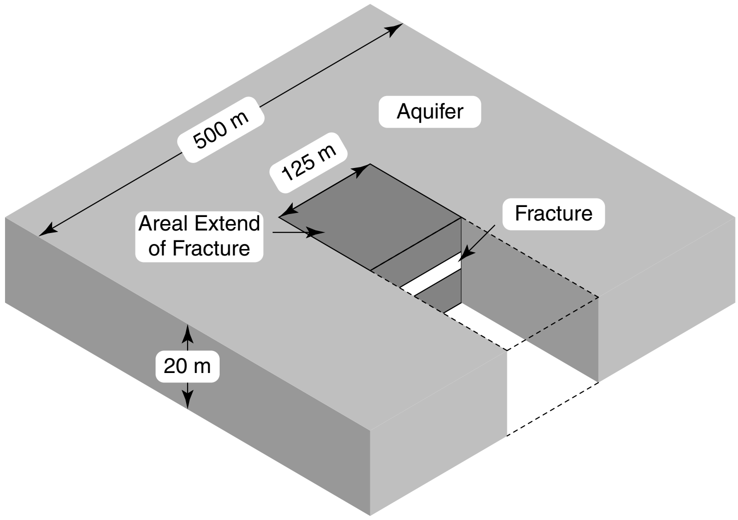

Fig. 10. Simplified geometry of the aquifer on the Campus Test Site used in the numerical model to estimate the sustainable yield of borehole UO5.

aquifer illustrated in Fig. 10 and the computer package Processing Modflow for Windows — PMWIN (Chiang and Kinzelbach, 1998). The spatial and temporal distributions of the piezometric heads, required by the three-dimensional model, were approximated with the drawdowns in Table 4.

The hydraulic parameters that are important in the study of a three-dimensional model for a fractured aquifer such as the one on the Campus Test Site are: the specific storativity (S_{sm}), horizontal and vertical hydraulic conductivities (K_{hm} and K_{vm}) of the matrix, the horizontal and vertical hydraulic conductivities (K_{hf} and K_{vf}) of the fracture and its specific storativity (S_{sf}). Since the flow in the fracture is in the horizontal direction and its ability to store water is insignificant, the last two parameters do not have a significant influence on the water levels and can be ignored in the model of the Site.

The first step in analysing the flow of water in an aquifer with a numerical model is to develop a suitable mesh or grid for the aquifer. The finite difference mesh used in the present model consisted of 66 rows, 93 columns and 23 layers of nodes. The horizontal extent of the model was chosen large enough to avoid boundary effects. The 12th layer of the model represented the fracture. The other layers represented the sandstone matrix above and below the fracture. The two layers directly above and below the fracture were each assigned a thickness of 0.5 m and the other layers a thickness of 1 m. This fine vertical discretization was necessary to account for the vertical drawdown in the rock matrix.

As mentioned above, the aperture of fracture zone was taken as 0.2 m, while its horizontal extent was estimated as 120 m × 120 m, based on the geological profiles of the boreholes. The three observation boreholes (UO6, UO15 and UO16) were also placed in the mesh at distances of 5, 22 and 32 m from UO5 within the fracture. It was also assumed that the aquifer is recharged at a rate of 15 mm a^{-1}.

The model was calibrated with the inverse model PEST98 of Doherty et al. (1994). The estimated parameters, obtained from this calibration, are listed in Table 6. The estimated value for the hydraulic conductivity of 3600 m d^{-1} for the fracture zone is in excellent agreement with the *T*-value of 720 m^2 d^{-1} obtained from tracer tests (Van Wyk, 1998). This estimate is reliable, however, the other values are correlated, because of the approximation of piezometric heads with water levels. The estimated parameters of the model are listed in Table 6, while the fit between the computed and observed drawdown is compared in Fig. 11. These parameters and a recharge rate of 15 mm a^{-1} was next used to compute a sustainable yield over a period of one year with PMWIN, This yielded a value of 0.4 l s^{-1}, which is

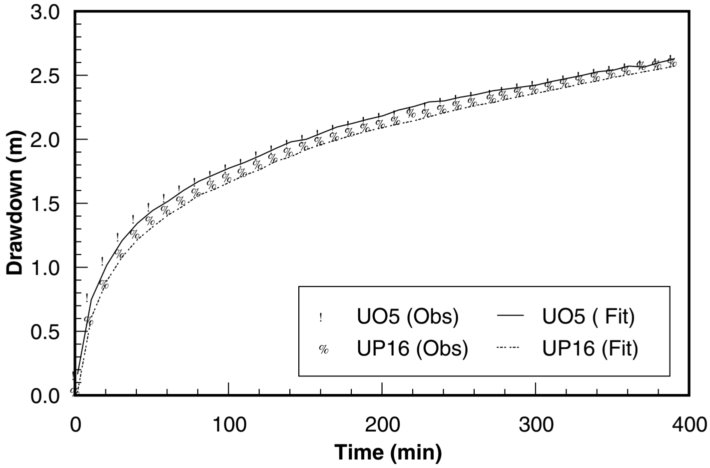

Fig. 11. Comparison of the computed and observed drawdowns for the constant rate test on borehole UO5 at the Campus Test Site.

Table 6
Hydraulic parameters for the aquifer on the campus test site as estimated with the three-dimensional model

Parameter	Estimated values
K_{hm} (m d^{-1})	0.158
K_{vm} (m d^{-1})	5.82×10^{-3}
S_{sm} (m^{-1})	5.65×10^{-5}
K_{hf} (m d^{-1})	3.6×10^{3}

very similar to the value of 0.36 l s^{-1} estimated with the FC-method.

The multiplication of S_{sm} in Table 6 and the thickness of the aquifer (20 m) shows that the storativity of the rock matrix is 1.1×10^{-3}, the same value obtained by Kirchner et al. (1991) in their study of Karoo aquifers.

7. Conclusions

There is no doubt that a hydraulic test in a fractured aquifer should be analysed with a three-dimensional numerical model from the theoretical point of view. However, the data required for such a model may not always be available. In such cases one may use either the numerical vertical flow model RPTsolv or the semi-analytical Method of Derivative Fitting described in this paper. The Flow Characteristic Method, which is based on extensions of the latter method is particularly useful when one is interested in estimating the sustainable yield of a production borehole. The Excel Workbook, which was developed specifically for this method and which is freely available on the website of the Institute for Groundwater Studies, simplifies the computations considerably.

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