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Troughs under threshold modeling of minimum flows in perennial streams

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Abstract

Troughs under threshold analysis has so far found little application in the modeling of minimum streamflows. In this study, all the troughs under a certain threshold level are considered in deriving the probability distribution of annual minima through the total probability theorem. For the occurrence of minima under the threshold, Poissonian, binomial or negative binomial processes are assumed. The magnitude of minima follows the generalized Pareto, exponential or power distribution. It is shown that asymptotic extreme value distributions for minima or the two-parameter Weibull distribution is obtained for the annual minima, depending on which models are assumed for the occurrence and magnitude of troughs under the threshold. Derived distributions can be used for modeling the minimum flows in streams which do not have zero flows. Expressions for the *T*-year annual minimum flow are obtained. An example illustrates the application of the troughs under threshold model to the minimum flows observed in a stream.

Keywords: Minimum flows; Low flows; Troughs under threshold; Annual minima; Extreme value distributions

1. Introduction

It is important to determine the probability distribution of annual minimum flows defined as 'the minimum average discharge in a year for a certain duration of *d* days', in the studies related to water supply planning, water quality management, minimum release policies, etc. Generally, observed series of annual minima are used in choosing the probability distribution functions that has the best fit to the observations. Two-parameter lognormal, Weibull and power distributions, and three-parameter lognormal, Weibull and logPearson type III distributions are among those that have been widely used as the probability function of minimum flows (Mc Mahon and Diaz, 1982; Vogel and Kroll, 1989; Önöz and Bayazit, 2001a). It is often difficult to choose the best function because of the low power of the available statistical goodness-of-fit tests.

In the similar subject of flood frequency analysis, the series of peaks over threshold (POT) have been used as an alternative to annual maximum flood series. A peak over threshold series consists of all the peaks above a certain threshold level, and therefore contains more information than the annual maximum flood series which has only one element each year. The probability distribution of the annual maximum floods is derived from the assumed distributions of the annual number of occurrences and the magnitudes of the peaks over threshold. The Poisson process is the model usually assumed for the occurrence times of flood peaks, whereas the exponential distribution is generally used for the magnitude of exceedances (Shane and Lynn, 1964), in which case the Gumbel

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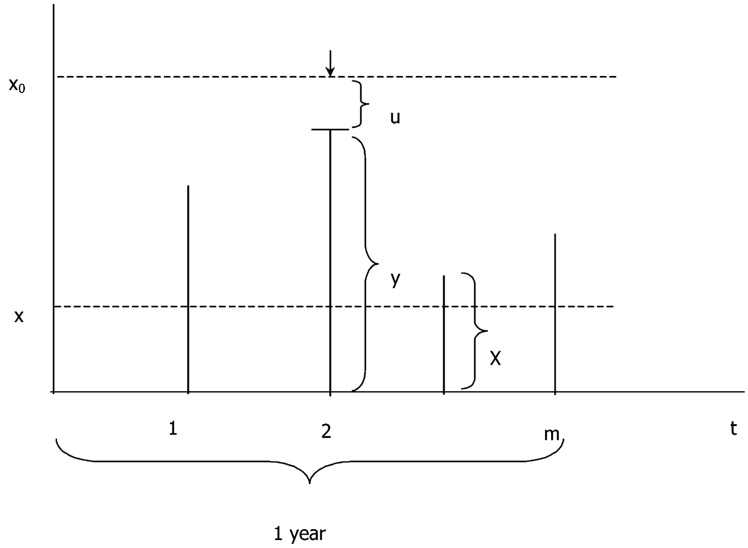

Fig. 1. Definition of the terms in TUT analysis.

distribution is found for the annual maxima (Zelenhasic, 1970). Rosbjerg et al. (1992) and Madsen et al. (1997) showed that the generalized extreme value (GEV) distribution is obtained for annual floods when the Poisson process is combined with the generalized Pareto distribution for the magnitude of exceedances. Lang et al. (1999) presented a state-of-the-art review of the POT modeling. In some cases it was seen that the number of peaks occurring each year was not a Poisson variate, its variance being significantly greater than its mean (NERC, 1975; Cunnane, 1979; Ben-Zvi, 1991). Vukmirovic and Petrovic (1997) and Lang et al. (1997) considered distributions other than the Poisson distribution (binomial and negative binomial) for the annual number of peaks. Recently, Önöz and Bayazit (2001b) analyzed the case when the binomial or negative binomial model for the occurrence of peaks is combined with the exponential distribution of peak magnitudes. They obtained expressions for the probability distribution of annual maxima, for the T-year flood and its sampling variance. Their results imply that the results are almost identical to those obtained using the Poisson model when the ratio variance/mean of the annual number of peaks over threshold is not much different from one, which is usually the case for floods.

In this paper, frequency analysis of minimum flows is performed by considering not only the annual minima but also all the minima (troughs) under a certain threshold level. This is to be called troughs under threshold (TUT) modeling. It will be assumed that the troughs are independent. Therefore, more than one minimum on the same recession curve should not be included in the analysis when the flow exceeds the threshold level for a short period of time during a long drought.

TUT modeling uses more information about the minimum flows than the annual minima modeling because it works with a larger number of observations. It can be argued that it has more physical relevance because it is based on models for the distribution of the annual number of troughs under threshold and of their magnitudes. For these reasons, it is expected to give good estimates of the T-year minima just as POT modeling does for the floods.

In this study the generalized Pareto distribution, which contains the exponential distribution as a special case, is chosen for the magnitude of the differences between the threshold level and troughs. Poisson model is assumed for the occurrence of minima under the threshold level x_0. The probability distribution $F_x(x)$ of the annual minima is derived applying the total probability theorem. The generalized Pareto distribution is also combined with the binomial or negative binomial models for the occurrence of the troughs under the threshold. As an alternative approach, the power distribution for the magnitude of troughs Y under the threshold is combined with the Poisson, binomial and negative binomial models, respectively, for the occurrence of minima under the threshold level. Expressions for the T-year annual minimum are obtained in each case. Finally, the use of the derived distributions is illustrated in an example.

The study is restricted to the case of streams with no zero flows, because the distributions assumed for the minima below the threshold cannot model zero minima properly. However, the derived distributions can be used for the conditional probability distribution of non-zero flows in ephemeral streams which are common in arid and semi-arid regions (Wang and Singh, 1995; Smakhtin, 2001).

2. Probability distribution of annual minima

The probability distribution function $F_x(x)$ of the annual minima X can be derived using the total probability theorem. If $F_Y(y)$ is the probability distribution function of the magnitude of troughs Y under the threshold level x_0 (Fig. 1), we can write

$$P(X > x) = 1 - F_x(x) = \sum_{i=0}^{\infty} P(m = i)[1 - F_Y(x)]^i \quad (1)$$

where $P(m = i)$ is the probability mass function of m, the annual number of troughs under threshold.

Eq. (1) expresses that when each of the i minima $Y_1, Y_2, \ldots, Y_i$ under the threshold in a year is greater than x, the smallest of them (annual minimum X) will exceed x. The probability of this event is $P[X > x] = P[Y_1 > x, Y_2 > x, \ldots, Y_i > x] = \prod_{m=1}^{i} P(Y_m > x) = [1 - F_Y(x)]^i$ because the troughs are assumed independent. Summing over i from 0 to ∞, the probability distribution $F_x(x)$ of the annual minima is obtained by the total probability theorem.

2.1. Generalized Pareto and Poisson distributions

Let us first assume that the occurrence of minima under the threshold is Poissonian:

$$P(m = i) = e^{-\mu}\mu^i/i! \qquad i = 0, 1, 2, \ldots \quad (2)$$

where $\mu = E(m)$ is the mean of m.

It will be assumed that $u = x_0 - y$, the difference between the threshold level and a trough, follows the generalized Pareto distribution:

$$F_U(u) = \begin{cases} 1 - \left(1 - \dfrac{ku}{\alpha}\right)^{1/k} & 0 \le u,\ k < 0 \\ 1 - \exp\left(-\dfrac{u}{\alpha}\right) & 0 \le u,\ k = 0 \\ 1 - \left(1 - \dfrac{ku}{\alpha}\right)^{1/k} & 0 \le u \le \dfrac{\alpha}{k},\ k > 0 \end{cases} \quad (3)$$

For $k \le 0$, u has no upper bound. For $k > 0$, it has an upper bound equal to α/k, which agrees well with the distribution of troughs under threshold because u cannot exceed x_0.

The distribution of the magnitude of the minima corresponding to Eq. (3) is:

$$F_Y(y) = 1 - F_U(u)$$

$$= \begin{cases} \left(1 - k\dfrac{x_0 - y}{\alpha}\right)^{1/k} & y \le x_0,\ k < 0 \\ \exp\left(-\dfrac{x_0 - y}{\alpha}\right) & y \le x_0,\ k = 0 \\ \left(1 - k\dfrac{x_0 - y}{\alpha}\right)^{1/k} & x_0 - \dfrac{\alpha}{k} \le y \le x_0,\ k > 0 \end{cases} \quad (4)$$

For $k = 0$, the exponential distribution is obtained for the magnitudes of minima. Combining Eqs. (1) and (2) and the second term of Eq. (4), we obtain the probability distribution of the annual minima when the occurrences of the minima are Poissonian and their magnitudes are exponential:

$$\begin{aligned} F_x(x) &= 1 - \sum_{i=0}^{\infty} e^{-\mu}\mu^i/i!\left[1 - \exp\left(-\frac{x_0 - x}{\alpha}\right)\right]^i \\ &= 1 - e^{-\mu}\exp\left\{\mu\left[1 - \exp\left(\frac{x - x_0}{\alpha}\right)\right]\right\} \\ &= 1 - \exp\left[-\exp\left(\frac{x - x_0'}{\alpha}\right)\right] \end{aligned} \quad (5)$$

$x \le x_0$

where $x_0' = x_0 - \alpha \ln \mu$. This distribution is known as the Type I extreme value distribution for minima

(Önöz and Bayazit, 1999). $1 - F_x(x_0) = \exp(-\mu) = P(m = 0)$ is the probability that the annual minimum is greater than x_0.

For $k \neq 0$, combining the first (or the third) term of Eq. (4) with Eqs. (1) and (2), the following distribution is obtained for the annual minima;

$$F_x(x) = 1 - \sum_{i=0}^{\infty} e^{-\mu}\mu^i/i!\left[1 - \left(1 - k\frac{x_0 - x}{\alpha}\right)^{1/k}\right]^i$$
$$= 1 - e^{-\mu}\exp\left\{\mu\left[1 - \left(1 - k\frac{x_0 - x}{\alpha}\right)^{1/k}\right]\right\}$$
$$= 1 - \exp\left[-\left(1 - k\frac{x_0'' - x}{\alpha'}\right)^{1/k}\right] \quad (6)$$

where $\alpha' = \alpha/\mu^k$ and $x_0'' = x_0 + \alpha/k(1/\mu^k - 1)$. For $k < 0$, this is known as the Type II extreme value distribution for minima, where $x \leq x_0$ (Önöz and Bayazit, 1999). Again, $1 - F_x(x_0) = \exp(-\mu) = P(m = 0)$ is the probability that the annual minimum exceeds x_0.

For $k > 0$, Type III extreme value distribution for minima is obtained, where $x_0 - \alpha/k \leq x \leq x_0$ (Önöz and Bayazit, 1999). For $k = 0$ and $k < 0$, x has no lower bound. This is not desirable because the annual minimum can have no negative value. On the other hand, the annual minimum has the lower bound $x_0 - \alpha/k$ for $k > 0$. This may not be desirable for streams which has very low minima.

2.2. Exponential and (negative) binomial distributions

The Poisson distribution has a variance equal to the mean. When the variance is significantly lower (higher) than the mean, the binomial (negative binomial) distributions may be used. The binomial distribution has the probability mass function

$$P(m = i) = \binom{\gamma}{i} p^i(1 - p)^{\gamma - i} \qquad i = 0, 1, 2, \ldots \quad (7)$$

where the mean of m is $E(m) = p\gamma$ and the variance of m is $\mathrm{Var}(m) = p(1 - p)\gamma$.

For the negative binomial distribution the probability mass function is

$$P(m = i) = \binom{\gamma + i - 1}{i} p^i(1 - p)^{\gamma} \qquad i = 0, 1, 2, \ldots \quad (8)$$

where the mean and variance of m are given by $E(m) = p\gamma/(1 - p)$ and $\mathrm{Var}(m) = p\gamma/(1 - p)^2$, respectively.

Combining the second term of Eq. (4) with Eq. (7) of the binomial distribution and Eq. (8) of the negative binomial distribution, respectively, following expressions are obtained for the distribution of annual minima:

$$F_x(x) = 1 - \sum_{i=0}^{\infty} \binom{\gamma}{i} p^i(1 - p)^{\gamma - i} \times\left[1 - \exp\left(-\frac{x_0 - x}{\alpha}\right)\right]^i$$
$$= 1 - (1 - p)^{\gamma} \times\left\{1 + \frac{p}{1 - p}\left[1 - \exp\left(-\frac{x_0 - x}{\alpha}\right)\right]\right\}^{\gamma} \quad (9)$$
$$x \leq x_0$$

using the binomial series expansion of $(1 + a)^{\gamma}$ where $1 + a$ is the quantity in brackets of Eq. (9), and

$$F_x(x) = 1 - \sum_{i=0}^{\infty} \binom{\gamma + i - 1}{i} p^i(1 - p)^{\gamma} \times\left[1 - \exp\left(-\frac{x_0 - x}{\alpha}\right)\right]^i$$
$$= 1 - (1 - p)^{\gamma}\left\{1 - p\left[1 - \exp\left(-\frac{x_0 - x}{\alpha}\right)\right]\right\}^{-\gamma}$$
$$x \leq x_0 \quad (10)$$

using the binomial series expansion of $(1 - b)^{-\gamma}$ where $(1 - b)$ is the quantity in brackets of Eq. (10). In both Eqs. (9) and (10), $1 - F_x(x_0) = (1 - p)^{\gamma}$ is the probability of $x > x_0$.

2.3. Generalized Pareto and (negative) binomial distributions

Generalized Pareto distribution for the difference between the threshold level and a trough can be combined with the binomial distribution for the occurrence of troughs below the threshold to obtain the probability distribution of annual minima. From Eqs. (1), (4) and (7):

$$F_x(x) = 1 - \sum_{i=0}^{\infty} \binom{\gamma}{i} p^i (1-p)^{\gamma-i} \times \left[1 - \left(1 - k\frac{x_0 - x}{\alpha}\right)^{1/k}\right]^i = 1 - (1-p)^{\gamma} \times \left\{1 + \frac{p}{1-p}\left[1 - \left(1 - k\frac{x_0 - x}{\alpha}\right)^{1/k}\right]\right\}^{\gamma} \quad (11)$$

Similarly, combining the generalized Pareto distribution with the negative binomial distribution, annual minima is found to have the following probability distribution, using Eqs. (1), (4) and (8):

$$F_x(x) = 1 - \sum_{i=0}^{\infty} \binom{\gamma + i - 1}{i} p^i (1-p)^{\gamma} \times \left[1 - \left(1 - k\frac{x_0 - x}{\alpha}\right)^{1/k}\right]^i = 1 - (1-p)^{\gamma} \times \left\{1 - p\left[1 - \left(1 - k\frac{x_0 - x}{\alpha}\right)^{1/k}\right]\right\}^{-\gamma} \quad (12)$$

For $k < 0$, $x \le x_0$ and for $k > 0$, $x_0 - \alpha/k \le x \le x_0$ in the above equations.

2.4. Power and Poisson distributions

Önöz and Bayazit (2001a) found that the power distribution has a good fit to the low streamflow data. This distribution is given by:

$$F_Y(y) = (y/x_0)^c \qquad 0 \le y \le x_0 \quad (13)$$

Assuming that the troughs under the threshold has the power distribution, and inserting Eqs. (2) and (13) in Eq. (1)

$$F_x(x) = 1 - \sum_{i=0}^{\infty} \mathrm{e}^{-\mu}\mu^i/i![1 - (x/x_0)^c]^i = 1 - \mathrm{e}^{-\mu}\exp\{\mu[1 - (x/x_0)^c]\} = 1 - \exp[-\mu(x/x_0)^c] = 1 - \exp[-(x/x'_0)^c] \quad (14)$$

where $x'_0 = x_0/\mu^{1/c}$. Eq. (14) is the two-parameter Weibull distribution that has been widely used for minimum flows. $1 - F_x(x_0) = \exp(-\mu) = P(m = 0)$ is the probability that the annual minimum is greater than x_0.

2.5. Power and (negative) binomial distributions

Combining Eqs. (1), (7) and (13), the probability distribution of the annual minima is derived as:

$$F_x(x) = 1 - \sum_{i=0}^{\infty} \binom{\gamma}{i} p^i (1-p)^{\gamma-i}[1 - (x/x_0)^c]^i = 1 - (1-p)^{\gamma}\sum_{i=0}^{\infty}\binom{\gamma}{i}\left(\frac{p}{1-p}\right)^i[1 - (x/x_0)^c]^i = 1 - (1-p)^{\gamma}\left\{1 + \frac{p}{1-p}[1 - (x/x_0)^c]\right\}^{\gamma}$$

$$0 \le x \le x_0 \quad (15)$$

Inserting Eqs. (8) and (13) in Eq. (1):

$$F_x(x) = 1 - \sum_{i=0}^{\infty} \binom{\gamma + i - 1}{i} p^i (1-p)^{\gamma}[1 - (x/x_0)^c]^i = 1 - (1-p)^{\gamma}\{1 - p[1 - (x/x_0)^c]\}^{-\gamma}$$

$$0 \le x \le x_0 \quad (16)$$

where $1 - F_x(x_0) = (1-p)^{\gamma} = P(m = 0)$ is the probability that the annual minimum is greater than x_0 in both Eqs. (15) and (16).

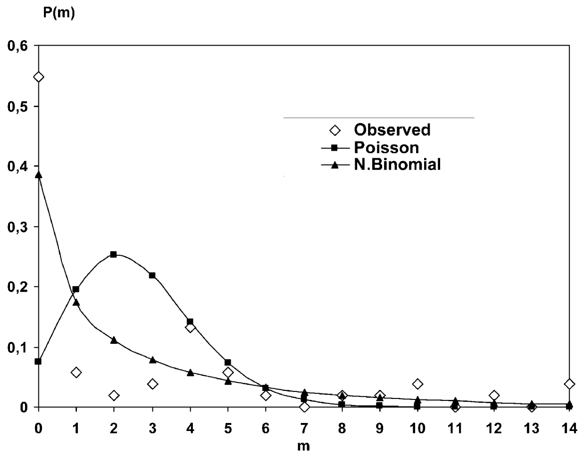

Fig. 2. Observed frequencies of the annual number of troughs, and probabilities of the fitted Poisson and negative binomial distributions.

3. *T*-year annual minimum flow

Expressions for x_T, the annual minimum flow corresponding to a return period of *T*-years, can be obtained from the equations derived for the probability distribution $F_x(x)$ of the annual minima as the value of x that corresponds to $F_x(x) = 1/T = 1 - T_1$.

When the generalized Pareto distribution for $k \neq 0$ is combined with the Poisson, binomial and negative binomial distributions, using, respectively, Eqs. (6), (11) and (12), the following results are obtained for the *T*-year annual minimum:

$$x_T = x_0 - \frac{\alpha}{\mu^k k}[1 - (-\ln T_1)^k] + \frac{\alpha}{k}\left(\frac{1}{\mu^k} - 1\right)$$

$$= x_0 - \frac{\alpha}{k}\left[1 - \left(-\frac{\ln T_1}{\mu}\right)^k\right] \quad (17)$$

$$x_T = x_0 + \frac{\alpha}{k}\left[\left(\frac{1 - T_1^{1/\gamma}}{p}\right)^k - 1\right] \quad (18)$$

$$x_T = x_0 + \frac{\alpha}{k}\left[\frac{(1-p)^k\left(T_1^{-1/\gamma} - 1\right)^k}{p^k} - 1\right] \quad (19)$$

In the case of the exponential distribution, using, respectively, Eqs. (5), (9) and (10), corresponding formulas are:

$$x_T = x_0 + \alpha \ln(-\ln T_1) - \alpha \ln \mu \quad (20)$$

$$x_T = x_0 + \alpha \ln\left(\frac{1 - T_1^{1/\gamma}}{p}\right) \quad (21)$$

$$x_T = x_0 + \alpha \ln\left[\frac{1-p}{p}\left(T_1^{-1/\gamma} - 1\right)\right] \quad (22)$$

Results for the Poisson, binomial and negative binomial distributions, using, respectively, Eqs. (14)–(16), are as follows when the magnitudes of minima have the power distribution

$$x_T = x_0(-\ln T_1)^{1/c}/\mu^{1/c} \quad (23)$$

$$x_T = x_0\left(1 - T_1^{1/\gamma}\right)^{1/c}/p^{1/c} \quad (24)$$

$$x_T = x_0\left(T_1^{-1/\gamma} - 1\right)^{1/c}(1-p)^{1/c}/p^{1/c} \quad (25)$$

4. An example

TUT modeling is applied to the one-day minimum flows of the flow gauging station No. 1203 on the River Porsuk in the Sakarya river basin in Turkey. The basin area is 3938 km^2, average discharge is

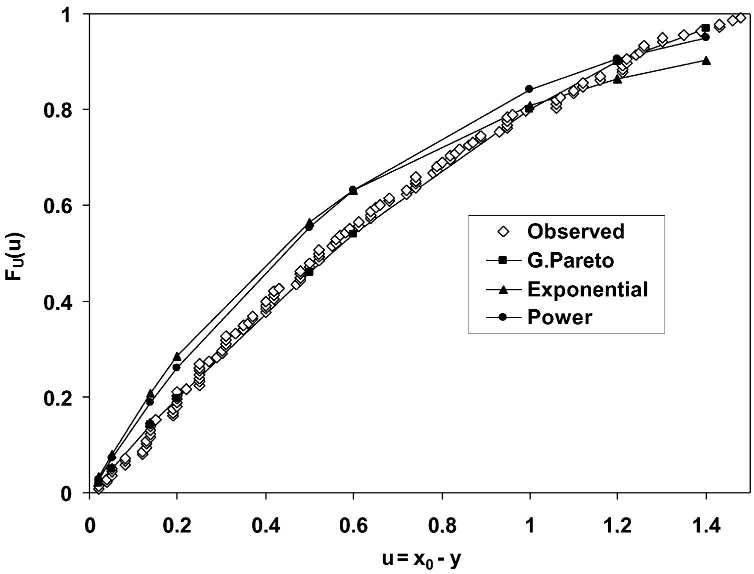

Fig. 3. Empirical probability plot of the observed data, and the fitted exponential, generalized Pareto and power distributions.

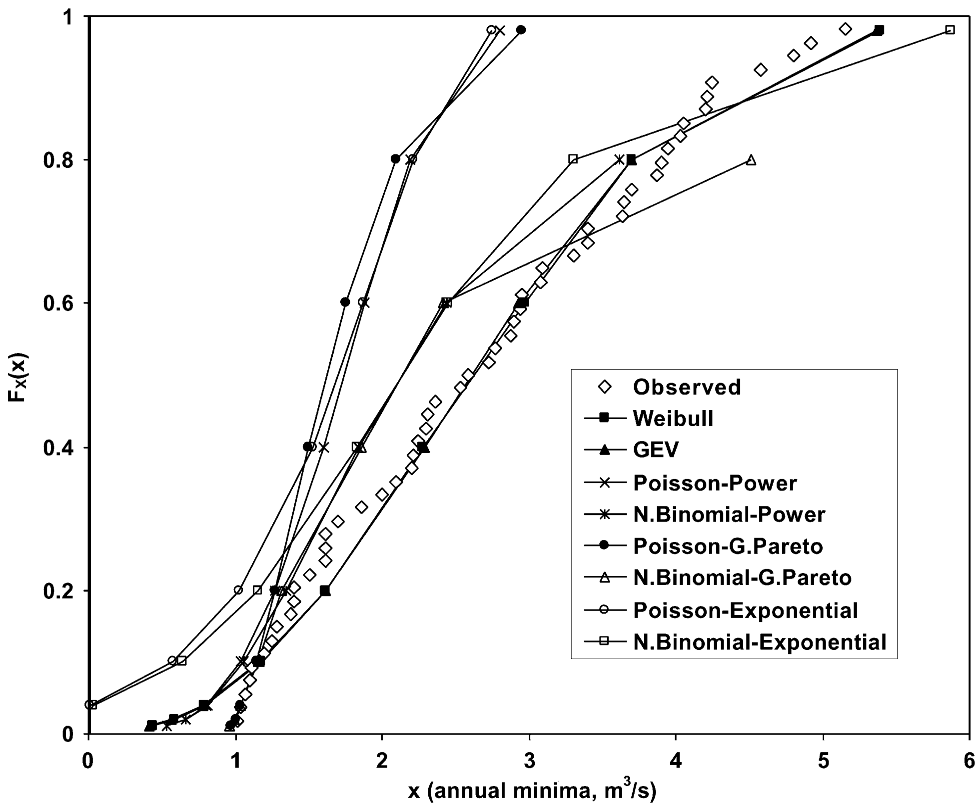

Fig. 4. Probability plots of the TUT and annual minima models fitted to the data, and the empirical distribution of the observed annual minima.

Table 1
T-year annual minima estimated by various distributions

x_T (m^3/s)	TUT modeling						Annual minima modeling	
	Generalized Pareto and Poisson	Generalized Pareto and negative binomial	Exponential and Poisson	Exponential and negative binomial	Power and Poisson	Power and negative binomial	Two-parameter Weibull	GEV for minima
x_5	1.27	1.32	1.03	1.16	1.27	1.35	1.61	1.61
x_{10}	1.14	1.15	0.58	0.64	1.04	1.06	1.18	1.17
x_{25}	1.04	1.04	0.01	0.03	0.8	0.81	0.79	0.78
x_{50}	1.00	0.99	–	–	0.66	0.66	0.59	0.57
x_{100}	0.97	0.96	–	–	0.54	0.54	0.44	0.42

8.7 m^3/s. No zero daily flows have been recorded. Fifty three-year portion of the flow record (1938–1990) is used in the analysis.

The threshold level is chosen as $x_0 = 2.5$ m^3/s. Local minima below this level have been selected for the analysis. Some of these, however, are on the same recession curve and, are therefore, not independent. In such cases, only the smallest minimum is preserved and the others are eliminated by the visual inspection of the hydrograph. Average lag-one autocorrelation coefficient of the troughs in years with at least four troughs is computed as 0.157. The number of independent minima used in the TUT modeling is 137, corresponding to an annual average of 2.59 troughs.

The variance of the annual number of troughs is 14.90, much larger than its mean. Therefore, the negative binomial model is more suitable than the Poisson model for the occurrence of troughs under threshold. In 29 yr of the observation period, annual minima are above the threshold level $x_0 = 2.5$ m^3/s. The probability of the annual minima being greater than x_0 is then $P(m = 0) = 29/53 = 0.55$.

The Poisson and negative binomial distributions are fitted to the annual number of occurrences of the troughs under the threshold level, with parameters estimated by the method of moments. Fig. 2 shows the frequencies $P(m = i)$ estimated from the observed annual number of troughs data, compared with the probability mass functions of the fitted Poisson and negative binomial distributions. Although neither of these distributions has a very good fit to the data, it is seen that the negative binomial distribution provides a much better fit. Although χ^2 statistic is much smaller for the negative binomial ($\chi^2 = 36.5$), χ^2 test rejects this distribution even at the 0.01 level of significance ($\chi^2_{0.01} = 26.2$).

The generalized Pareto and exponential distributions are fitted to the difference between the threshold level and a trough. The power distribution is fitted to the magnitude of the troughs. Parameters are estimated in each case by the L-moments (Hosking and Wallis, 1997; Önöz and Bayazit, 2001a). Fitted exponential, generalized Pareto and power distributions are plotted in Fig. 3 together with the empirical frequencies of the observed data. It is clearly seen that the generalized Pareto distribution has a much better fit than the other two distributions. Kolmogorov–Smirnov (K–S) statistic is $\Delta = 0.04$ for the generalized Pareto, 0.09 for the power, and 0.11 for the exponential distribution. K–S test rejects only the exponential distribution at the 0.10 level ($\Delta_{0.10} = 0.10$).

Fig. 4 shows the plots of the probability distributions of annual minima obtained by the TUT modeling with various assumptions for the distributions of the magnitude of troughs and of the annual number of troughs. Furthermore, the two-parameter Weibull distribution and GEV distribution for the minima are fitted directly to the observed annual minima (L-moments are used in estimating the parameters of the distributions, Önöz and Bayazit, 1999). Probability plots of these two distributions and the empirical distribution of the observed annual minima are also shown in Fig. 4.

The Weibull and GEV distributions whose parameters are estimated directly from the annual minima, have the best overall fit to the observed frequencies. But their fit is not good for $F_x(x) < 0.1$ ($T > 10$ yr). In this region, which is very important for practical applications, TUT models Poisson-generalized Pareto and negative binomial-generalized Pareto have the best fit. The Poisson-generalized Pareto TUT model deviates significantly from the empirical distribution for $F_x(x) > 0.1$, but the negative binomial-generalized Pareto model provides a reasonable fit up to $F_x(x) = 0.75$, above which it estimates too large values of annual minima. As it would be expected from the results of Fig. 2, TUT models with the exponential or power assumptions lead to estimates that are very far from the observations in this example.

In order to achieve a better comparison in the range of higher return periods, T-year annual minimum flows are estimated for $T = 5$,10, 25, 50 and 100 yr, on the basis of various TUT models as well as the Weibull and GEV distribution for the minima fitted directly to the observed annual minima. Results are shown in Table 1.

As was explained earlier, exponential distribution is not a good choice for the magnitudes in this example. It leads to negative values for x_{50} and x_{100}. The generalized Pareto and power distributions are seen to give close estimates for $T = 5$ yr, but for larger return periods estimated x_T values are much higher in the case of the generalized Pareto. The reason is that for $k > 0$ the annual minimum has a lower bound $x_0 -$

α/k. In this case $k = 0.61$, $\alpha = 0.97$, and the lower bound is 0.91. None of the observed minima is below this value, and the TUT models based on the generalized Pareto distribution have a very good fit at the lower tail.

It was shown that when the power and Poisson distributions are combined in the TUT modeling, the two-parameter Weibull distribution is obtained. When the x_T estimates of this case are compared with the estimates given by the direct two-parameter Weibull modeling of the annual minima, it is seen that the TUT estimates are lower for small T but higher for large T.

The GEV distribution for the minima is obtained when the generalized Pareto and Poisson distribution are combined in the TUT modeling. This distribution is compared with the GEV distribution whose parameters are estimated from the annual minima. Similar to the case of the Weibull distribution, estimates of x_T are much lower for small T ($T < 10$ yr) and much higher for large T ($T \geq 25$ yr) when the parameters are estimated by TUT modeling.

Although the variance of the observed number of annual troughs under the threshold is much larger than its mean, the use of the negative binomial distribution rather than the Poisson distribution has little effect on the x_T estimates for $T > 10$ yr. This was also observed in the case of POT modeling for floods (Önöz and Bayazit, 2001b). However, there is a marked difference for smaller return periods. For $T < 10$ yr, the negative binomial assumption leads to much larger x_T values.

Estimates of x_{10} given by different models are close to each other, with the exception of the exponential model. x_5 values are much lower when TUT modeling is used. On the other hand, the generalized Pareto assumption leads to rather high estimates for large return periods, because of the lower bound as explained above. The x_T estimates of the two-parameter Weibull and GEV distributions whose parameters are estimated from the annual minima observations are very close to each other, and higher for $T < 10$ but much lower for $T \geq 25$ yr than the estimates of the TUT models.

The observed lowest annual minima in the 53 yr series is 1.02 m^3/s, close to the x_{50} estimate of the generalized Pareto assumption. But $x_{100} = 0.97$, not much different from $x_{50} = 1.00\ m^3/s$ for the same distribution because of the lower bound effect. The GEV distribution applied directly to the annual minima has a negative lower bound equal to -0.05 in this case. It gives an estimate of $x_{100} = 0.42\ m^3/s$, almost the same as that of the two-parameter Weibull distribution ($x_{100} = 0.44\ m^3/s$), which is somewhat smaller than the TUT model estimate where the power and Poisson (or negative binomial) distributions are assumed.

It may be concluded that for this stream the negative binomial-generalized Pareto TUT model has the best fit for return periods larger than 10 yr. However, in this region it could be preferred to estimate the x_T values either by using the annual minima distributions or through the TUT model with the assumption of the power distribution for the magnitude of the troughs under the threshold, because they provide lower estimates.

For small return periods less than 10 yr, on the other hand, the annual minima distributions have the best fit to the observations. In this region, TUT models lead to lower estimates. The choice between the negative binomial-generalized Pareto TUT model and the annual minima model depends on whether it is desired to achieve the best fit to the observations or to obtain safer estimates for low flows.

5. Conclusions

It is shown that the GEV distribution for the minima is obtained when the Poisson model for the occurrence of minima under threshold is combined with the generalized Pareto distribution for their magnitudes. Thus the theoretical asymptotical distributions of the minima can also be obtained using the TUT approach. Type I and Type II distributions are not bounded from below, which may lead to negative estimates for the minima. On the other hand, Type III distribution has a lower bound that is usually positive, which may not be appropriate for some streams because the estimates of annual minima will always remain above this value.

As an alternative, the power distribution can be assumed for the magnitude of troughs under the threshold. This leads to the two-parameter Weibull distribution for the annual minima when it is combined with the Poisson model. In this case, the lower bound of the minima is zero, which is a reasonable value. However, this model cannot be used in streams which have zero flows, because zero probability is assigned to the flow being zero.

The two-parameter Weibull and GEV for minima distributions for the annual minima can give quite different x_T estimates depending on whether their parameters are estimated by the TUT model or from the observed annual minima. The final choice should be made considering both the bounds of the distributions and the physical characteristics of the low flows in each case.

The variance of the annual number of minima under threshold is usually much larger than its mean. In this case, the negative binomial distribution is more suitable than the Poisson distribution for the occurrences of minima. The negative binomial distribution is combined with the power and the generalized Pareto distributions for the magnitudes of minima.

It is seen that the choice of the negative binomial distribution has no significant effect on the x_T estimates for high return periods even when the variance/mean ratio of the annual number of minima below the threshold is much higher than one. However, the negative binomial distribution leads to higher x_T estimates for T less than 10 yr.

Distributions derived in this study for the annual minima based on the TUT model can also be used in streams with zero flows, to represent the conditional probability distribution of non-zero flows.

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