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A contribution to data combination in ill-posed downward continuation problems

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Abstract

The spaceborne gravity field missions CHAMP, GRACE and GOCE, to be realized in the coming years, will improve our knowledge of the Earth's gravity field in the short, medium and long wavelength parts. Nevertheless, it is necessary to supplement this gravity field information with available terrestrial data. While the combination of data of different origin is comparably well-understood in well-posed problems there are some open questions in combining satellite derived and terrestrial data sets in the case of improperly posed problems. In this investigation, the future satellite missions are shortly characterized. The downward continuation process and its regularization based on Tikhonov's regularization method is reviewed. The test strategies to detect the contributions of satellite derived and terrestrial gravity field information applied in this investigation are summarized. A detailed simulation based on a GRACE-type SST mission scenario and additional terrestrial data with the task to derive parameters of a gravity field representation by local base functions is performed. The investigation tries to tackle a clarification of various questions related to the data combination by numerical tests.

1. Gravity field from space with future satellite techniques

The spaceborne gravity field mission concepts SST (satellite-to-satellite tracking) and SGG (satellite gravity gradiometry) are considered to have the potential to improve our knowledge of the Earth's gravity field in the short, medium and long wavelength parts. In both alternatives, the relative motions of test masses are measured, as relative distances and velocities or as relative accelerations. In the case of SST the relative motion is measured along the line-of-sight(s) of two

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(or more) satellites. The concept is possible either in the so-called low–low or in the high–low mode. In the former case the satellites have approximately the same altitude (200–400 km). In this case both satellites are equally sensitive to gravity field irregularities. In the latter case only one (the gravity field sensitive) satellite is placed into a low orbit while the other (observing) satellite(s) describe orbits with high altitudes. In the case of SGG the elements of the gravity gradient or linear combinations, thereof, are intended to be measured simultaneously, depending on the sensitivity axes realized in the gradiometer instrument.

The basic physical principle of these various techniques is more or less the same. Because of the different distances between the test masses, it can be shown that the observations in these three cases can be related to the gravitational potential V, in case of high–low SST, to the gradient of the potential ∇V, in case of low-low SST, and to the gradient of the gradient of the gravitational potential $\nabla\nabla V$ (gravitational tensor), in case of SGG (Fig. 1).

In July 2000, the German geoscientific small satellite CHAMP (challenging mini-satellite payload for geophysical research and application) was brought into a nearly circular orbit with an inclination of 83° at an altitude of about 400 km. In a five years' mission the observation system will provide earth system related data with yet unattained accuracy. Besides various other tasks, the measurement of the stationary and time variable part of the global gravity field in the medium and long wavelength frequency range is envisaged. The American–German mission GRACE (gravity recovery and climate experiment) will open the door to a further improvement of the gravity field with respect to accuracy and resolution. GRACE is a follow-on mission to CHAMP. The mission will consist of two identical CHAMP-type satellites. The intersatellite range-rates in along-track direction between the low satellites (at an altitude of about 400 km) will be measured with μm-accuracy. GOCE (gravity field and steady-state ocean circulation explorer misson) was planned to become the first ESA Earth explorer core mission. It is a high resolution gravity field mission concept and will open a completely new range of spatial scales of the Earth's gravitational field spectrum down to 100 km wavelength. GOCE is planned to be launched in a nearly circular sun-synchronous orbit with an inclination of ≈97° at an altitude of around 250 km, carrying as the main instrument a three-axis gravity gradiometer with a precision of about $3 \times 10^{-3}\mathrm{E}/\sqrt{\mathrm{Hz}}$ (for a review of various mission proposals see: Sneeuw and Ilk, 1997).

2. The downward continuation problem and its regularization

Global gravity field recovery approaches based on future satellite missions are usually aimed at the computation of high resolution spherical harmonics coefficients. The base functions of these gravity models, the solid spherical harmonics, have global support. In this investigation an alternative approach is applied. The gravitational potential is modelled by base functions of (almost) local support. An example of such a model is the representation of the gravity field by gravity anomalies defined within certain blocks. This regional approach has advantages and disadvantages compared to the global approach. For more details about the recovery model refer to Ilk et al. (1995), Thalhammer (1994) and Rudolph, (2000). But there is no difference in the basic characteristics of a linearized model $\mathbf{A{:}X}\rightarrow\mathbf{Y}$ connecting gravity field parameters $\mathbf{x}\in\mathbf{X}$ defined at the Earth's surface and (error free) observations $\mathbf{y} \in Y$ collected at satellite altitude. $\mathbf{X}$ and $\mathbf{Y}$ are properly chosen Hilbert

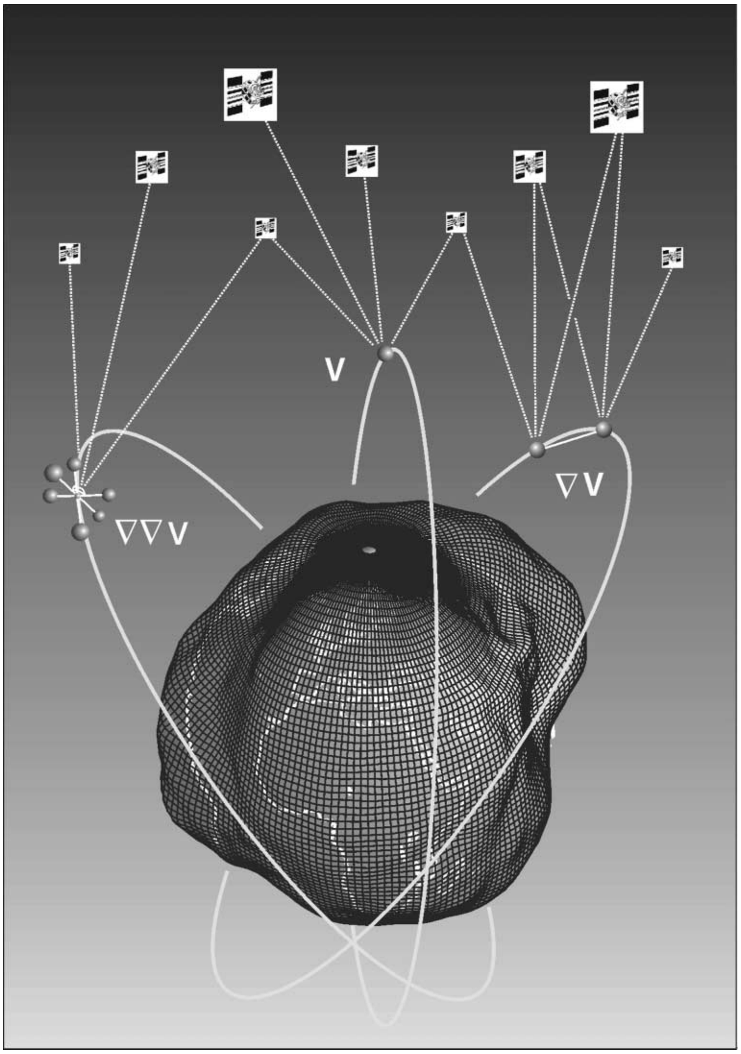

Fig. 1. Various future scenarios of gravity field measurement: SGG, high–low SST and high–low/low–low SST (from left to right).

spaces. Both models lead to so-called improperly posed problems according to the definition of Hadamard. In the present application the unknown parameters as the solution (e.g. Ilk, 1993)

$$\mathbf{x} = \mathbf{A}^{+}\mathbf{y} \text{ with the (generalized) inverse } \mathbf{A}^{+} \tag{1}$$

do not continuously depend on the observations $\mathbf{y}$. The consequence is that error-contaminated data $\mathbf{y}^{\varepsilon}$ result in excessive oscillations in the solution.

$$\mathbf{x}^{\varepsilon} = \mathbf{A}^{+}\mathbf{y}^{\varepsilon} \tag{2}$$

A way out of this problem are regularizations of $\mathbf{A}^+$. These are families of mappings

$$\{\mathbf{T}_\gamma\}_{\gamma>0}, \quad \mathbf{T}_\gamma : \mathbf{Y} \rightarrow \mathbf{X}, \tag{3}$$

with a regularization parameter $\gamma = \gamma(\varepsilon, \mathbf{y}^\varepsilon)$, $\mathbf{y}^\varepsilon \in \mathbf{Y}$, $\|\mathbf{y} - \mathbf{y}^\varepsilon\| \leqslant \varepsilon$. The regularizations converge pointwise to $\mathbf{A}^+$ at its domain $\mathrm{D}(\mathbf{A}^+)$

$$\lim_{\substack{\varepsilon \to 0 \\ \mathbf{y}^\varepsilon \to \mathbf{y}}} \mathbf{T}_{\gamma(\varepsilon,\mathbf{y}^\varepsilon)}\mathbf{y}^\varepsilon = \mathbf{A}^+\mathbf{y} \ \text{ for } \ \mathbf{y} \in \mathrm{D}(\mathbf{A}^+). \tag{4}$$

For the regularization parameters the relation holds

$$\lim_{\substack{\varepsilon \to 0 \\ \mathbf{y}^\varepsilon \to \mathbf{y}}} \gamma(\varepsilon, \mathbf{y}^\varepsilon) = 0. \tag{5}$$

The regularization parameter can be chosen a priori in case of $\gamma(\varepsilon)$ or a posteriori in case of $\gamma(\varepsilon, \mathbf{y}^\varepsilon)$. Because of the fact that only disturbed observations $\mathbf{y}^\varepsilon$ are available only an approximation for the unknown parameters $\mathbf{x}$,

$$\mathbf{x}_\gamma^\varepsilon = \mathbf{T}_\gamma \mathbf{y}^\varepsilon, \tag{6}$$

can be computed. The total error of this approximation is composed by two parts

$$\mathbf{x}_\gamma^\varepsilon - \mathbf{x} = \mathbf{T}_\gamma(\mathbf{y}^\varepsilon - \mathbf{y}) + \left(\mathbf{T}_\gamma - \mathbf{A}^+\right)\mathbf{y}. \tag{7}$$

Obviously the first part is the data error,

$$\mathbf{x}_\varepsilon := \mathbf{T}_\gamma(\mathbf{y}^\varepsilon - \mathbf{y}). \tag{8}$$

The second part is the regularization error or bias,

$$\mathbf{x}_b := \left(\mathbf{T}_\gamma - \mathbf{A}^+\right)\mathbf{y}, \tag{9}$$

which can be written as

$$\mathbf{x}_b = \left(\mathbf{T}_\gamma \mathbf{A} - \mathbf{I}\right)\mathbf{x}. \tag{10}$$

The difference

$$\mathbf{T}_\gamma \mathbf{A} - \mathbf{I} \text{ with the unit operator } \mathbf{I} \tag{11}$$

is the regularization error or bias of the regularized mapping $\mathbf{T}_\gamma$. For example, it can be shown that—typical for improperly posed problems—the total error can not be made arbitrarily small. Usually, data error and regularization error act in a contrary direction. To which extend this holds depends on the special application. In general, it is possible to minimize the total error by a proper balancing of data and regularization error. But it is not possible to exploit the full data accuracy in improperly posed problems and the regularization error cannot be reduced arbitrarily. The degree of ill-posedness is a property of the operator $\mathbf{A}$ together with the solution space $\mathbf{X}$ and the data space $\mathbf{Y}$. Therefore, stability of a problem could be restored also by changing the spaces $\mathbf{X}$ and $\mathbf{Y}$ and their norms. But this seems to be possible only in special cases. A regularization method successfully applied in downward continuation problems is due to Tikhonov–Phillips. The method consists essentially in minimizing a smoothing functional of the type

$$\mathcal{F}_\gamma = \left\|\mathbf{Ax} - \mathbf{y}\right\|_2^2 + \gamma^2 \Omega^{(n)}(\mathbf{x}) \tag{12}$$

$\mathcal{F}_\gamma$ is called a Tikhonov–Phillips functional and $\Omega^{(n)}(\mathbf{x})$ is the so-called problem stabilizer. $\gamma^2 > 0$ is the regularization parameter which can be defined in various ways. In many cases a problem stabilizer of type $n = -1$ is applied,

$$\Omega^{(-1)}(\mathbf{x}) = \|\mathbf{P}_x \mathbf{x}\|^2, \tag{13}$$

corresponding to the smoothing functional

$$\mathcal{F}_\gamma = \left\|\mathbf{Ax} - \mathbf{y}\right\|_2^2 + \gamma^2 \|\mathbf{P}_x \mathbf{x}\|^2. \tag{14}$$

The regularized solution in this case reads

$$\mathbf{x}_\gamma^\varepsilon = \mathbf{T}_\gamma \mathbf{y}^\varepsilon, \text{ with the regularization } \mathbf{T}_\gamma = \left(\mathbf{A}^T\mathbf{A} + \gamma^2 \mathbf{P}_x^T \mathbf{P}_x\right)^{-1} \mathbf{A}^T. \tag{15}$$

In the following we understand these quantities in discretized form so that

$$\mathbf{Ax} = \mathbf{y}^\varepsilon, \text{ with } \mathbf{y}^\varepsilon \in \mathbf{M}_{m1}(\mathbb{R}), \ \mathbf{x} \in \mathbf{M}_{n1}(\mathbb{R}), \ \mathbf{A} \in \mathbf{M}_{nn}(\mathbb{R}), \ m > n. \tag{16}$$

$\mathbf{M}_{mn}(\mathbb{R})$ is the set of all matrices with m rows and n columns and elements from $\mathbb{R}$, $m, n \in \mathbb{N}$. To find a best approximation for the unknowns $\mathbf{x}$ all available prior information has to be taken into account. These are

- the a priori variance–covariance matrix of the observations $\mathbf{C}_y \in \mathbf{M}_{mm}(\mathbb{R})$,
- the a priori variance–covariance matrix of the unknowns $\mathbf{C}_x \in \mathbf{M}_{nn}(\mathbb{R})$,
- the a priori estimates of the unknowns $\mathbf{x}$, $\overset{p}{\mathbf{x}} \in \mathbf{M}_{n1}(\mathbb{R})$.

If the a priori variance–covariance matrices are not know then instead weight matrices can be used which are related to the variance–covariance matrices by the relations:

$$\mathbf{P}_y = \mathbf{C}_y^{-1}, \quad \mathbf{P}_{\overset{p}{x}} = \mathbf{C}_x^{-1}, \tag{17}$$

Following the regularization procedure the regularized solution reads:

$$\mathbf{x}_\gamma^\varepsilon = \left(\mathbf{A}^T\mathbf{P}_y\mathbf{A} + \gamma^2\mathbf{P}_{\overset{p}{x}}\right)^{-1}\left(\mathbf{A}^T\mathbf{P}_y\mathbf{y}^\varepsilon + \gamma^2\mathbf{P}_{\overset{p}{x}}\overset{p}{\mathbf{x}}\right) \tag{18}$$

with the error variance–covariance matrix of the regularized (biased) estimate

$$\mathbf{E}_x = \sigma_0^2\left(\mathbf{A}^T\mathbf{P}_y\mathbf{A} + \gamma^2\mathbf{P}_{\overset{p}{x}}\right)^{-1}\mathbf{A}^T\mathbf{P}_y\mathbf{A}\left(\mathbf{A}^T\mathbf{P}_y\mathbf{A} + \gamma^2\mathbf{P}_{\overset{p}{x}}\right)^{-1} \tag{19}$$

and a variance factor σ_0. If the regularized estimate is viewed as (Bayesian) estimate with a priori variance–covariance matrix $(\gamma^2\mathbf{P}_{\overset{p}{x}})^{-1}$ then we arrive at an error variance–covariance matrix for the regularized unknowns

$$\mathbf{E}_x = \sigma_0^2\left(\mathbf{A}^T\mathbf{P}_y\mathbf{A} + \gamma^2\mathbf{P}_{\overset{p}{x}}\right)^{-1}. \tag{20}$$

3. Combination of satellite derived solutions and terrestrial data

If we want to combine the solution for the gravity field parameters $\overset{s}{\mathbf{x}} := \mathbf{x}_\gamma^\varepsilon$ derived from satellite observations with gravity field parameters derived from terrestrial measurements $\overset{t}{\mathbf{x}}$ then we come up with the combined result by using the respective weight matrices $\mathbf{P}_{\overset{s}{x}}$ and $\mathbf{P}_{\overset{t}{x}}$,

$$\mathbf{x} = \left(\mathbf{P}_{\overset{s}{x}} + \mathbf{P}_{\overset{t}{x}}\right)^{-1}\left(\mathbf{P}_{\overset{s}{x}}\overset{s}{\mathbf{x}} + \mathbf{P}_{\overset{t}{x}}\overset{t}{\mathbf{x}}\right), \tag{21}$$

with the error variance–covariance matrix

$$\mathbf{E}_x = \sigma_0^2\left(\mathbf{P}_{\overset{s}{x}} + \mathbf{P}_{\overset{t}{x}}\right)^{-1}. \tag{22}$$

We can write the result (21) in the form (e.g. Koch, 2000)

$$\mathbf{x}-\overset{t}{\mathbf{x}} = \mathbf{P}^{-1}_{\overset{t}{x}}\left(\mathbf{P}^{-1}_{\overset{t}{x}} + \mathbf{P}^{-1}_{\overset{s}{x}}\right)^{-1}\left(\overset{s}{\mathbf{x}} - \overset{t}{\mathbf{x}}\right), \tag{23}$$

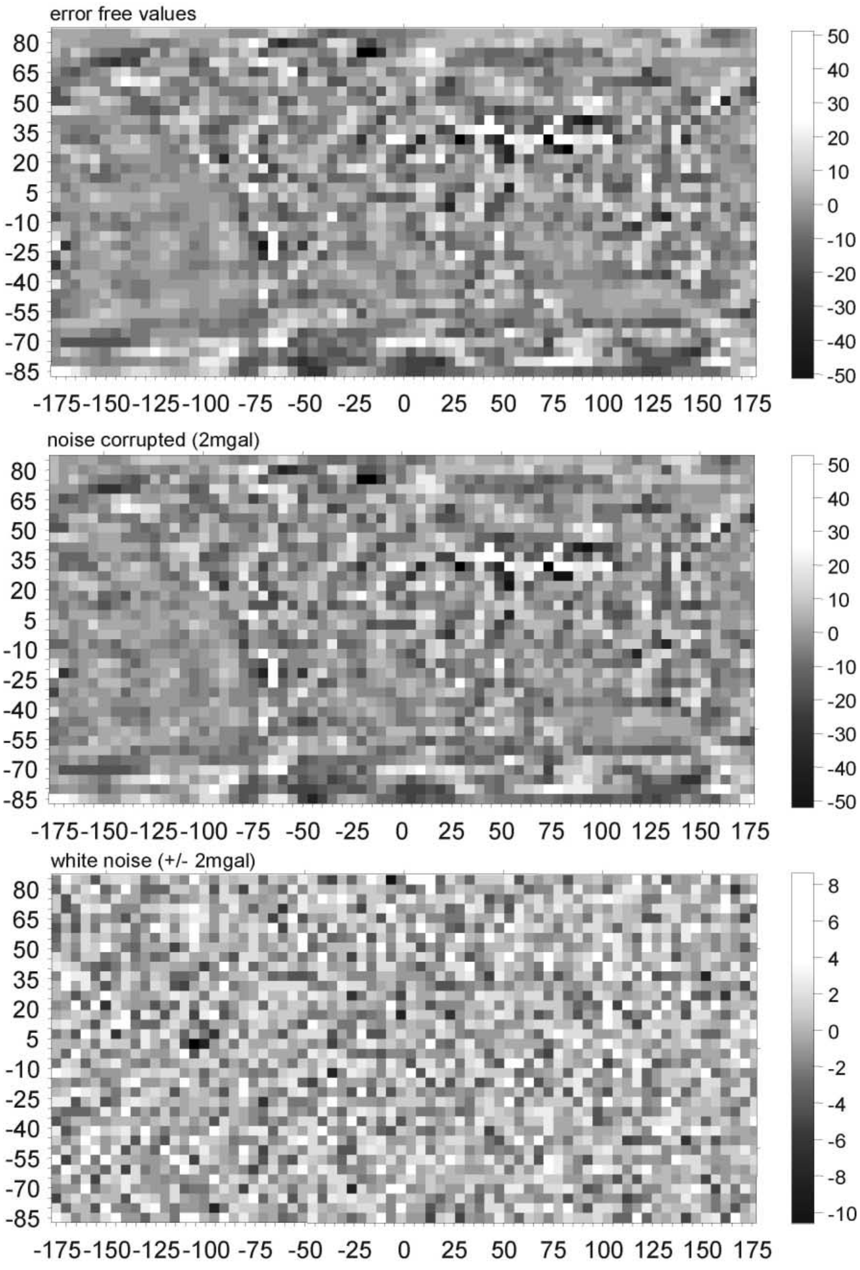

Fig. 2. Terrestrial data: error-free, noise corrupted, white noise (top to bottom, gray-scale bar values in mGal).

or alternatively,

$$\mathbf{x}-\overset{s}{\mathbf{x}}=\boldsymbol{P}^{-1}_{\overset{s}{x}}\left(\mathbf{P}^{-1}_{\overset{s}{x}}+\mathbf{P}^{-1}_{\overset{t}{x}}\right)^{-1}\left(\overset{t}{\mathbf{x}}-\overset{s}{\mathbf{x}}\right). \tag{24}$$

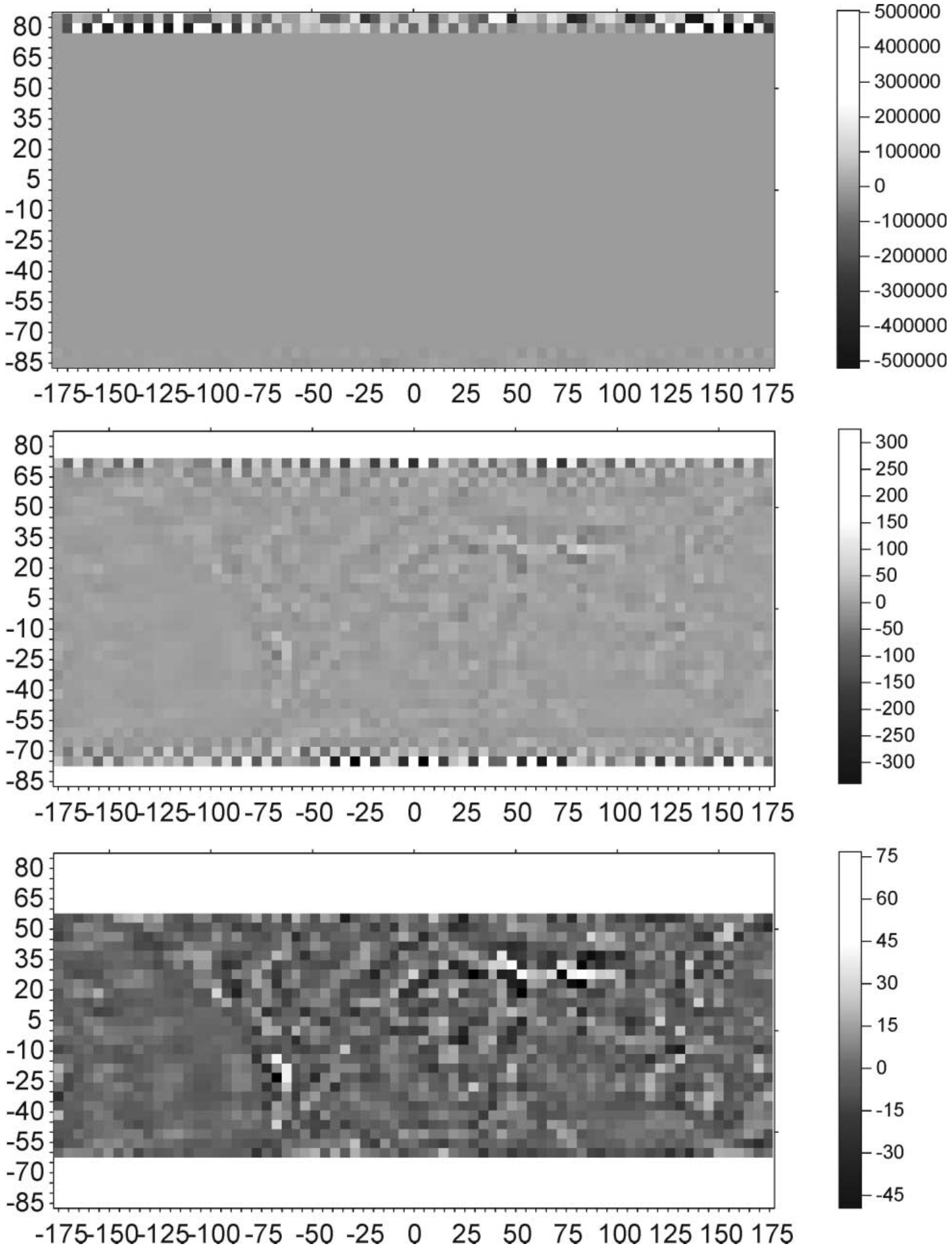

Fig. 3. Unregularized satellite solution with various magnifications (gray-scale bar values in mGal).

Formula (23) shows how the terrestrial solution, considered as prior information, will be changed if the satellite solution is being added. Correspondingly, formula (24) shows how the satellite solution, considered as prior information, will be changed if the terrestrial solution is being added.

Another aspect of data combination arises if Eq. (18) is inserted in Eq. (21) by setting $\overset{s}{\mathbf{x}} = \mathbf{x}_{\gamma}^{\varepsilon}$ and Eq. (20) in Eq. (21) by setting $\mathbf{P}_{\overset{s}{x}}^{-1} = \mathbf{E}_x/\sigma_0^2$. Then we get as combined solution

$$\mathbf{x} = \left(\mathbf{A}^T\mathbf{P}_y\mathbf{A} + \gamma^2\mathbf{P}_{\overset{p}{x}} + \mathbf{P}_{\overset{t}{x}}\right)^{-1}\left(\mathbf{A}^T\mathbf{P}_y\mathbf{y}^{\varepsilon} + \gamma^2\mathbf{P}_{\overset{p}{x}}\overset{p}{\mathbf{x}} + \mathbf{P}_{\overset{t}{x}}\overset{t}{\mathbf{x}}\right). \tag{25}$$

Obviously in case of improperly posed problems the a priori estimates $\overset{p}{\mathbf{x}}$ of the solution $\mathbf{x}$ with the weight matrix $\mathbf{P}_{\overset{p}{x}}$ are treated in a different way in the combination procedure than the solution $\overset{t}{\mathbf{x}}$ derived from terrestrial data with the weight matrix $\mathbf{P}_{\overset{t}{x}}$. The a priori data set is included in the regularization procedure while the terrestrial data set is not. This difference in treating additional information about the unknowns does not occur in case of properly posed problems where a regularization is not necessary, that means, in those cases where the regularization factor is set

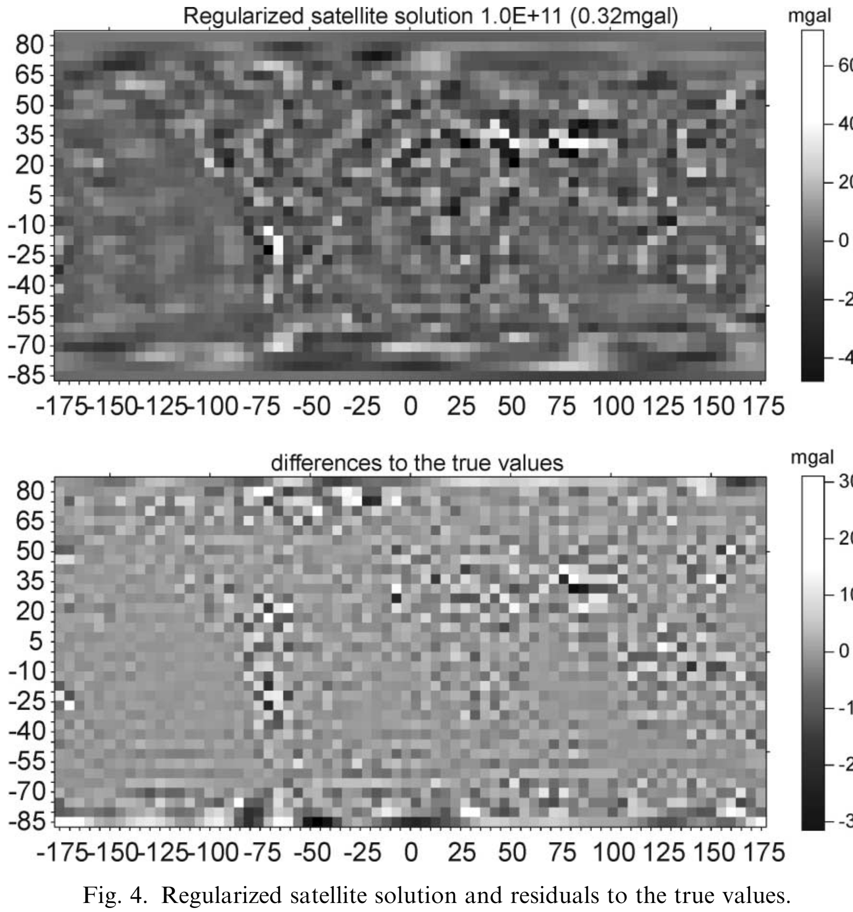

Fig. 4. Regularized satellite solution and residuals to the true values.

to one. Besides this arbitrariness of handling the terrestrial data, Eq. (25) shows that the terrestrial data will only influence the final result in dependency of the size of an optimal regularization parameter. If a very large regularization parameter is necessary then the terrestrial data will not have much effect on the combined solution. This holds also in case of missing prior information, that means, in those cases where the prior estimates $\mathbf{P}_{\overset{p}{x}}$ are set to zero.

Another possibility to get an idea of the effect of terrestrial data on the final combined result can be derived by using the so-called partial redundancy numbers related to the terrestrial data (Schwintzer, 1990). The partial redundancy of each local gravity field parameter i reflects the contribution of the terrestrial (likewise satellite) information to the corresponding result derived from satellite (likewise terrestrial) data. These numbers can be easily computed using the diagonal elements of the matrices $\mathbf{P}_{\overset{t}{x}}$ and $\mathbf{E}_x$ [Eq. (22)]:

$$(f_{\overset{t}{x}})_i = 1 - (\mathbf{E}_x)_i\left(\mathbf{P}_{\overset{t}{x}}\right)_i. \tag{26}$$

A low partial redundancy number $(f_{\overset{t}{x}})_i \in [0, 1]$ indicates a poor sensitivity of the satellite data to the gravity field parameter. In case of $(f_{\overset{t}{x}})_i = 0$ no information at all comes from the satellite

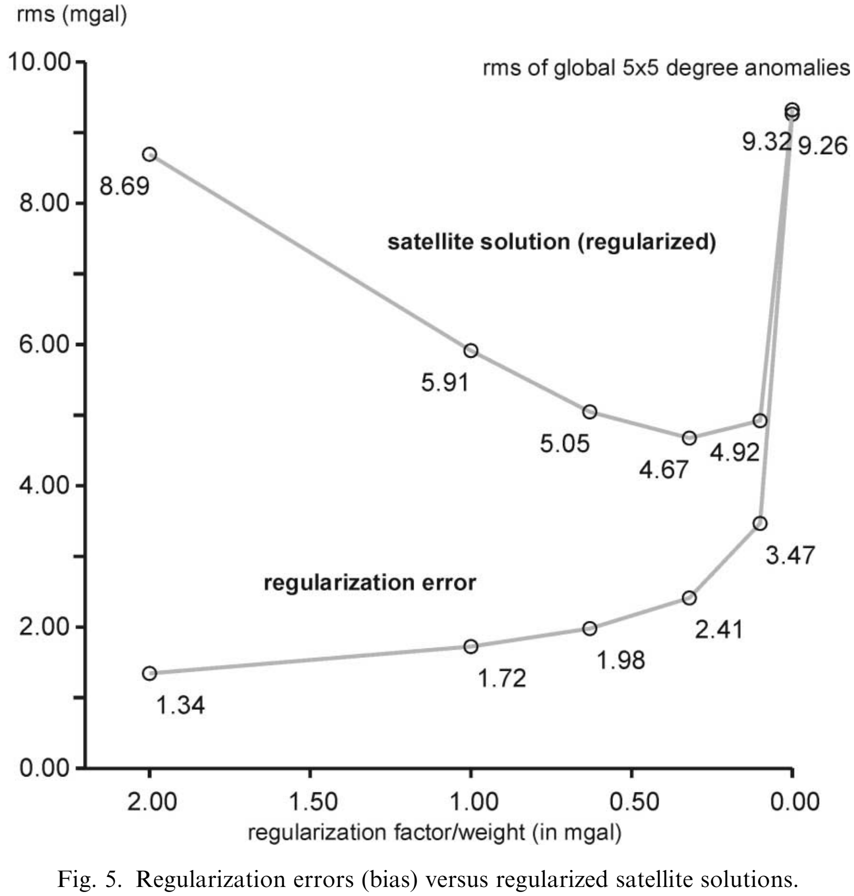

Fig. 5. Regularization errors (bias) versus regularized satellite solutions.

observations. With increasing values $(f_{\underset{x}{t}})_i$ the influence of the terrestrial data decreases because of the growing sensitivity of the satellite data to the respective gravity field parameter. In case of $(f_{\underset{x}{t}})_i = 1$ the terrestrial gravity field value contributes no information to the resulting gravity field parameter.

4. Simulation computations

4.1. Simulation scenarios

To investigate the characteristics of a combination of satellite derived gravity anomalies with terrestrial gravity anomalies a GRACE low–low SST mission scenario has been simulated. The satellite configuration consists of two low orbiting satellites with a baseline of 460 km between the satellites. The two satellites are placed in the same orbit with an inclination of about 83° and a

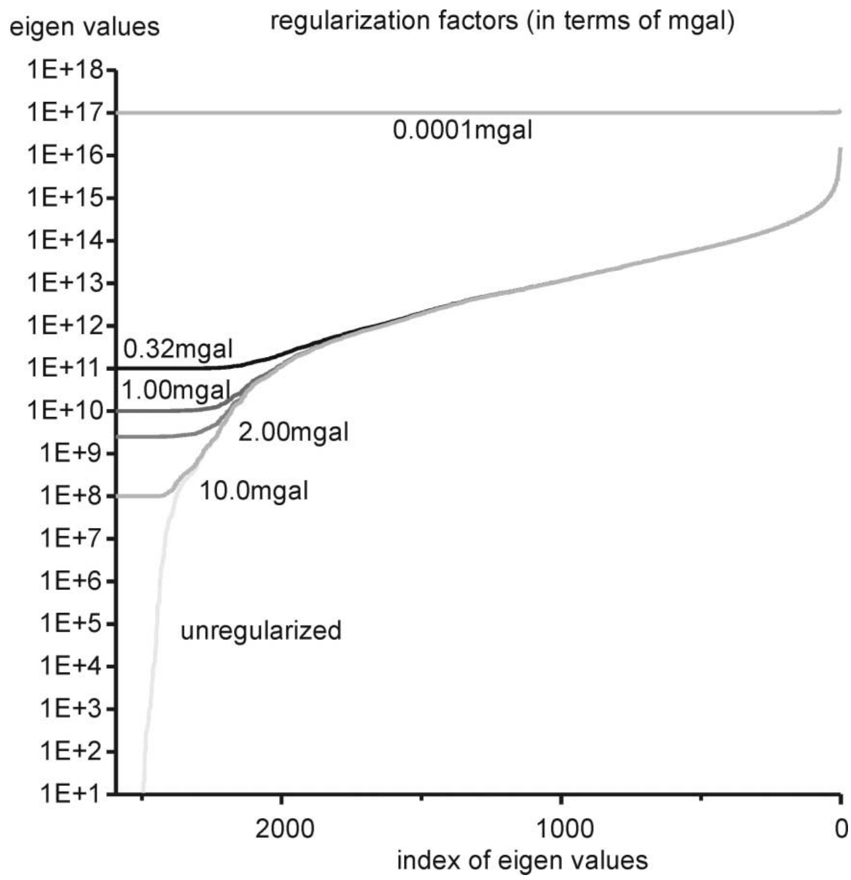

Fig. 6. Regularization of satellite solutions, reflected in the eigenvalue spectra.

numerical excentricity of 0.001, following each other along-track. Consequently, the orbits are approximately circular with an altitude above ground of about 400 km. The orbit characteristics of the GRACE mission have been changed recently so that the problems caused by a non-polar orbit are obsolete. Nevertheless, the examples shown in this investigation are important in so far as

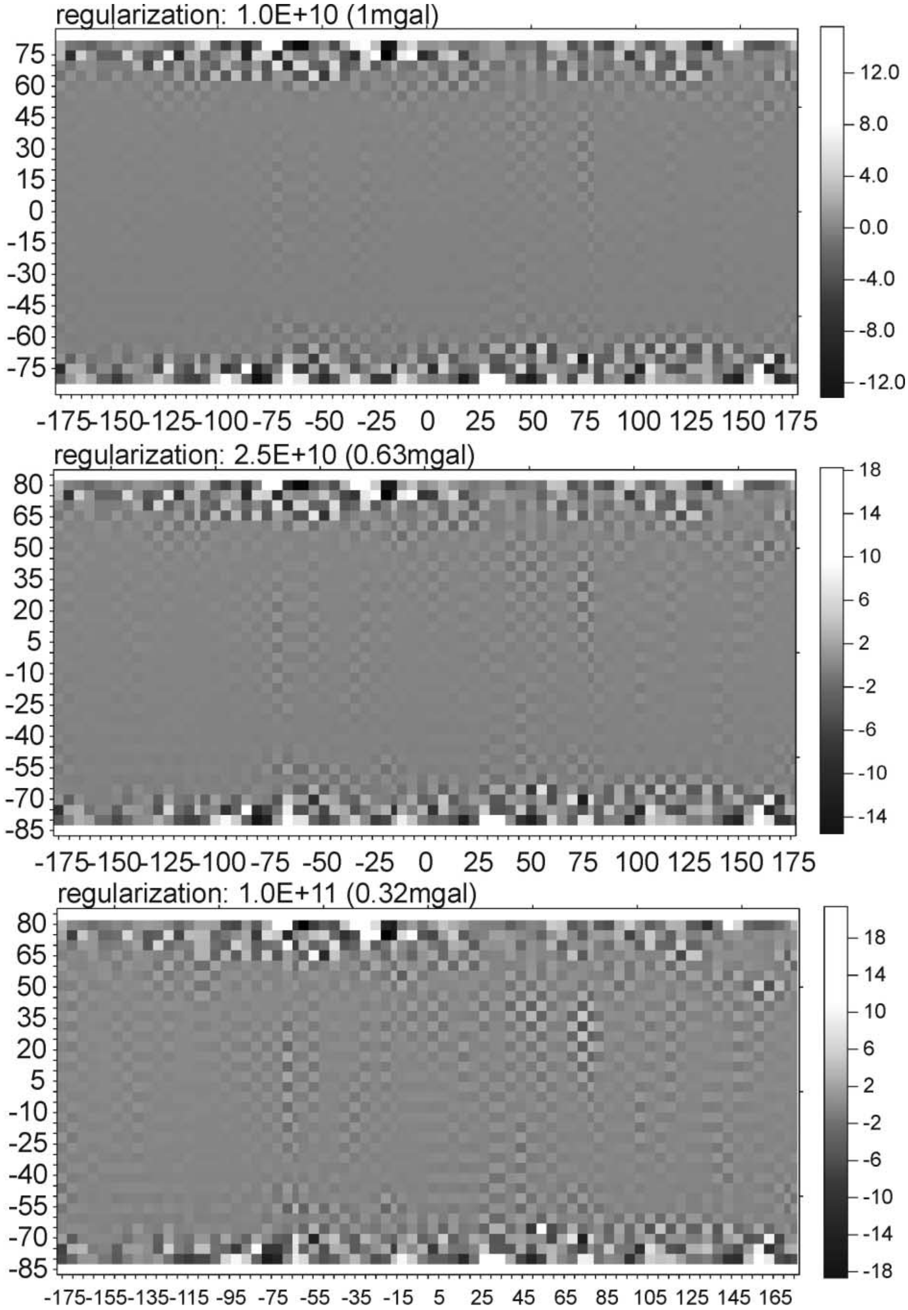

Fig. 7. Geographical distribution of the bias in case of various regularizations.

they show the instability problems of non-polar low–low SST mission scenarios and the characteristics of a combination with gravity field information at ground level. To restrict the computation effort, the mission period was considered to be about 31 days providing 535, 000 simulated pseudo-observations. As original observations range-rate measurements are used which are transferred into pseudo observations in the spectral domain. A white noise level of 1 μm/s for the SST-link was assumed. The pseudo-observations were arranged in the column matrix **y** of Eq. (1). All orbits were generated by numerical integration using the EGM96 gravity model. The recovery area covers the whole Earth, discretized by $5^\circ \times 5^\circ$ block anomalies. Based on this gravity field representation and the pseudo-observations the linearized model **A** has been established. The least-squares adjustment results (with varying regularizations) are computed and denoted in the following by $\overset{s}{\mathbf{x}}$. Also the terrestrial gravity $\overset{t}{\mathbf{x}}$ data have been derived based on the EGM96 gravity model and corrupted by white noise of varying standard deviations as indicated in the examples (Fig. 2 shows the terrestrial data: error-free, white noise corrupted, white noise). For the a priori weight matrix of the unknowns we assumed the unit matrix: $\mathbf{P}_{\overset{p}{x}} = \mathbf{I}$. The a priori values are set to zero: $\overset{p}{\mathbf{x}} = \boldsymbol{0}$.

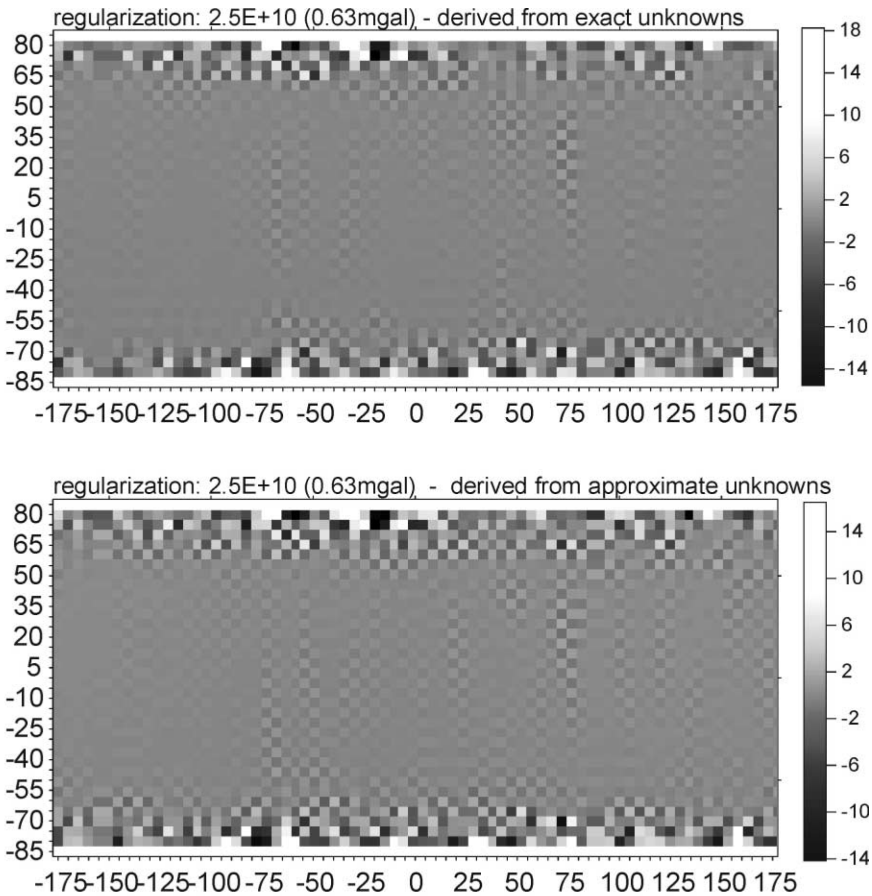

Fig. 8. Geographical distribution of the bias using true parameters and estimates.

4.2. Discussion of the simulation results

As expected, the unregularized solution shows strong oscillation effects especially in the polar regions caused by the polar observation gaps but also by the smaller compartments near the polar areas in very high (respectively low) latitudes because of converging meridians. In Fig. 3 the global recovery area is shown as well as various magnifications excluding the critical polar areas. The bottom graphs make clear that increased oscillations also occur in areas of a rough gravity field (Asia, South America). The following questions shall be investigated in the following:

- what is the mechanism of regularization and what is the size of the regularization error?
- what is the effect of regionally restricted terrestrial data sets?
- what are the contributions of terrestrial and satellite data in a combined solution?

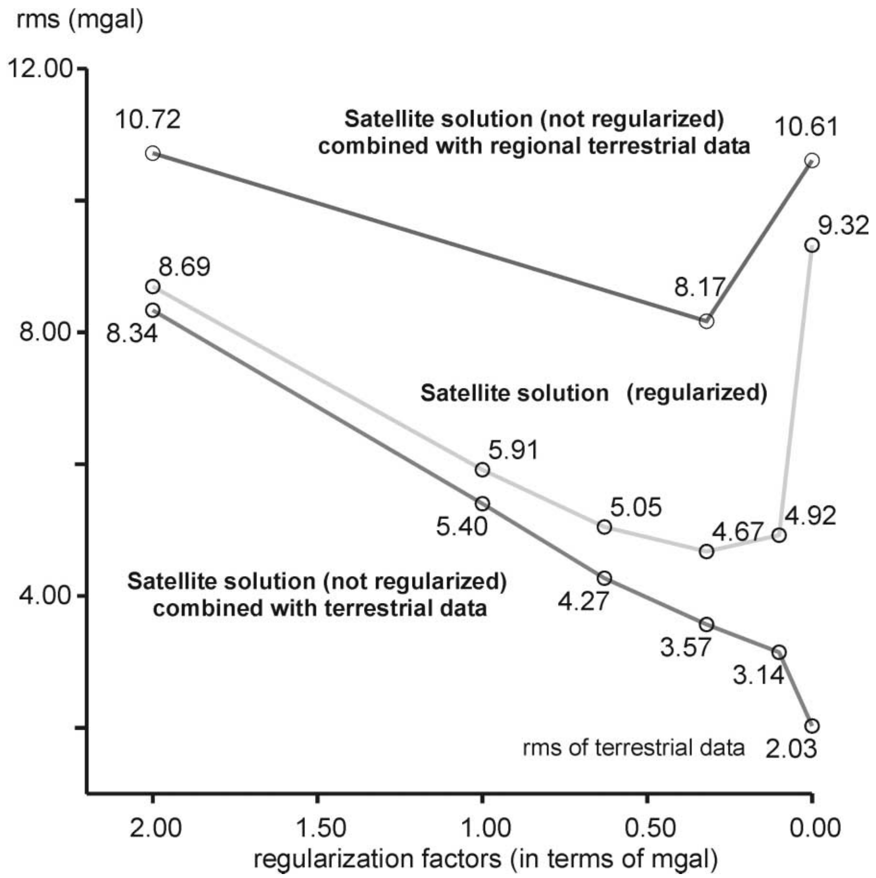

Fig. 9. Effect of regional and global terrestrial data on the unregularized satellite solution (regularization factor corresponds to the noise level).

4.2.1. Effect of regularization and regularization error

The basic effect of regularization has been explained already: the regularization factor balances the effects of data error and regularization error. If the regularization factor is set to zero then the data noise will be strongly amplified and the residuals to the true values become very large, resulting in a very high root mean square (rms) error for the set of unknowns. An increasing regularization factor acts as a smoothing filter, the data errors will be damped and the differences to the true values become smaller and smaller until they reach a minimum. The corresponding value of the regularization factor is usually considered as the "optimal" regularization factor (which is not easy to find if no true values are available!). Fig. 4 Fig. shows these recovery results; there are still some oscillations left in the unstable regions. Further increase of the regularization factor continues in damping the signal until the unknowns become zero within the recovery area. The rms-value of the differences to the true values assume the rms-value of the true values within the recovery area. The situation is shown in Fig. 5, where the rms-value of the true values is around 9.3 mgal. The rms-value of the regularization error starts with a small number in case of a moderate regularization and assumes again the rms-value of the true values. It should be mentioned that the regularization factor in the figures of this paper is expressed in mgal to make the

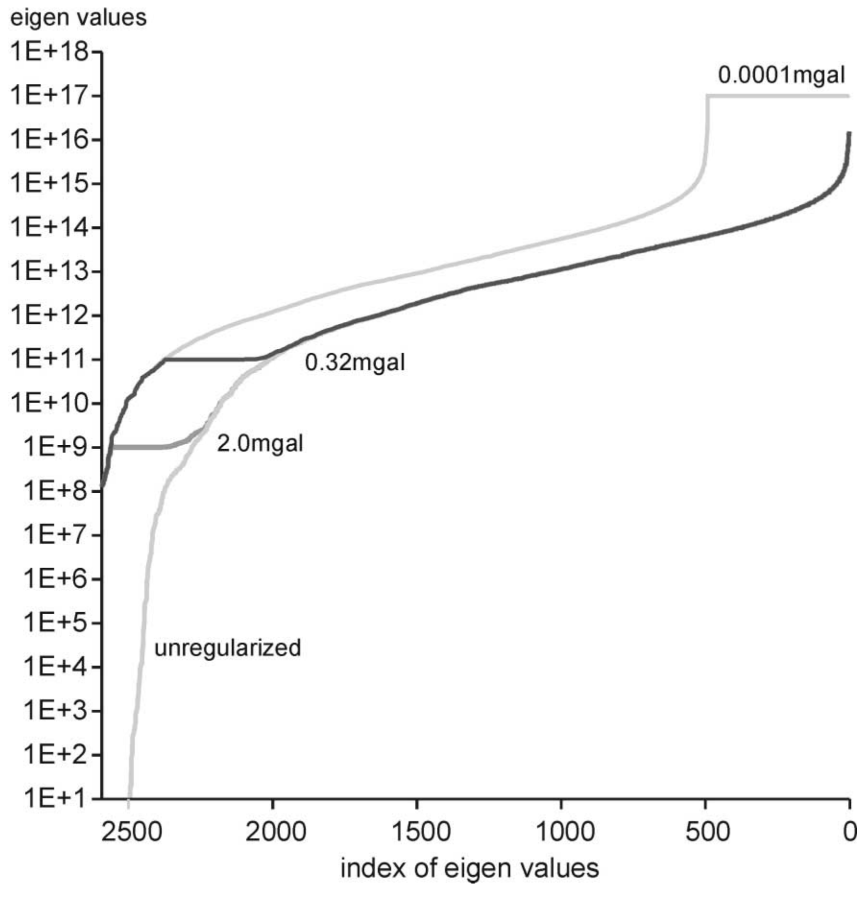

Fig. 10. Combinations of regional terrestrial data with satellite data, reflected in the eigenvalue spectra.

corresponding contributions from the regularization and the a priori values of the unknowns as well as from the terrestrial data comparable in formula (25). Therefore, in case of $\gamma^2 \rightarrow \infty$ the corresponding value in terms of mgal tends to zero.

As is well-known, this behavior is reflected also in the eigenvalue spectra of the normal matrix as shown in Fig. 6. While in case of the unregularized solution the condition number as quotient

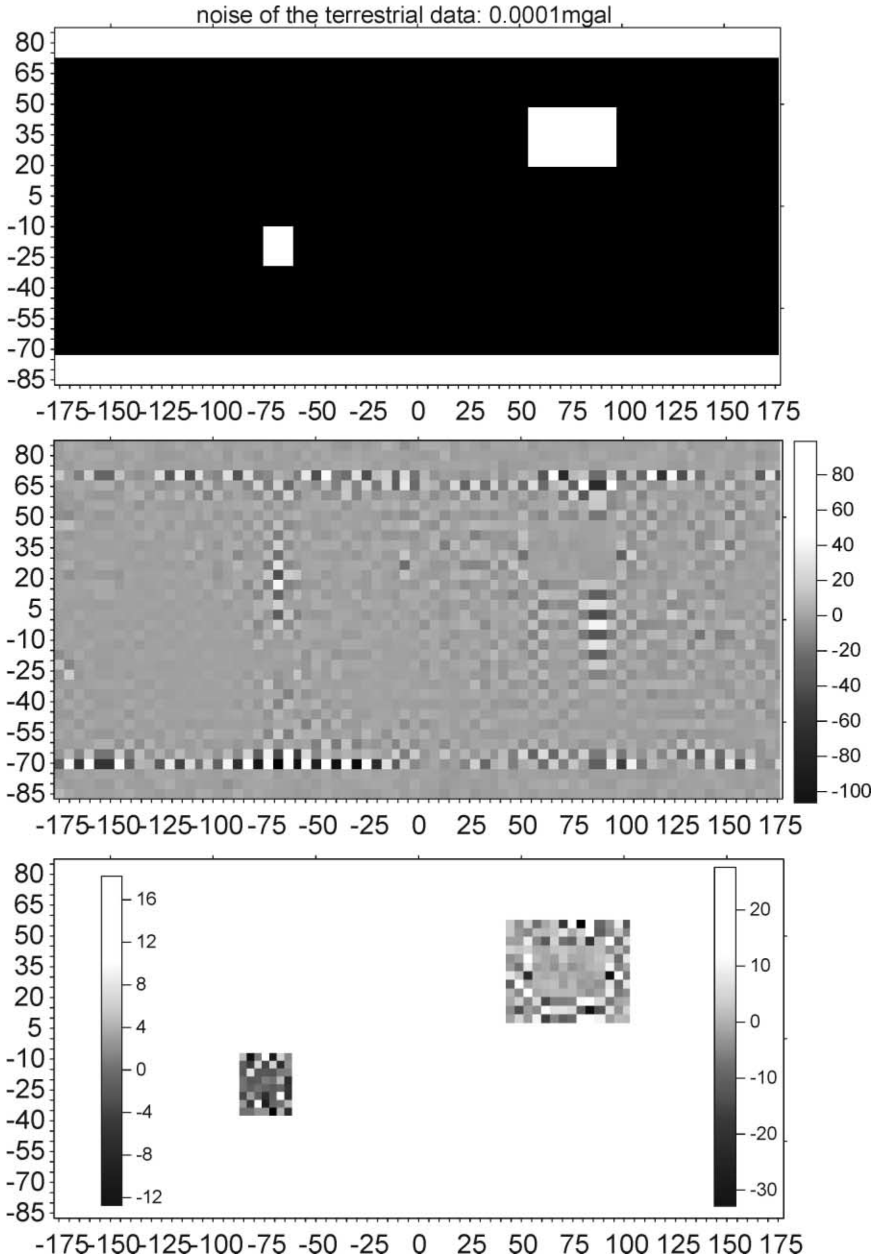

Fig. 11. Effect of regional terrestrial data (±0.0001 mGal) on the unregularized satellite solution.

between the largest and the smallest eigenvalue is very large, it is reduced by regularization, where the lower end of the spectra is shifted due to the regularization factor. The condition number approaches the value one in the case of extreme damping of the parameters. So far, the results of Figs. 5 and 6 are as expected. More interesting is the geographical distribution of the regularization error or bias. Fig. 7 shows the results of three regularization cases computed due to Eq. (10).

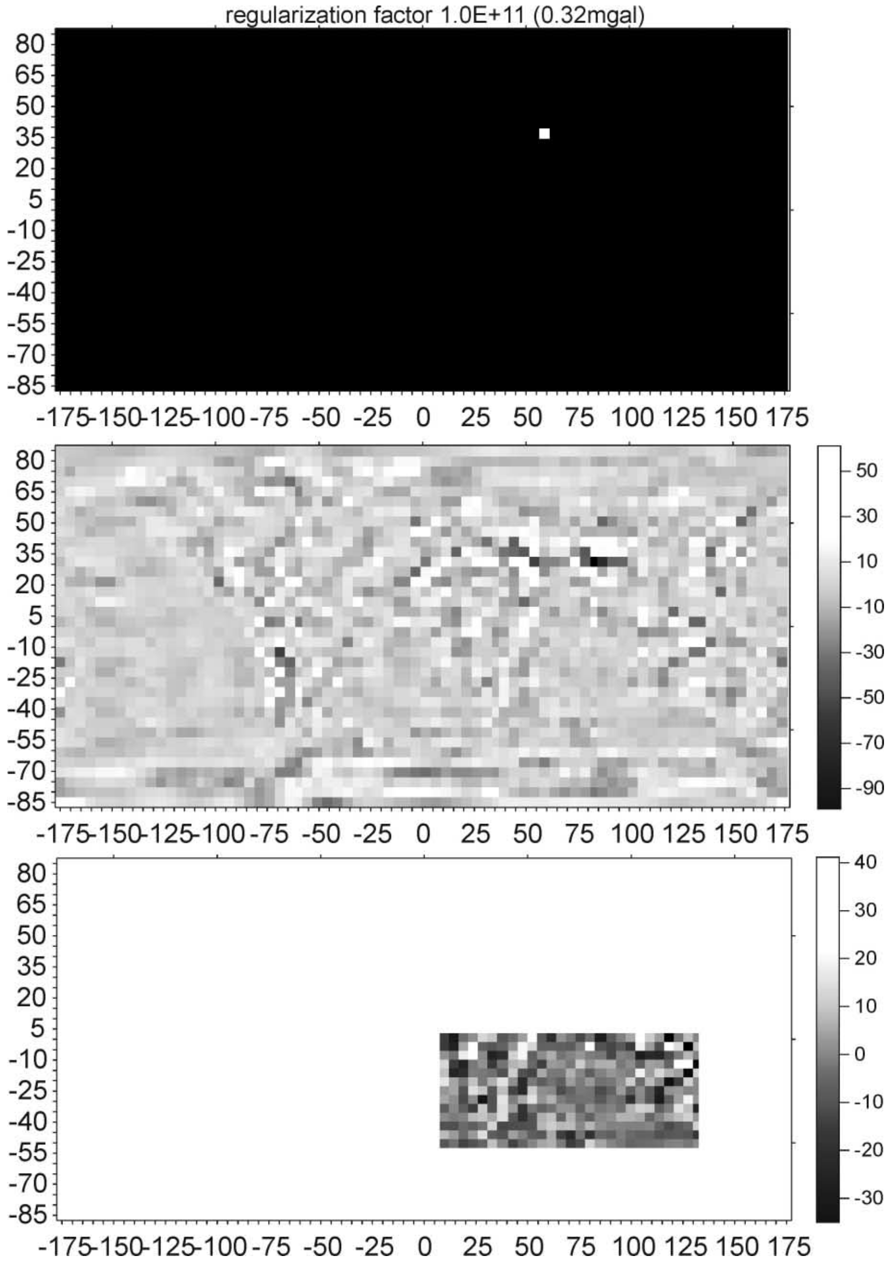

Fig. 12. Effect of a fixed terrestrial gravity anomaly (±0.0001 mGal) on the regularized satellite solution.

It may be surprising that the bias plays an important role only in those areas where the unknowns are very weakly determined.

These are first of all the polar regions but also those areas with very rough gravity field features. There is no large effect on the gravity field parameters in the areas of stable computation. It seems that this behavior is typical for base functions with local support and may not hold by using surface spherical harmonics. But further investigations are necessary. The computation of the bias according to formula (10) is only possible, if the true values of the parameters are known. This is usually not the case, but also by using the estimates instead of the true values one obtains more or less the same result (see Fig. 8). Therefore, in this way it is basically possible to get an impression of the effect of bias on the result.

4.2.2. *Effect of regional terrestrial data*

The regionally restricted instabilities, detected in the last chapter and represented in Figs. 4 and 7 and, may suggest to introduce additional terrestrial data in those regions where the gravity field parameters are only weakly determined. To stabilize the satellite derived anomalies in the polar regions and in two regions with rough gravity field features, terrestrial data with varying weights

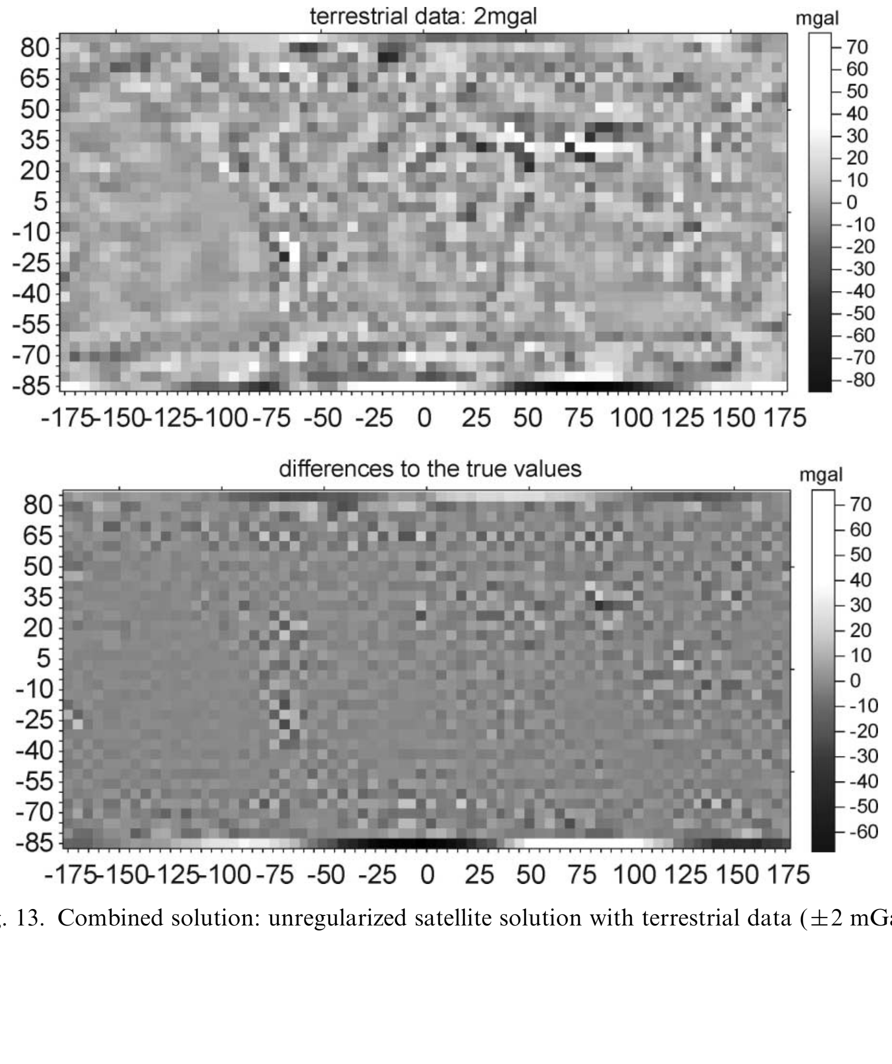

Fig. 13. Combined solution: unregularized satellite solution with terrestrial data (±2 mGal).

are included in the adjustment procedure. The graph at the top of Fig. 9 shows the rms-values of not regularized solutions combined with regional terrestrial data in case of three different noise levels (2.0, 0.32, 0.0001 mGal). Fig. 11 shows at the top these regions with the terrestrial data (marked by white areas). The center graph shows the residuals to the true values. In the bottom graph of Fig. 11 the residuals within the two supported regions are shown, slightly magnified to see the details. Unexpectedly, the overall results are not better than the optimally regularized solutions without terrestrial data in the unstable areas. The results are even much worse, caused by the strong oscillations around the supported regions as Fig. 11 shows for the error-free case of the terrestrial data (±0.001 mGal). Only the combined results within the supported regions correspond to the noise level of the terrestrial data. These results are again reflected in the eigenvalue spectra of the normal matrices (Fig. 10). In contrast to the regularized cases only those eigenvalues corresponding to the number of terrestrial anomalies included are changed but the condition number is still very large—it seems that the stronger the constraints the worse is the result. Fig. 9, upper graph, shows a minimum in case of a noise level of 0.32 mGal for the terrestrial data. Nevertheless, even in this case the result is far away from the optimally regularized satellite only solution. In Fig. 12 the effect of a single fixed gravity anomaly on a regularized satellite solution is

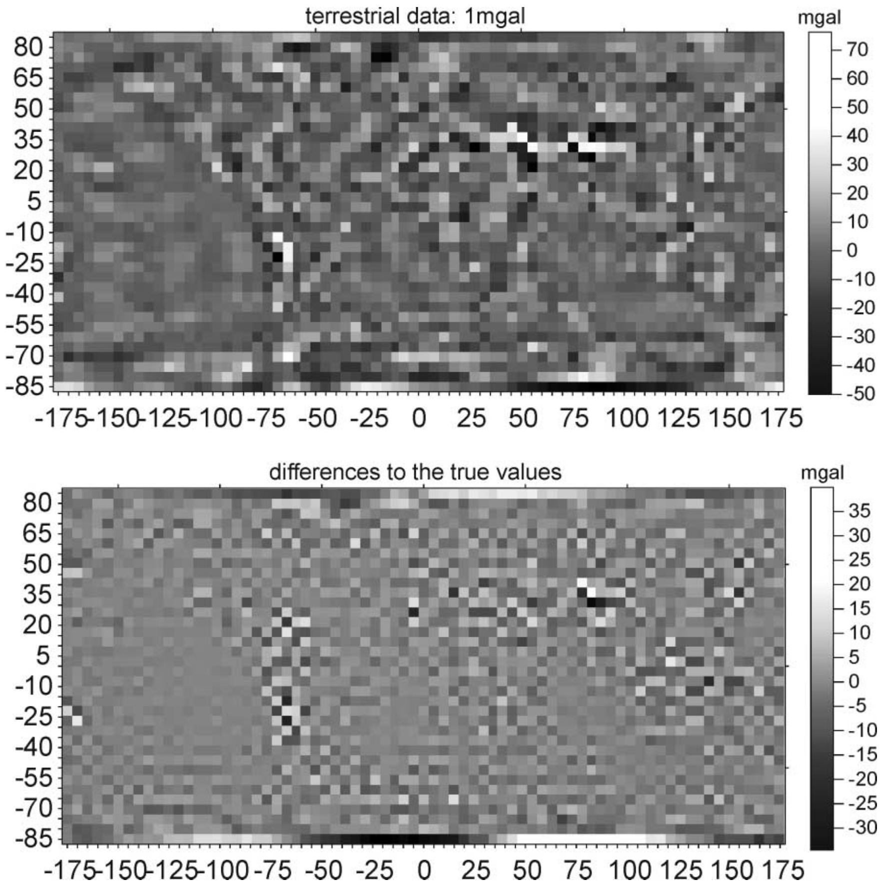

Fig. 14. Combined solution: unregularized satellite solution with terrestrial data (±1 mGal).

presented. Again, the quality of the solution is not improved—the oscillations around the fixed location are rather large leading to a worse total result compared to the regularized satellite case without the fixed gravity anomaly (Fig. 4).

4.2.3. Contributions of terrestrial and satellite data in a combined solution

In contrast to the terrestrial data restricted to certain regions the support of a data set of terrestrial data covering the complete recovery area is able to improve the combined solution. This

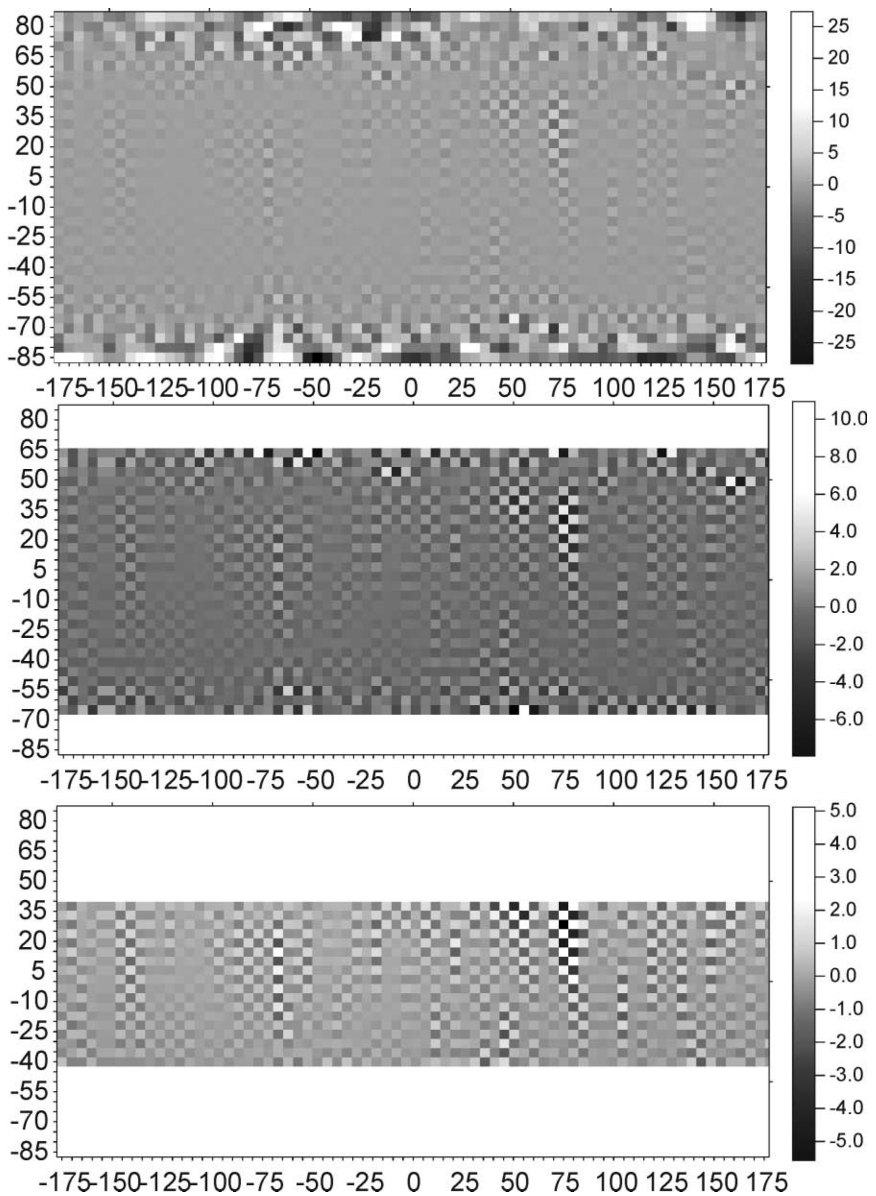

Fig. 15. Difference between regularized satellite solution and unregularized satellite solution combined with terrestrial data.

can be expected, because the normal matrix of the terrestrial data acts in the same way as the regularization matrix [Eq. (27)].

$$\mathbf{x} = \left(\mathbf{A}^T\mathbf{P}_y\mathbf{A} + \gamma^2\mathbf{I} + \mathbf{P}_{\overset{t}{x}}\right)^{-1}\left(\mathbf{A}^T\mathbf{P}_y\mathbf{y}^{\varepsilon} + \mathbf{P}_{\overset{t}{x}}\overset{t}{\mathbf{x}}\right). \tag{27}$$

Therefore, we can expect a similar error variance-covariance matrix as in case of a satellite-only solution. This does not hold for the local gravity field parameters itself. In case of increasing accuracy of the terrestrial data, tantamount to a weight matrix with increasing weights, the combination solution approaches the terrestrial data included. The rms-errors derived from the differences to the true values become smaller and smaller with increasing weights until it coincides with the white noise of the terrestrial data used—in our case 2 mGal (Fig. 9). The question arises: what is the adequate weight for the terrestrial data to be combined with unregularized satellite solutions. If the terrestrial data are provided exactly with a weight corresponding to the white noise level of 2 mGal of the terrestrial data—then still large oscillations remain in the unstable regions (Fig. 13). The oscillations still remain even if the terrestrial data are provided with a higher weight corresponding to a white noise level of 1 mGal (Fig. 14). Only if the terrestrial data

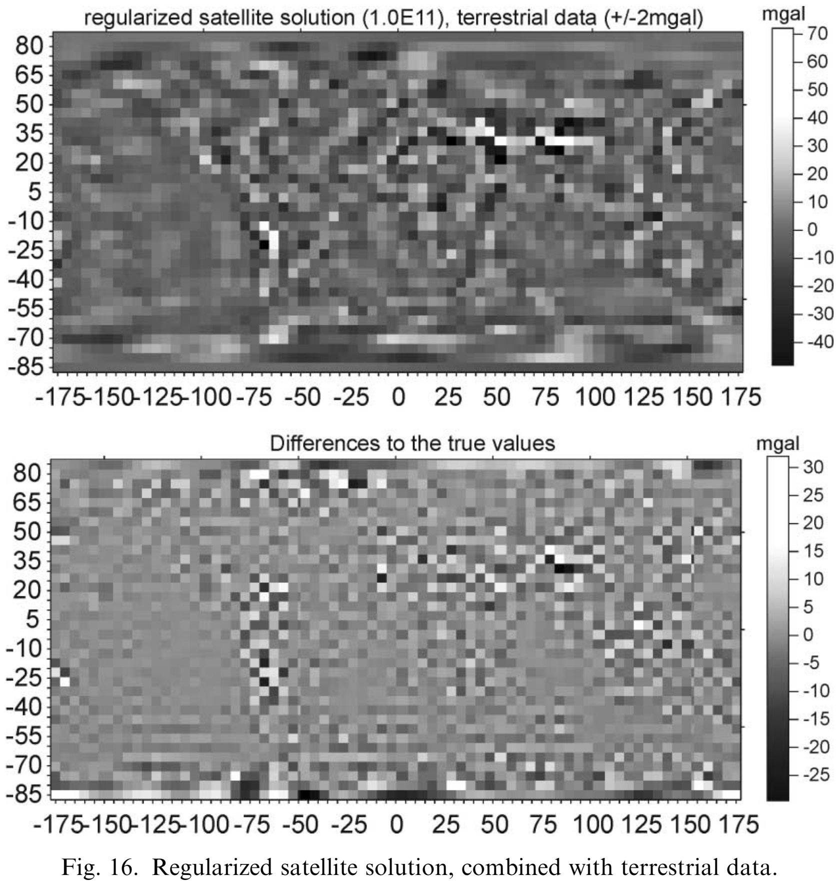

Fig. 16. Regularized satellite solution, combined with terrestrial data.

are provided with an unrealistic high weight corresponding e.g. to 0.32 mGal then the combination with unregularized satellite data results in a solution comparable with the satellite solution, regularized with an optimal regularization factor of 0.32 mGal (Fig. 15).

If the terrestrial data (white noise: 2 mGal) are combined with the regularized satellite solution (regularization factor: 0.32 mGal) then the main oscillations vanish and the result turns out to be acceptable (Fig. 16). But in this case we realize that the terrestrial data do not have a remarkable effect on the combined solution. The reason is that if the regularization matrix is interpreted as a priori information then the regularization factor corresponds to a very low standard deviation of about 0.32 mGal (corresponding to a high weight) while the terrestrial data are used with a weight which is considerably smaller. From these results we conclude that the terrestrial data can improve the combined result only in those cases where the optimal regularization parameter is comparable with the weight used also for the terrestrial data.

To get a more detailed information of the use of satellite or terrestrial data to a combined result we applied Eqs. (23) and (24), respectively. First we considered either the satellite solution or the

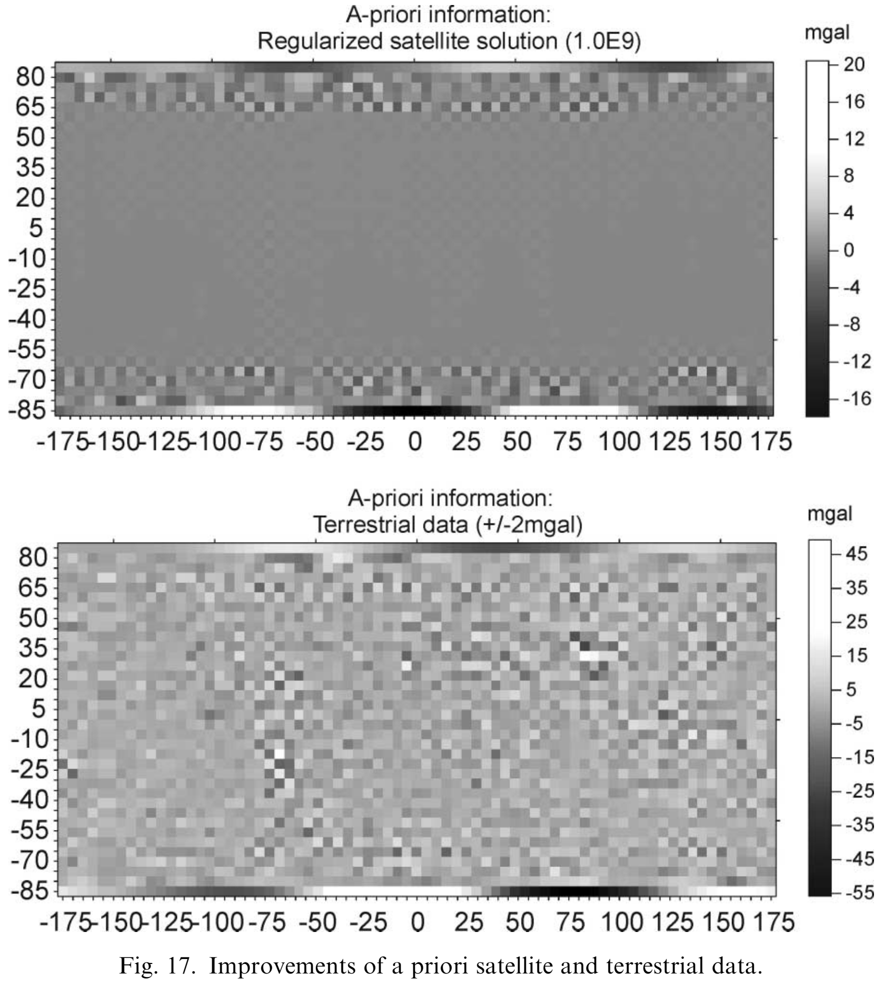

Fig. 17. Improvements of a priori satellite and terrestrial data.

terrestrial data as a priori information and computed the improvements by the terrestrial data and the satellite derived data respectively. The regularized satellite solution can be improved only in the weakly determined polar regions, while the terrestrial data are improved in the whole global solution domain. Fig. 17 shows the results in case of a regularization factor corresponding to 5 mGal and Fig. 18 in case of a regularization factor corresponding to 0.32 mGal. In both cases the terrestrial data are used with a weight corresponding to 2 mGal. The characteristic properties are identical in both cases.

Finally we computed the partial redundancies as measures for the contribution of the corresponding terrestrial data to the combined adjustment results. The test presented in Fig. 19 confirms the results arrived before.

Even if the unregularized satellite solution is combined with terrestrial data (white noise: 2 mGal), the contribution reaches 100% only in the critical polar regions. If the satellite solution is properly regularized (regularization factor corresponding to 0.32 mGal) then the terrestrial data contribute to the polar area only by 5% and nearly nothing within the rest of the solution domain.

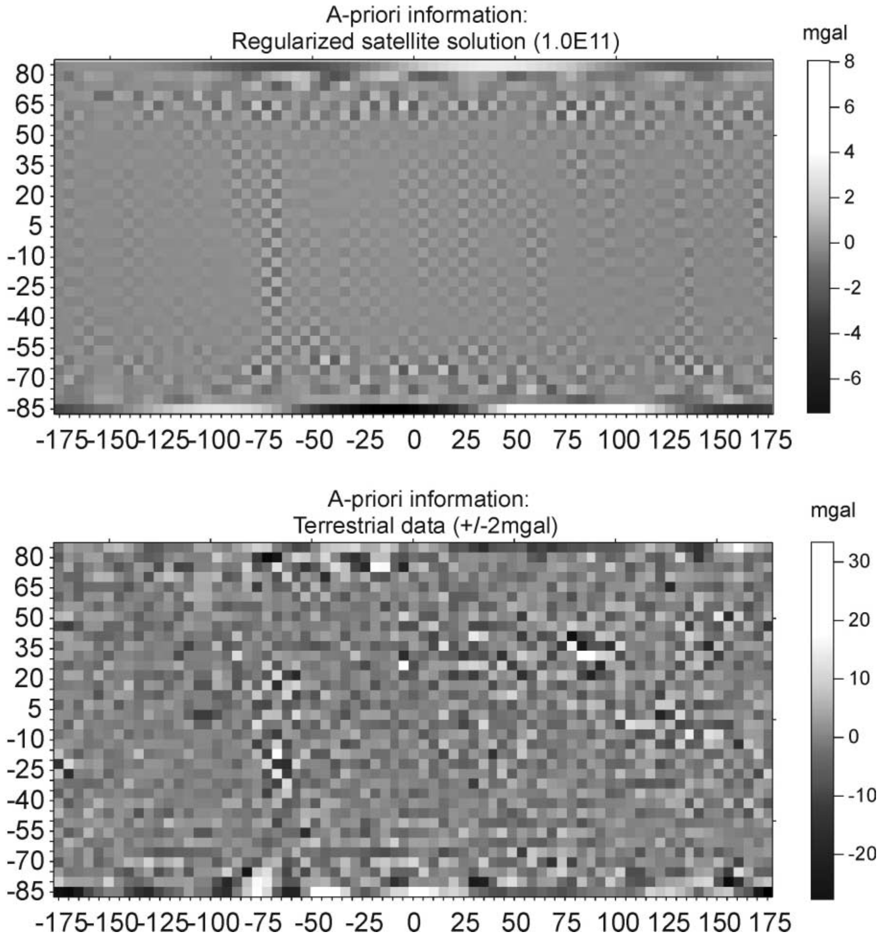

Fig. 18. Improvements of a priori satellite and terrestrial data.

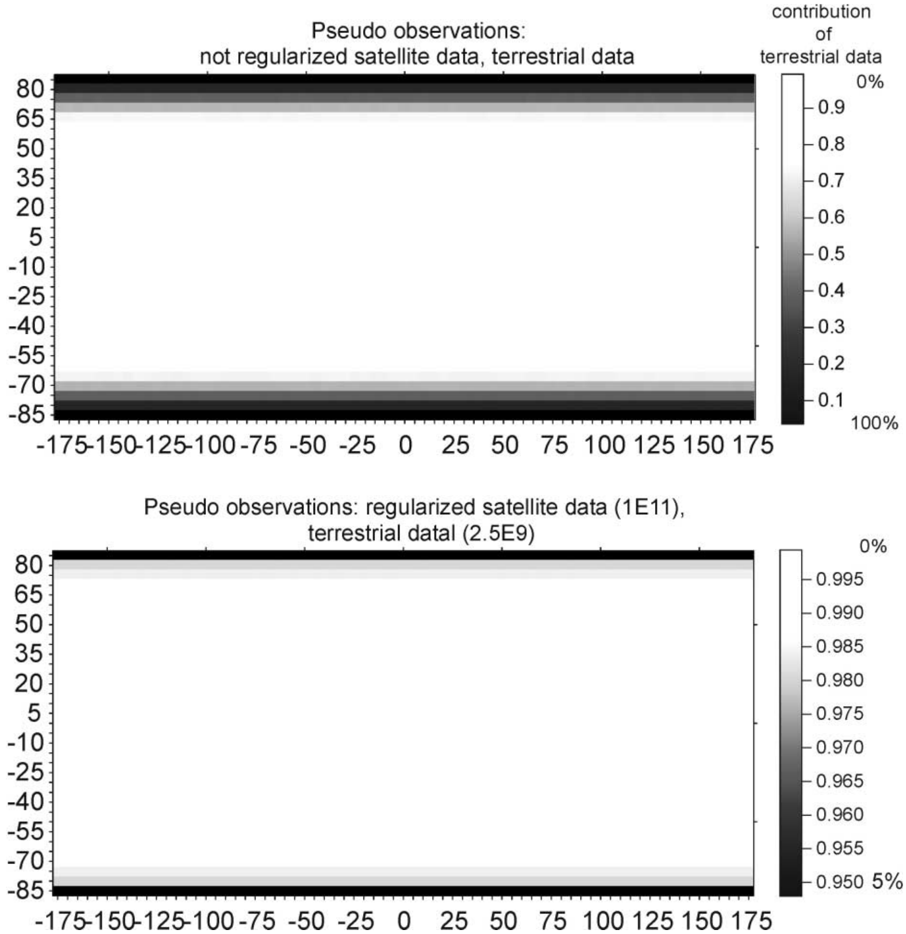

Fig. 19. Partial redundancy numbers of terrestrial data: contribution to unregularized and regularized satellite solutions.

5. Summary

Data combination usually is no problem in case of stable normal equations as long as the weight matrices or the variance-covariance matrices of the various observation types are known. The situation changes dramatically if one group of observations creates a singular or badly conditioned normal equation matrix. This is the case if a downward continuation process is contained in the mathematical model. All operators which map satellite borne observational functionals into gravity field parameters but also airborne gravimetry or gradiometry belong to these unstable problems. From the simulation results of this study we conclude:

- regularization of satellite data by using an appropriate regularization factor is indispensable,
- regularization errors (bias) should cause no real problems; usually they are restricted to the unstable areas and are within the unavoidable effects of data errors,

- combination of regularized solutions with terrestrial data by using proper weights which may not correspond to the real data accuracy can improve the combined results,
- combination of not regularized satellite data with terrestrial data in general does not improve the merged solution, except very accurate terrestrial data are available for the whole recovery area,
- regional terrestrial data cannot stabilize the unstable satellite derived solutions; this holds independent of its accuracy, independent of its regional distribution and independent of the roughness of the gravity field features.

Further investigations are necessary related to alternative types of observational functionals as gradiometry or high–low SST data. It seems that the present results hold especially for gravity field parametrizations based on space-localizing base functions. The situation may be different by using frequency-localizing base functions. The present investigation defines mainly problem areas related to the recovery procedure applied. A sophisticated technique for a proper combination procedure is still pending.

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