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Traveltime approximation for a reflected wave in a homogeneous anisotropic elastic layer

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SUMMARY

An approximation to the traveltime field is calculated for an elastic wave that propagates in a homogeneous anisotropic layer and is reflected at a plane boundary. The traveltime is approximated by a Taylor series expansion with the third derivative of the traveltime being taken into account. The coefficients of the series refer to the seismic ray, which is locally the fastest ray. Simple formulae are obtained for orthorhombic media in the crystal coordinate system, which relate the traveltimes of the reflected waves to the elastic constants of the medium. A numerical example is presented for wave propagation in orthorhombic olivine, which is a constituent of the Earth's mantle. A second example is given by an isotropic host rock with a set of parallel cracks, which is an important model for wave propagation in the Earth's crust. The elastic parameters can be determined by measuring the reflection times as a function of source–receiver offset. The approximate traveltime–distance curves are compared with traveltimes obtained from seismic ray tracing.

Key words: anisotropy, body waves, ray theory, seismic reflection, traveltime.

1 INTRODUCTION

The analysis of traveltime–distance curves is used in seismology and exploration geophysics to obtain information concerning the distribution of seismic velocities in the Earth (Slotnick 1959; Pilant 1979; Bullen 1985; Goldin 1990; Sheriff & Geldart 1995). The X^2–T^2 method is a classical method of velocity analysis that is applied to determine the subsurface velocity by measuring the arrival times of reflected waves as a function of source–receiver offset. A propagating wave in a homogeneous isotropic layer that is reflected at a horizontal interface in the subsurface and then recorded at the Earth's surface at different offsets, leads to a hyperbolic traveltime–distance curve. In this case, the function τ^2–r^2 is a straight line with slope v^{-2} and intercept $2hv^{-1}$, where τ is the traveltime, r is the offset, v is the velocity and h is the reflector depth. This method has already been used by Green (1938). A model of several layers separated by plane parallel interfaces has been treated by Dix (1955) who showed how to calculate the velocities of the layers by using the information obtained from τ^2–r^2 plots. Taner & Koehler (1969) and Al-Chalabi (1973) calculated higher-order coefficients in the expansion $\tau^2 = \sum c_n r^{2n}$ for the multi-layer case. A generalization to layers with arbitrary dips and reflector curvature has been derived by Krey and Hubral (Krey 1976; Krey & Hubral 1980). Analytical traveltime–distance formulae are used for another important concept: the so-called 'normal moveout' correction, which is the difference in traveltime for reflections recorded at offset r and at zero offset. Seismic data that are collected in a common-midpoint gather are corrected by this traveltime difference before they are stacked (Sheriff & Geldart 1995).

Seismic anisotropy is the directional dependence of seismic velocities (e.g. Crampin 1981). Anisotropy can be caused by fine-scale layering of sediments, by preferred orientation of non-spherical grains or by the presence of parallel oriented cracks in the host rock (Crampin *et al.* 1980; Johnston & Christensen 1995). Many minerals, such as olivine, are intrinsically anisotropic (Backus 1965; Anderson 1989). Analytical results for the traveltimes of reflected waves in anisotropic media have been derived for transversely isotropic media (Lyakhovitsky & Nevsky 1971; Hake *et al.* 1984; Thomsen 1986), for wave propagation in symmetry planes (Tsvankin 1995) and for weak anisotropy (Sena 1991; Li & Crampin 1993). Sayers (1995) uses an expansion of the function v^{-2} into spherical harmonics and treats the case of monoclinic symmetry with the symmetry plane parallel to the interface.

In this paper a formula is derived for the traveltime of a reflected wave that propagates in a homogeneous anisotropic medium described by 21 elastic constants. The reflector plane is parallel to the plane of observation. In general, the direction of the group velocity deviates from

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the vertical plane through the source point and the point of observation leading to 3-D geometry of the ray path. The traveltime of the reflected wave is a function of two space coordinates, e.g. the lateral distance between source and receiver and the azimuth angle of the source–receiver line. Simple formulae are obtained for orthorhombic media for cases in which one of the symmetry planes of the medium is co-planar to the plane of observation and to the reflector plane. These formulae are also valid for transversely isotropic media with a horizontal orientation of the axis of symmetry. In these cases, the formulae directly relate the reflection traveltimes to the elastic constants of the medium. A numerical example is given for orthorhombic olivine representing wave propagation in the Earth's upper mantle, where the olivine crystals are partially oriented by the flow field (Ringwood 1975; Anderson 1989). A second example is presented for a model of a system of vertically oriented cracks in an isotropic background medium. The approximate traveltimes are compared with traveltimes obtained by seismic ray tracing.

2 STATEMENT OF THE PROBLEM

A plane elastic wave propagates in a homogeneous anisotropic layer. The wave is reflected at a horizontal interface that is parallel to the plane of observation (Fig. 1). Since the medium is homogeneous, the ray path consists of two straight lines. A right-handed Cartesian coordinate system with the basis vectors ($\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$) is introduced. The surface of observation is represented by the plane $x_3 = 0$ and the horizontal reflector is represented by the plane $x_3 = h$. An approximation to the traveltime field in form of the following expansion is calculated:

$$\tau(x_1, x_2) \approx \tau(0,0) + \frac{1}{2}\frac{\partial^2\tau}{\partial x_1^2}(0,0)x_1^2 + \frac{\partial^2\tau}{\partial x_1 \partial x_2}(0,0)x_1x_2 + \frac{1}{2}\frac{\partial^2\tau}{\partial x_2^2}(0,0)x_2^2 + \frac{1}{6}\frac{\partial^3\tau}{\partial x_1^3}(0,0)x_1^3 + \frac{1}{2}\frac{\partial^3\tau}{\partial x_1^2 \partial x_2}(0,0)x_1^2x_2$$
$$+ \frac{1}{2}\frac{\partial^3\tau}{\partial x_1 \partial x_2^2}(0,0)x_1x_2^2 + \frac{1}{6}\frac{\partial^3\tau}{\partial x_2^3}(0,0)x_2^3. \quad (1)$$

The coefficients in this equation refer to the 'central ray'. The incident and the reflected central ray propagate along the same path in opposite directions, and the ray finally arrives at the origin of the plane of observation where it started. The following calculation shows that the first derivative of the traveltime equals zero in the limit of the central ray, which means that this ray refers to an extremum of the traveltime surface.

The traveltime and the points on the plane of observation are represented by the parametric equations

$$\tau(\gamma_1, \gamma_2) = \frac{h}{\xi_3^+(\gamma_1, \gamma_2)} - \frac{h}{\xi_3^-(\gamma_1, \gamma_2)} \quad (2)$$

and

$$\mathbf{x}(\gamma_1, \gamma_2) = \frac{h}{\xi_3^+(\gamma_1, \gamma_2)}\boldsymbol{\xi}^+(\gamma_1, \gamma_2) - \frac{h}{\xi_3^-(\gamma_1, \gamma_2)}\boldsymbol{\xi}^-(\gamma_1, \gamma_2), \quad (3)$$

where γ_1 and γ_2 are the ray parameters or ray coordinates. The upper index '+' denotes the incident wave that travels in the positive x_3-direction towards the reflector plane and the index '−' denotes the reflected wave that travels in the negative x_3-direction back to the plane of observation.

Some properties of plane waves are used in the following when the coefficients of the traveltime series expansion are calculated (see, e.g., Synge 1956, 1957; Fedorov 1968; Musgrave 1970; Helbig 1994).

The group velocity vector $\boldsymbol{\xi}$ of a plane wave is given by

$$\xi_i = \sum_{k,l,m=1}^{3} \lambda_{iklm} p_l A_k A_m, \qquad (i = 1, 2, 3), \quad (4)$$

where $\mathbf{p}$ is the slowness vector, $\mathbf{A}$ is the polarization vector and λ_{iklm} are the density-reduced elastic constants c_{iklm}/ρ. The group velocity is equal to the ratio of the time averages of energy flux (Poynting or Umov vector) and energy density.

The slowness vector is given by

$$\mathbf{p} = \frac{1}{v}\mathbf{n}, \quad (5)$$

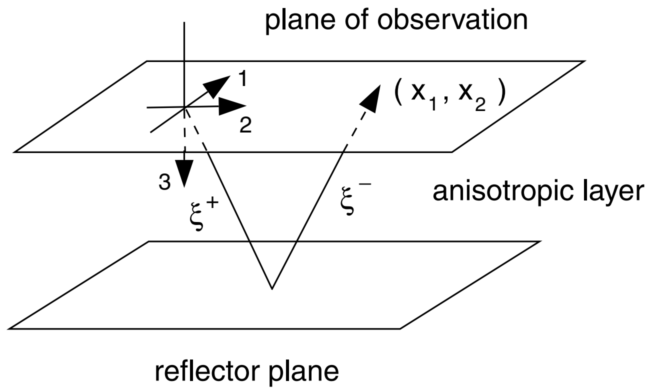

Figure 1. A seismic ray is reflected at a plane interface which is parallel to the plane of observation.

where v is the phase velocity and $\mathbf{n}$ is the phase normal vector, which is introduced here for the incident wave as a function of the ray parameters by

$$\mathbf{n}(\gamma_1, \gamma_2) = \left(\gamma_1, \gamma_2, \sqrt{1 - \gamma_1^2 - \gamma_2^2}\right)^{\mathrm{T}}. \tag{6}$$

The polarization vector and the phase velocity are obtained by solving Christoffel's equation:

$$\sum_{k,l,m=1}^{3} \lambda_{iklm} n_k n_l A_m = v^2 A_i, \qquad (i = 1, 2, 3). \tag{7}$$

This equation is a real, symmetric and positive-definite eigenvalue problem. Its three solutions describe three types of plane waves. Using eq. (5) the last equation takes the following form:

$$\sum_{k,l,m=1}^{3} \lambda_{iklm} p_k p_l A_m = A_i, \qquad (i = 1, 2, 3). \tag{8}$$

3 CALCULATION OF THE COEFFICIENTS OF THE TRAVELTIME SERIES

To determine the coefficients of the traveltime series (1) it is necessary to relate some properties of the central ray before and after reflection at the boundary. These properties are the slowness, polarization and group velocity vectors and their derivatives with respect to the ray parameters. The coefficients are calculated as a function of the ray coordinates γ_1 and γ_2. These are then transformed into Cartesian coordinates x_1 and x_2.

3.1 Traveltime of the central ray

The incident central ray has the ray parameters $(\gamma_1, \gamma_2) = (0, 0)$. Its phase normal is given by the vector

$$\mathbf{n}(0, 0) = \mathbf{i}_3. \tag{9}$$

The phase velocity $v(0, 0)$ is determined from Christoffel's equation and the slowness vector is given by

$$\mathbf{p}^+(0, 0) = \frac{1}{v(0, 0)} \mathbf{n}(0, 0). \tag{10}$$

Snell's law states that the projection of the slowness vector on the interface is preserved in the reflection process. This is described by the equation

$$\mathbf{p}^- - \mathbf{p}^+ = q\mathbf{m}, \tag{11}$$

where $\mathbf{m} = \mathbf{i}_3$ is the unit vector normal to the interface and q is a scalar.

The incident wave can be any of three types of waves that propagate in anisotropic media. The reflected wave is assumed to be of the same type as the incident wave. The following calculation uses the point symmetry of the slowness surface with respect to the origin of the coordinate system. Owing to the fact that eq. (8) is a quadratic function of the slowness, it is possible to conclude for the slowness vector of incident and reflected central ray:

$$\mathbf{p}^-(0, 0) = -\mathbf{p}^+(0, 0), \tag{12}$$

and for the polarization and group velocity vectors:

$$\mathbf{A}^-(0, 0) = \mathbf{A}^+(0, 0) \tag{13}$$

and

$$\boldsymbol{\xi}^-(0, 0) = -\boldsymbol{\xi}^+(0, 0). \tag{14}$$

The last equation shows that the central ray is the ray that is reflected in itself (Fig. 2). By using Christoffel's equation and the definition of the group velocity, one can prove the following equation:

$$\mathbf{p} \cdot \boldsymbol{\xi} = 1. \tag{15}$$

Substituting eq. (10) into this equation, it follows that the third component of the group velocity of the central ray equals its phase velocity:

$$\xi_3^+(0, 0) = v(0, 0). \tag{16}$$

The traveltime of the central ray is obtained from eqs (2) and (14):

$$\tau(0, 0) = \frac{2h}{\xi_3^+(0, 0)} = \frac{2h}{v(0, 0)}. \tag{17}$$

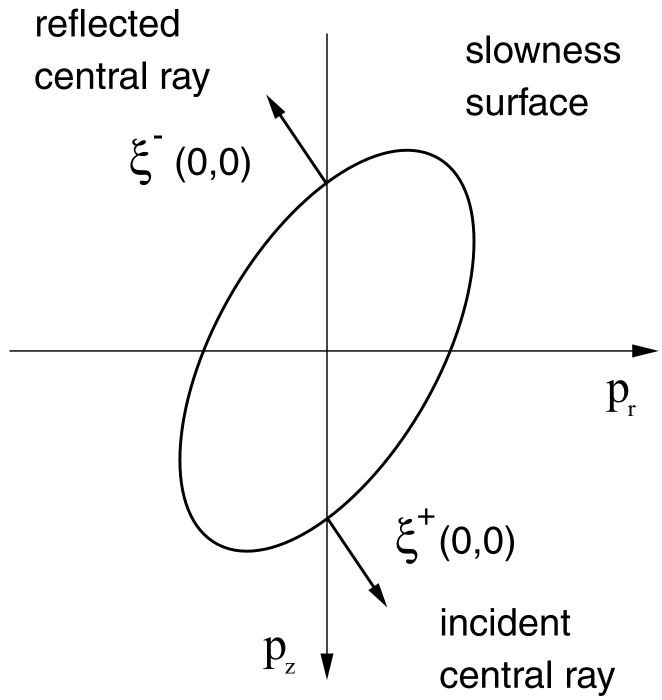

Figure 2. Slowness surface and group velocity vector of the incident and the reflected central ray.

3.2 First derivative of the traveltime of the central ray

Next, it is shown that the first derivative of the traveltime is zero for the central ray, which means that the central ray is locally the fastest ray.

By differentiating eq. (15) it follows that

$$\frac{\partial \boldsymbol{\xi}}{\partial \gamma_s} \cdot \mathbf{p} + \boldsymbol{\xi} \cdot \frac{\partial \mathbf{p}}{\partial \gamma_s} = 0, \qquad (s = 1, 2). \tag{18}$$

If eq. (8) is differentiated and the scalar product with the polarization vector is formed, then it can be shown that

$$\boldsymbol{\xi} \cdot \frac{\partial \mathbf{p}}{\partial \gamma_s} = 0, \qquad (s = 1, 2). \tag{19}$$

Using the last two equations one obtains

$$\frac{\partial \boldsymbol{\xi}}{\partial \gamma_s} \cdot \mathbf{p} = 0, \qquad (s = 1, 2). \tag{20}$$

Using eqs (10) and (12) it follows in the limit of the central ray that

$$\frac{\partial \xi_3^+}{\partial \gamma_s}(0, 0) = \frac{\partial \xi_3^-}{\partial \gamma_s}(0, 0) = 0, \qquad (s = 1, 2). \tag{21}$$

Differentiating the equation for the traveltime and using eqs (14) and (21) leads to

$$\frac{\partial \tau}{\partial \gamma_s}(0, 0) = 0, \qquad (s = 1, 2). \tag{22}$$

With the help of the transformation formula from ray coordinates to Cartesian coordinates,

$$\frac{\partial \tau}{\partial x_s} = \frac{\partial \tau}{\partial \gamma_1}\frac{\partial \gamma_1}{\partial x_s} + \frac{\partial \tau}{\partial \gamma_2}\frac{\partial \gamma_2}{\partial x_s}, \qquad (s = 1, 2), \tag{23}$$

it follows that

$$\frac{\partial \tau}{\partial x_s}(0, 0) = 0, \qquad (s = 1, 2). \tag{24}$$

3.3 Second derivative of the traveltime of the central ray

The coefficients of the second-order terms of the traveltime series expansion are calculated next. Differentiating eq. (2) leads to

$$\frac{\partial^2 \tau}{\partial \gamma_r \partial \gamma_s}(0, 0) = \frac{h}{v^2(0, 0)}\left[\frac{\partial^2 \xi_3^-}{\partial \gamma_r \partial \gamma_s}(0, 0) - \frac{\partial^2 \xi_3^+}{\partial \gamma_r \partial \gamma_s}(0, 0)\right], \qquad (r, s = 1, 2). \tag{25}$$

Snell's law, eq. (11), is differentiated and the scalar product with the group velocity is taken on both sides of the resulting equation. With the help of eqs (14) and (19) one obtains the following relation between the derivatives of the slowness vectors of the reflected and the incident central ray:

$$\frac{\partial \mathbf{p}^-}{\partial \gamma_s}(0,0) = \frac{\partial \mathbf{p}^+}{\partial \gamma_s}(0,0), \qquad (s = 1, 2). \tag{26}$$

By differentiating eq. (8) and using eqs (12), (13) and (26), one obtains for the derivative of the polarization vector:

$$\frac{\partial \mathbf{A}^-}{\partial \gamma_s}(0,0) = -\frac{\partial \mathbf{A}^+}{\partial \gamma_s}(0,0), \qquad (s = 1, 2). \tag{27}$$

Differentiating eq. (4) and using eqs (12), (13), (26) and (27) results in

$$\frac{\partial \boldsymbol{\xi}^-}{\partial \gamma_s}(0,0) = \frac{\partial \boldsymbol{\xi}^+}{\partial \gamma_s}(0,0), \qquad (s = 1, 2), \tag{28}$$

for the first derivative of the group velocity.

From eq. (5) one obtains

$$\frac{\partial \mathbf{p}}{\partial \gamma_s} = \frac{1}{v}\frac{\partial \mathbf{n}}{\partial \gamma_s} - \frac{1}{v^2}\left(\boldsymbol{\xi} \cdot \frac{\partial \mathbf{n}}{\partial \gamma_s}\right)\mathbf{n}, \qquad (s = 1, 2), \tag{29}$$

where

$$\frac{\partial v}{\partial \gamma_s} = \boldsymbol{\xi} \cdot \frac{\partial \mathbf{n}}{\partial \gamma_s}, \qquad (s = 1, 2) \tag{30}$$

is used.

The first derivative of the phase normal vector of the incident central ray is obtained by differentiating eq. (6):

$$\frac{\partial \mathbf{n}}{\partial \gamma_s}(0,0) = \mathbf{i}_s, \qquad (s = 1, 2). \tag{31}$$

This leads to

$$\frac{\partial \mathbf{p}^+}{\partial \gamma_s}(0,0) = \frac{1}{v(0,0)}\mathbf{i}_s - \frac{\xi_s^+(0,0)}{v^2(0,0)}\mathbf{i}_3, \qquad (s = 1, 2). \tag{32}$$

By differentiating eq. (20) and using eqs (10), (12), (21), (26), (28) and (32) one can prove that

$$\frac{\partial^2 \xi_3^-}{\partial \gamma_r \partial \gamma_s}(0,0) = -\frac{\partial^2 \xi_3^+}{\partial \gamma_r \partial \gamma_s}(0,0) = \frac{\partial \xi_r^+}{\partial \gamma_s}(0,0), \qquad (r, s = 1, 2). \tag{33}$$

Then one obtains the following formula for the second derivative of the traveltime:

$$\frac{\partial^2 \tau}{\partial \gamma_r \partial \gamma_s}(0,0) = \frac{2h}{v^2(0,0)}\frac{\partial \xi_r^+}{\partial \gamma_s}(0,0), \qquad (r, s = 1, 2). \tag{34}$$

The first derivative of the group velocity is calculated in the following (Kashtan *et al.* 1983; Yeatts 1984). Differentiating eq. (4) leads to

$$\frac{\partial \xi_r}{\partial \gamma_s} = \sum_{k,l,m=1}^{3}\left[\lambda_{rklm}\frac{\partial p_l}{\partial \gamma_s}A_k A_m + (\lambda_{rklm} + \lambda_{rmlk})p_l A_m \frac{\partial A_k}{\partial \gamma_s}\right], \qquad (r, s = 1, 2). \tag{35}$$

The polarization vector $\mathbf{A}$ is a unit vector, and its derivative is orthogonal to the vector itself. The derivative can be written as a linear combination of the vectors $\mathbf{A}^{(2)}$ and $\mathbf{A}^{(3)}$, which are the other two eigenvectors of Christoffel's equation corresponding to the eigenvalues v_2 and v_3:

$$\frac{\partial \mathbf{A}}{\partial \gamma_s} = c_2^{(\gamma_s)}\mathbf{A}^{(2)} + c_3^{(\gamma_s)}\mathbf{A}^{(3)}, \qquad (s = 1, 2). \tag{36}$$

The unknown constants $c_2^{(\gamma_s)}$ and $c_3^{(\gamma_s)}$ can be determined by differentiating eq. (8):

$$c_{2,3}^{(\gamma_s)} = \frac{v^2}{v^2 - v_{2,3}^2}\sum_{i,k,l,m=1}^{3}(\lambda_{iklm} + \lambda_{ilkm})\frac{\partial p_k}{\partial \gamma_s}p_l A_m A_i^{(2,3)}, \qquad (s = 1, 2). \tag{37}$$

In the limit of the central ray one obtains with the help of the eqs (7), (10) and (32):

$$\frac{\partial \xi_r^+}{\partial \gamma_s}(0,0) = \frac{\partial \xi_s^+}{\partial \gamma_r}(0,0) = \frac{1}{v(0,0)}\left[\sum_{k,m=1}^{3}\lambda_{rksm}A_k(0,0)A_m(0,0) - \xi_r(0,0)\xi_s(0,0) + \left(v^2(0,0) - v_2^2(0,0)\right)c_2^{(\gamma_r)}(0,0)c_2^{(\gamma_s)}(0,0)\right.$$
$$\left. + \left(v^2(0,0) - v_3^2(0,0)\right)c_3^{(\gamma_r)}(0,0)c_3^{(\gamma_s)}(0,0)\right], \qquad (r, s = 1, 2), \tag{38}$$

where

$$c_{2,3}^{(\gamma_s)}(0,0) = \frac{1}{v^2(0,0) - v_{2,3}^2(0,0)} \sum_{i,m=1}^{3} (\lambda_{is3m} + \lambda_{i3sm}) A_m(0,0) A_i^{(2,3)}(0,0), \qquad (s = 1,2). \tag{39}$$

Finally, eq. (34) is transformed from ray coordinates to Cartesian coordinates with help of the inverse function of

$$\frac{\partial x_r}{\partial \gamma_s}(0,0) = \frac{2h}{v(0,0)} \frac{\partial \xi_r^+}{\partial \gamma_s}(0,0), \qquad (r, s = 1,2). \tag{40}$$

The last equation is derived from eq. (3). One obtains:

$$\frac{\partial^2 \tau}{\partial x_1^2}(0,0) = \frac{1}{2h\Delta(0,0)} \frac{\partial \xi_2^+}{\partial \gamma_2}(0,0), \tag{41}$$

$$\frac{\partial^2 \tau}{\partial x_2^2}(0,0) = \frac{1}{2h\Delta(0,0)} \frac{\partial \xi_1^+}{\partial \gamma_1}(0,0), \tag{42}$$

$$\frac{\partial^2 \tau}{\partial x_1 \partial x_2}(0,0) = \frac{-1}{2h\Delta(0,0)} \frac{\partial \xi_1^+}{\partial \gamma_2}(0,0), \tag{43}$$

where the determinant $\Delta(0,0)$ is given by

$$\Delta(0,0) = \frac{\partial \xi_1^+}{\partial \gamma_1}(0,0) \frac{\partial \xi_2^+}{\partial \gamma_2}(0,0) - \frac{\partial \xi_2^+}{\partial \gamma_1}(0,0) \frac{\partial \xi_1^+}{\partial \gamma_2}(0,0). \tag{44}$$

This result shows that the second derivative of the traveltime depends on the first derivative of the group velocity. Some simplification is achieved for wave propagation in symmetry planes and for the *qP* wave in weakly anisotropic media (see the appendices).

3.4 Third derivative of the traveltime of the central ray

The third derivative of the traveltime is given by

$$\frac{\partial^3 \tau}{\partial \gamma_r \partial \gamma_s \partial \gamma_t}(0,0) = \frac{h}{v^2(0,0)} \left(\frac{\partial^3 \xi_3^-}{\partial \gamma_r \partial \gamma_s \partial \gamma_t}(0,0) - \frac{\partial^3 \xi_3^+}{\partial \gamma_r \partial \gamma_s \partial \gamma_t}(0,0) \right), \qquad (r, s, t = 1,2), \tag{45}$$

where eq. (21) is used. Differentiating eq. (20) and using eqs (10), (12), (26) and (28) results in

$$\begin{aligned} \frac{\partial^3 \tau}{\partial \gamma_r \partial \gamma_s \partial \gamma_t}(0,0) = \frac{h}{v(0,0)} \Bigg[&\left(\frac{\partial^2 \boldsymbol{\xi}^+}{\partial \gamma_r \partial \gamma_s}(0,0) + \frac{\partial^2 \boldsymbol{\xi}^-}{\partial \gamma_r \partial \gamma_s}(0,0) \right) \cdot \frac{\partial \mathbf{p}^+}{\partial \gamma_t}(0,0) + \left(\frac{\partial^2 \boldsymbol{\xi}^+}{\partial \gamma_s \partial \gamma_t}(0,0) + \frac{\partial^2 \boldsymbol{\xi}^-}{\partial \gamma_s \partial \gamma_t}(0,0) \right) \cdot \frac{\partial \mathbf{p}^+}{\partial \gamma_r}(0,0) \\ &+ \left(\frac{\partial^2 \mathbf{p}^+}{\partial \gamma_r \partial \gamma_t}(0,0) + \frac{\partial^2 \mathbf{p}^-}{\partial \gamma_r \partial \gamma_t}(0,0) \right) \cdot \frac{\partial \boldsymbol{\xi}^+}{\partial \gamma_s}(0,0) \Bigg], \qquad (r, s, t = 1,2). \end{aligned} \tag{46}$$

It is shown in the following that the first and second terms on the right-hand side of the last equation can be written in the form of the third term.

The second derivative of the slowness vector for the incident central ray can be calculated by using the general formula

$$\begin{aligned} \frac{\partial^2 \mathbf{p}}{\partial \gamma_r \partial \gamma_s} = &-\frac{1}{v^2}\left(\boldsymbol{\xi} \cdot \frac{\partial \mathbf{n}}{\partial \gamma_r}\right) \frac{\partial \mathbf{n}}{\partial \gamma_s} - \frac{1}{v^2}\left(\boldsymbol{\xi} \cdot \frac{\partial \mathbf{n}}{\partial \gamma_s}\right) \frac{\partial \mathbf{n}}{\partial \gamma_r} + \frac{1}{v} \frac{\partial^2 \mathbf{n}}{\partial \gamma_r \partial \gamma_s} + \frac{2}{v^3}\left(\boldsymbol{\xi} \cdot \frac{\partial \mathbf{n}}{\partial \gamma_r}\right)\left(\boldsymbol{\xi} \cdot \frac{\partial \mathbf{n}}{\partial \gamma_s}\right)\mathbf{n} \\ &- \frac{1}{v^2}\left(\frac{\partial \boldsymbol{\xi}}{\partial \gamma_r} \cdot \frac{\partial \mathbf{n}}{\partial \gamma_s}\right)\mathbf{n} - \frac{1}{v^2}\left(\boldsymbol{\xi} \cdot \frac{\partial^2 \mathbf{n}}{\partial \gamma_r \partial \gamma_s}\right)\mathbf{n}, \qquad (r, s = 1,2). \end{aligned} \tag{47}$$

For the reflected central ray one obtains using Snell's law, eqs (11) and (19):

$$\frac{\partial^2 \mathbf{p}^-}{\partial \gamma_r \partial \gamma_s}(0,0) = \frac{\partial^2 \mathbf{p}^+}{\partial \gamma_r \partial \gamma_s}(0,0) - \frac{2}{v(0,0)} \left[\frac{\partial^2 \mathbf{p}^+}{\partial \gamma_r \partial \gamma_s}(0,0) \cdot \boldsymbol{\xi}^+(0,0) \right] \mathbf{m}, \qquad (r, s = 1,2). \tag{48}$$

It then follows that

$$\frac{\partial^2 \mathbf{p}^\pm}{\partial \gamma_r \partial \gamma_s}(0,0) = \frac{1}{v^2(0,0)} \left[-\xi_r^+(0,0)\mathbf{i}_s - \xi_s^+(0,0)\mathbf{i}_r + \left(\frac{2}{v(0,0)} \xi_r^+(0,0)\xi_s^+(0,0) \mp \frac{\partial \xi_r^+}{\partial \gamma_s}(0,0) \right) \mathbf{i}_3 \right], \qquad (r, s = 1,2). \tag{49}$$

For the second derivative of the polarization vector in the limit of the central ray one obtains

$$\frac{\partial^2 \mathbf{A}^+}{\partial \gamma_r \partial \gamma_s}(0,0) - \frac{\partial^2 \mathbf{A}^-}{\partial \gamma_r \partial \gamma_s}(0,0) = \frac{-2}{v(0,0)} \left[\xi_r^+(0,0) \frac{\partial \mathbf{A}^+}{\partial \gamma_s}(0,0) + \xi_s^+(0,0) \frac{\partial \mathbf{A}^+}{\partial \gamma_r}(0,0) \right], \qquad (r, s = 1,2), \tag{50}$$

and for the second derivative of the group velocity:

$$\frac{\partial^2 \xi_i^+}{\partial\gamma_r\partial\gamma_s}(0,0)+\frac{\partial^2 \xi_i^-}{\partial\gamma_r\partial\gamma_s}(0,0)=\sum_{k,l,m=1}^{3}(\lambda_{iklm}+\lambda_{ilkm})\left[\frac{\partial^2 A_k^+}{\partial\gamma_r\partial\gamma_s}(0,0)-\frac{\partial^2 A_k^-}{\partial\gamma_r\partial\gamma_s}(0,0)\right]A_l^+(0,0)p_m^+(0,0)$$
$$+\sum_{k,l,m=1}^{3}\lambda_{iklm}A_k^+(0,0)A_l^+(0,0)\left[\frac{\partial^2 p_m^+}{\partial\gamma_r\partial\gamma_s}(0,0)+\frac{\partial^2 p_m^-}{\partial\gamma_r\partial\gamma_s}(0,0)\right],\qquad (i=1,2,3;r,s=1,2). \tag{51}$$

Now the following equation can be proven:

$$\left[\frac{\partial^2 \boldsymbol{\xi}^+}{\partial\gamma_r\partial\gamma_s}(0,0)+\frac{\partial^2 \boldsymbol{\xi}^-}{\partial\gamma_r\partial\gamma_s}(0,0)\right]\cdot\frac{\partial \mathbf{p}^+}{\partial\gamma_t}(0,0)=\left[\frac{\partial^2 \mathbf{p}^+}{\partial\gamma_r\partial\gamma_s}(0,0)+\frac{\partial^2 \mathbf{p}^-}{\partial\gamma_r\partial\gamma_s}(0,0)\right]\cdot\frac{\partial \boldsymbol{\xi}^+}{\partial\gamma_t}(0,0)$$
$$=\frac{-2}{v^2(0,0)}\left[\xi_r^+(0,0)\frac{\partial\xi_t^+}{\partial\gamma_s}(0,0)+\frac{\partial\xi_r^+}{\partial\gamma_s}(0,0)\xi_t^+(0,0)\right],\qquad (r,s,t=1,2), \tag{52}$$

and one obtains for the third derivative of the traveltime the formula:

$$\frac{\partial^3\tau}{\partial\gamma_r\partial\gamma_s\partial\gamma_t}(0,0)=\frac{-4h}{v^3(0,0)}\left[\xi_r^+(0,0)\frac{\partial\xi_s^+}{\partial\gamma_t}(0,0)+\xi_s^+(0,0)\frac{\partial\xi_t^+}{\partial\gamma_r}(0,0)+\xi_t^+(0,0)\frac{\partial\xi_r^+}{\partial\gamma_s}(0,0)\right],\qquad (r,s,t=1,2). \tag{53}$$

This equation shows that the third term in the traveltime series expansion is equal to zero when the group velocity of the central ray is pointing in the x_3-direction: $\boldsymbol{\xi}^+(0,0)=v(0,0)\mathbf{i}_3$. This is the case when the anisotropic medium has high symmetry with respect to the x_3-direction. An example is discussed in the following section.

For the transformation from ray coordinates to Cartesian coordinates one needs the formula:

$$\frac{\partial^2\mathbf{x}}{\partial\gamma_r\partial\gamma_s}(0,0)=\frac{h}{v(0,0)}\left[\frac{\partial^2 \boldsymbol{\xi}^+}{\partial\gamma_r\partial\gamma_s}(0,0)+\frac{\partial^2 \boldsymbol{\xi}^-}{\partial\gamma_r\partial\gamma_s}(0,0)\right],\qquad (r,s=1,2), \tag{54}$$

which is obtained by differentiating eq. (3). Since the transformation formulae are lengthy, but easily derived, they are skipped here.

4 A COMPARISON OF APPROXIMATE AND EXACT TRAVELTIMES FOR HEXAGONAL AND ORTHORHOMBIC MEDIA IN THE NATURAL COORDINATE SYSTEM

The approximate traveltimes derived in Sections 2 and 3 are compared with exact traveltimes computed by seismic ray tracing.

The ray tracing in the homogeneous, anisotropic medium is performed in the following way. Two ray parameters are chosen and Christoffel's equation is solved for the phase velocity and the polarization vector. Then the group velocity is calculated and the reflection point is determined. Snell's law is applied (Petrashen & Kashtan 1984) and the group velocity of the reflected ray is calculated. This procedure is repeated for another choice of ray parameters until the ray is found that reaches the receiver position: the boundary value problem is solved by varying the initial values.

To calculate the approximate traveltime Christoffel's equation has to be solved only once for the central ray. If the medium is orthorhombic and if its symmetry planes coincide with the coordinate planes, then the Christoffel matrix is a diagonal matrix, the polarization vectors are directed along the coordinate axes, and the group velocity vector of the central ray is pointing in th x_3-direction. In this case the following simple formulae for the traveltimes of the three waves are derived from the general result of Sections 2 and 3:

$$\tau^2(r,\varphi;P)=\frac{4h^2}{\lambda_{3333}}+r^2\left\{\left[\lambda_{1313}+\frac{(\lambda_{1313}+\lambda_{1133})^2}{\lambda_{3333}-\lambda_{1313}}\right]^{-1}\cos^2\varphi+\left[\lambda_{2323}+\frac{(\lambda_{2323}+\lambda_{2233})^2}{\lambda_{3333}-\lambda_{2323}}\right]^{-1}\sin^2\varphi\right\}, \tag{55}$$

$$\tau^2(r,\varphi;S1)=\frac{4h^2}{\lambda_{1313}}+r^2\left\{\left[\lambda_{1111}-\frac{(\lambda_{1313}+\lambda_{1133})^2}{\lambda_{3333}-\lambda_{1313}}\right]^{-1}\cos^2\varphi+\frac{1}{\lambda_{1212}}\sin^2\varphi\right\}, \tag{56}$$

$$\tau^2(r,\varphi;S2)=\frac{4h^2}{\lambda_{2323}}+r^2\left\{\frac{1}{\lambda_{1212}}\cos^2\varphi+\left[\lambda_{2222}-\frac{(\lambda_{2323}+\lambda_{2233})^2}{\lambda_{3333}-\lambda_{2323}}\right]^{-1}\sin^2\varphi\right\}. \tag{57}$$

Here, r is the lateral distance between source and receiver and φ is the azimuth of the line of observation. $S1$ and $S2$ denote the shear waves with polarization vectors $\mathbf{i}_1$ and $\mathbf{i}_2$ in the limit of the central ray. These equations show that by measuring the reflection times of all three waves, it is possible to determine eight of the nine elastic constants of the orthorhombic medium, if the thickness of the layer and the density are known.

Eq. (55) can be applied to determine the traveltime of the qP wave in a transversely isotropic medium with the symmetry axis orthogonal to the reflector plane. In this case the equations for the traveltime of the shear waves eqs (56) and (57) cannot be used, owing to the zero denominator in eq. (37). If the symmetry axis of the transversely isotropic medium is lying in the reflector plane, then eqs (55)–(57) can be used to determine the five elastic constants of the medium and the thickness of the layer.

Table 1. Elastic constants (GPa) and density (g cm^{-3}) of olivine ($(Mg_{91.7}\,Fe_{8.3})_2SiO_4$.

c_{1111}	c_{2222}	c_{3333}	c_{2323}	c_{1313}	c_{1212}	c_{1122}	c_{1133}	c_{2233}	ρ
324	198	249	66.7	81	79	59	79	78	3.324

As an example, traveltimes are calculated for orthorhombic olivine, which is an important constituent of the Earth's upper mantle (Anderson 1989). The elastic constants are given in Table 1 (Landolt-Börnstein 1982). The a-, b- and c-axes of olivine are assumed to be oriented along the x_1-, x_2- and x_3-coordinate axes. Traveltime–distance curves for the three waves are shown in Fig. 3 for three azimuths of observation: in the a-axis direction with reflection in the a–c plane ($\varphi = 0°$), in the b-axis direction with reflection in the b–c plane ($\varphi = 90°$) and for an azimuth angle of $\varphi = 45°$. The reflector depth is set to $h = 1$ km. The solid lines are the traveltimes obtained by ray tracing and the dots are the approximate traveltimes. Cut planes through the group velocity surface are also shown in Fig. 3. They show the incident and reflected wavefronts. In the symmetry planes, $\varphi = 0°$ and 90°, one wave is a pure SH mode and the intersection of the group velocity surface with the symmetry plane is an ellipse. It is easy to show that in this case the traveltime–distance curve is a straight line in τ^2–r^2 coordinates (see, e.g., Hake *et al.* 1984), and the approximate formula gives the exact result. In the case of the faster shear wave in the a–c plane of olivine ($\varphi = 0°$) the group velocity curve deviates from the elliptical shape, and the approximate traveltimes differ strongly from the exact traveltimes.

The second example is a hexagonal medium with the symmetry axis in the x_1-direction. The isotropic host rock has compressional and shear wave velocities $v_p = 4$ km s^{-1} and $v_s = 2.3$ km s^{-1} and density $\rho = 2.5$ g cm^{-3}. The effective anisotropy is caused by a system of penny-shaped parallel oriented cracks filled with water (Hudson 1981). The crack density is 0.1 and the aspect ratio is 0.001. The elastic constants for this model are given in Table 2. Eqs (55)–(57) can be simplified using the following relations that reflect the high elastic symmetry of the hexagonal medium:

$$\lambda_{1122} = \lambda_{1133}, \qquad \lambda_{1313} = \lambda_{1212}, \qquad \lambda_{2222} = \lambda_{3333}, \qquad \lambda_{2233} = \lambda_{2222} - 2\lambda_{2323}. \tag{58}$$

There are only five independent elastic moduli. Cuts through the group velocity surfaces and traveltime–distance curves for the three waves are shown in Fig. 4. The intersection singularity, which is shown in the group velocity surface for source–receiver azimuths $\varphi = 0°$ and 45°,

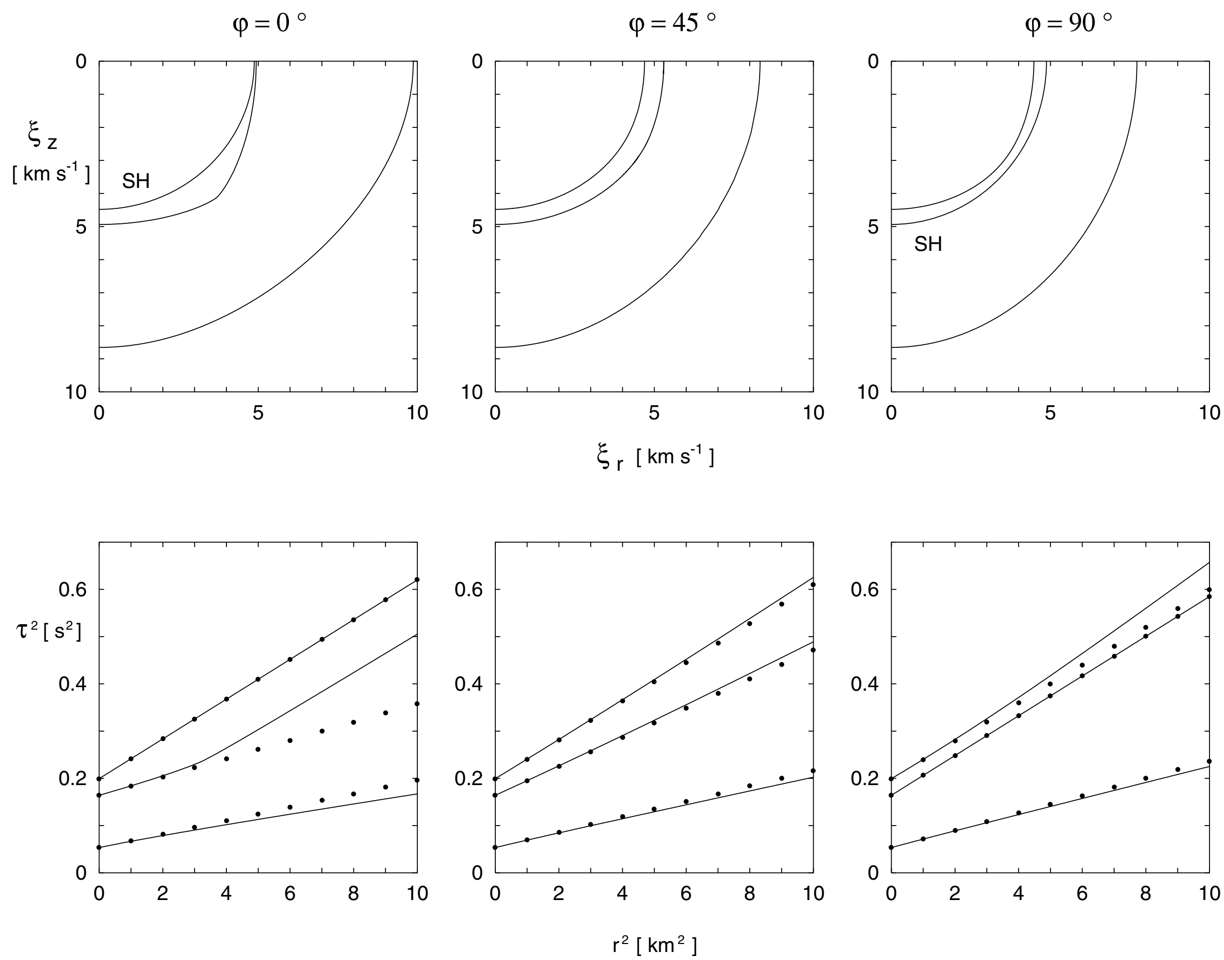

Figure 3. (a) Group velocity surfaces (wavefronts) of the three waves which propagate in orthorhombic olivine. Three azimuths of observation. (b) Traveltime curves determined by ray tracing (solid lines) and by using the traveltime series approximation (dots).

Table 2. Elastic constants (GPa) and density (g cm^{-3}) of a medium with a set of parallel oriented cracks.

c_{1111}	c_{2222}	c_{2323}	c_{1313}	c_{1122}	c_{2233}	ρ
39.71	39.97	13.23	10.21	13.45	13.52	2.5

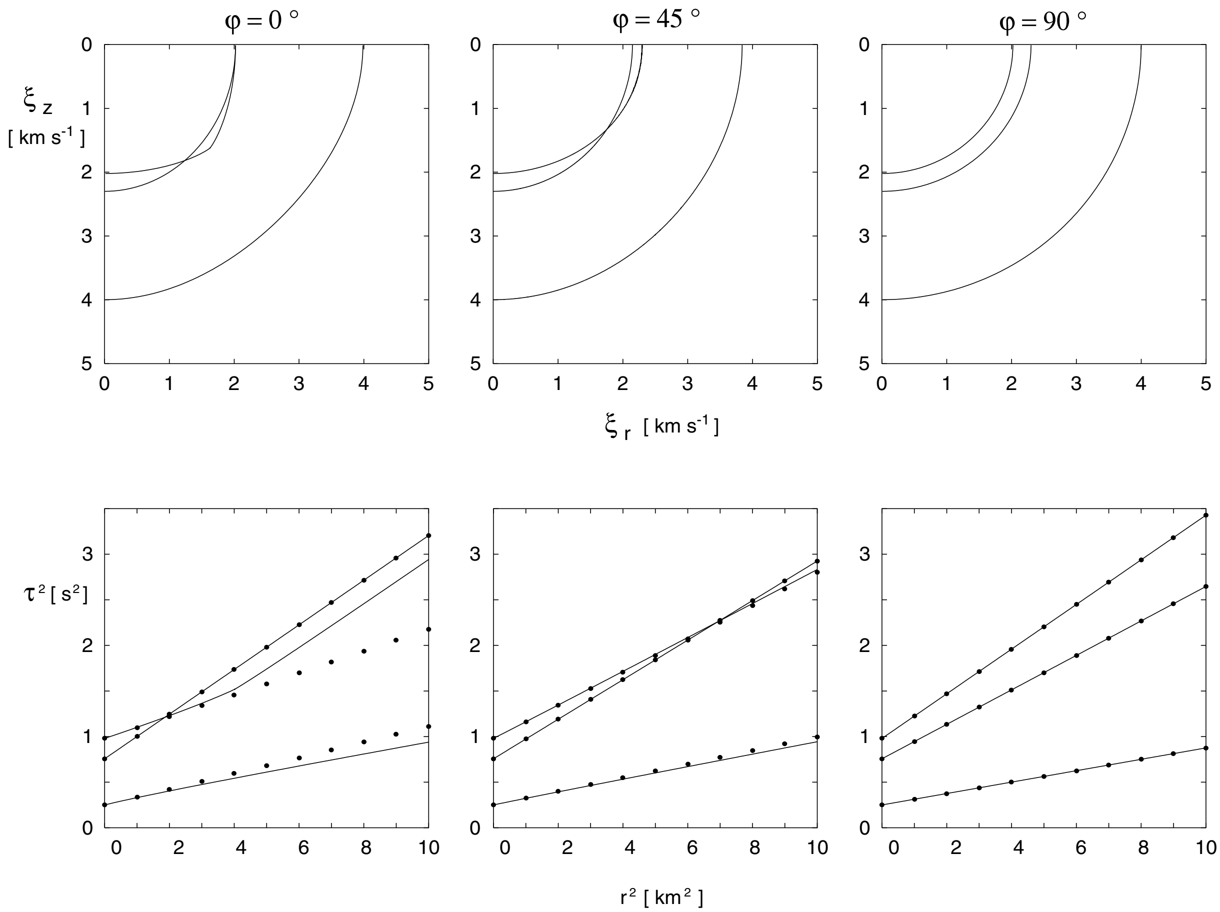

Figure 4. (a) Group velocity surfaces (wavefronts) of the three waves which propagate in a hexagonal medium. Three azimuths of observation. (b) Traveltime curves determined by ray tracing (solid lines) and by using the traveltime series approximation (dots).

and the kiss singularity at $\varphi = 0^\circ$ do not coincide with the direction of the phase normal of the central ray. Therefore eqs (55)–(57) can be applied.

5 DISCUSSION AND CONCLUSIONS

The traveltime formulae derived in Sections 2–4 should be applied if seismic measurements indicate that the reflection traveltimes depend on the azimuth angle of the source–receiver profile. Eqs (1), (17), (24), (41)–(44) and (53) represent a fast solution for computing reflection traveltimes because they involve only a small number of multiplications, whereas ray tracing needs a numerical technique to solve the boundary value problem. For the model of a single homogeneous layer, the assumption of general anisotropy leads to a 3-D problem with a non-hyperbolic traveltime-offset function and a non-linear τ^2–r^2 function. Non-hyperbolic traveltime curves also result from a stack of isotropic layers, but in this case there is no dependence on source–receiver azimuth (Taner & Koehler 1969). In a future investigation we intend to use the result for the single anisotropic layer in combination with a Dix-type formula to calculate interval velocities for a stack of general anisotropic layers. Hake *et al.* (1984) and Sena (1991) treated this problem for the simpler case of anisotropic media of high elastic symmetry.

Eqs (55)–(57) directly link reflection traveltimes to the elastic parameters of orthorhombic media. There are several models that are important for applications in geophysics: olivine is orthorhombic owing to its crystal structure (Anderson 1989); orthorhombic symmetry is also given in the case of horizontal fine layering combined with a set of vertically oriented cracks and in the case of an isotropic host rock with two orthogonal sets of vertical cracks (Schoenberg & Helbig 1997). Eqs (55)–(57) represent the traveltime series truncated after the second-order term. In this case a straight line is obtained in τ^2–r^2 coordinates, the slope of which varies with the azimuth angle of the seismic profile. The intercept and the slope contain information concerning the reflector depth and the elastic parameters of the medium. Measuring traveltimes as a function of offset and azimuth angle allows the determination of some or all elastic parameters of the medium and the reflector depth. In the case of a hexagonal medium with a horizontal symmetry axis all five density-reduced elastic moduli and the reflector depth can be determined independently. In the case of an orthorhombic medium, eight of the nine elastic moduli can be determined; the reflector depth, however, cannot be determined independently. In this case additional information concerning the depth of the reflector is required, and other measurement configurations, e.g. the direct wave in a VSP configuration, could be used to obtain additional information concerning the elastic parameters (Mensch & Farra 1999).

The accuracy and limitations of the reflection traveltime formulae are illustrated in Figs 3 and 4, where approximate traveltimes are compared with exact (ray-tracing) traveltimes. Because the traveltime formulae are derived as a Taylor series expansion around the central ray, their validity is restricted to the vicinity of the central ray. This ray is also called the zero-offset ray because it is the ray that comes back to the source point at the surface after being reflected at the interface in the subsurface.

The source–receiver offset range in Figs 3 and 4 ($r \sim 3h$) includes directions far from the central ray; smaller offsets are often used in reflection seismics. The figures indicate that the approximation can be used with good accuracy for offsets equal to the receiver depth: $r \sim h$. The cut planes through the group velocity surfaces in Figs 3 and 4 show that the approximation fails if the wavefront curvature changes significantly. In this case, the exact τ^2–r^2 curve is no longer a straight line. Calculating the higher-order terms of the series would improve the accuracy of the approximation for large offsets, but the derivation of the cubic coefficient is already quite complicated. Strong anisotropy may even lead to cusps (triplications) in shear wave traveltime-offset curves. This case cannot be described by the Taylor series expansion of eq. (1) and ray tracing should be used instead. SH waves have ellipsoidal group velocity surfaces. In this special case, the exact τ^2–r^2 function is linear (Hake *et al.* 1984). For this type of wave the approximate solution equals the exact one (Fig. 4). If the wavefronts deviate only slightly from an elliptical shape, then eqs (55)–(57) represent a good approximation even for large offsets.

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APPENDIX A: APPROXIMATE TRAVELTIME OF REFLECTED WAVES IN SYMMETRY PLANES OF ANISOTROPIC MEDIA

In this appendix the case of wave propagation in symmetry planes of elastic media is considered. The problem depends only on one ray parameter, which can be chosen as the polar angle θ. The phase normal vector is given by

$$\mathbf{n} = \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}. \tag{A1}$$

The symmetry plane is the x_1–x_3 coordinate plane. Just as for the general problem discussed in the main body of the text, an analogous calculation results in

$$\frac{\partial^2 \tau^2}{\partial x_1^2}(0) = \frac{2}{v(0)}\left[\frac{\partial \xi_1^+}{\partial \theta}(0)\right]^{-1}, \tag{A2}$$

where

$$v(0) = \xi_3^+(0). \tag{A3}$$

If one differentiates the equation

$$\frac{\partial \boldsymbol{\xi}}{\partial \theta} \cdot \mathbf{n} = 0 \tag{A4}$$

with respect to θ, one obtains in the limit of the central ray:

$$\frac{\partial^2 \xi_3^+}{\partial \theta^2}(0) = -\frac{\partial \xi_1^+}{\partial \theta}(0). \tag{A5}$$

Differentiating

$$\boldsymbol{\xi} \cdot \mathbf{n} = v \tag{A6}$$

and using eqs (A3) and (A5) gives

$$\frac{\partial \xi_1^+}{\partial \theta}(0) = v(0) + \frac{\partial^2 v}{\partial \theta^2}(0). \tag{A7}$$

Then eq. (A2) can be written in the form provided by Tsvankin (1995):

$$\frac{\partial^2 \tau^2}{\partial x_1^2}(0) = \frac{2}{v(0)}\left[v(0) + \frac{\partial^2 v}{\partial \theta^2}(0)\right]^{-1}. \tag{A8}$$

APPENDIX B: APPROXIMATE TRAVELTIME OF THE REFLECTED *qP* WAVE IN WEAKLY ANISOTROPIC MEDIA

The approximation to the traveltime field can be combined with perturbation theory to give a simple formula for the traveltime of the reflected qP wave in a weakly anisotropic medium.

The density-reduced elastic moduli are represented by an isotropic term of zeroth order and an anisotropic term of first order:

$$\lambda_{iklm} = \left(v_p^2 - 2v_s^2\right)\delta_{ik}\delta_{lm} + v_s^2(\delta_{il}\delta_{km} + \delta_{im}\delta_{kl}) + \epsilon\lambda_{iklm}^{(1)}, \tag{B1}$$

where ϵ is a small dimensionless parameter.

Applying perturbation theory to the Christoffel equation, the phase velocity and the polarization vector are obtained to first order in ϵ by Backus (1965):

$$v = v^{(0)} + \epsilon v^{(1)} = v_p + \epsilon\frac{1}{2v_p}\sum_{i,k,l,m=1}^{3}\lambda_{iklm}^{(1)} n_i n_k n_l n_m \tag{B2}$$

and

$$A_i = A_i^{(0)} + \epsilon A_i^{(1)} = n_i + \epsilon\frac{1}{v_p^2 - v_s^2}\left[\sum_{k,l,m=1}^{3}\lambda_{iklm}^{(1)} n_k n_l n_m - \sum_{t,u,v,w=1}^{3}\lambda_{tuvw}^{(1)} n_t n_u n_v n_w n_i\right], \qquad (i = 1, 2, 3). \tag{B3}$$

The group velocity of the qP wave is given by

$$\xi_i = \xi_i^{(0)} + \epsilon\xi_i^{(1)} = v_p n_i + \epsilon\frac{1}{v_p}\left[2\sum_{k,l,m=1}^{3}\lambda_{iklm}^{(1)} n_k n_l n_m - \frac{3}{2}\sum_{t,u,v,w=1}^{3}\lambda_{tuvw}^{(1)} n_t n_u n_v n_w n_i\right], \qquad (i = 1, 2, 3). \tag{B4}$$

In the limit of the central ray one obtains for the first derivatives of the group velocity to zeroth order

$$\frac{\partial \boldsymbol{\xi}^{+(0)}}{\partial \gamma_s}(0,0) = v_p \mathbf{i}_s, \qquad (s = 1, 2), \tag{B5}$$

and to first order

$$\frac{\partial \xi_1^{+(1)}}{\partial \gamma_1}(0,0) = \frac{1}{v_p}\left[2\lambda_{1133}^{(1)} + 4\lambda_{1313}^{(1)} - \frac{3}{2}\lambda_{3333}^{(1)}\right], \tag{B6}$$

$$\frac{\partial \xi_2^{+(1)}}{\partial \gamma_2}(0,0) = \frac{1}{v_p}\left[2\lambda_{2233}^{(1)} + 4\lambda_{2323}^{(1)} - \frac{3}{2}\lambda_{3333}^{(1)}\right], \tag{B7}$$

$$\frac{\partial \xi_2^{+(1)}}{\partial \gamma_1}(0,0) = \frac{\partial \xi_1^{+(1)}}{\partial \gamma_2}(0,0) = \frac{1}{v_p}\left[2\lambda_{1233}^{(1)} + 4\lambda_{2313}^{(1)}\right]. \tag{B8}$$

According to eqs (17) and (41)–(44), the traveltime of the central ray and its second partial derivatives are given by

$$\tau(0,0) = \frac{2h}{v_p}\left[1 - \epsilon\frac{\lambda_{3333}^{(1)}}{2v_p^2}\right], \tag{B9}$$

$$\frac{\partial^2 \tau}{\partial x_1^2}(0,0) = \frac{1}{2hv_p}\left\{1 - \epsilon\frac{1}{v_p^2}\left[2\lambda_{1133}^{(1)} + 4\lambda_{1313}^{(1)} - \frac{3}{2}\lambda_{3333}^{(1)}\right]\right\}, \tag{B10}$$

$$\frac{\partial^2 \tau}{\partial x_2^2}(0,0) = \frac{1}{2hv_p}\left\{1 - \epsilon\frac{1}{v_p^2}\left[2\lambda_{2233}^{(1)} + 4\lambda_{2323}^{(1)} - \frac{3}{2}\lambda_{3333}^{(1)}\right]\right\}, \tag{B11}$$

$$\frac{\partial^2 \tau}{\partial x_1 \partial x_2}(0,0) = -\epsilon\frac{1}{hv_p^3}\left[\lambda_{1233}^{(1)} + 2\lambda_{2313}^{(1)}\right]. \tag{B12}$$