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Unsaturated flow in heterogeneous soils with spatially distributed uncertain hydraulic parameters

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Abstract

Uncertain soil properties are often modeled as random fields. This renders the unsaturated flow equations stochastic. Determining statistics of pressure head statistics, ψ, is nontrivial, since the Richards equation is highly nonlinear. The prevalent approach is to linearize relative hydraulic conductivity, $K_r(\psi)$, around the ensemble mean pressure head, $\langle\psi\rangle$, which often leads to significant errors. Recently, an approach has been proposed to avoid such a linearization for the Gardner model, $K_r = \exp(\alpha\psi)$, with the soil parameter α being a random variable. We generalize this approach by allowing α to be a random field. This is achieved by means of a partial mean-field approximation with respect to $\alpha(\mathbf{x})$. Using two-dimensional infiltration into a heterogeneous soil with uncertain hydraulic parameters as an example, we demonstrate that our predictions of the mean pressure head and its variance remain accurate for moderately variable αs. The robustness of our solutions increases with the correlation length of α.

Keywords: Stochastic; Random; Moment equations; Porous media; Nonlinear

1. Introduction

Even within a given soil type, hydraulic properties and parameters often vary significantly from point to point in a manner that cannot be described with certainty. Therefore, it seems appropriate to model saturated hydraulic conductivity and the parameters of constitutive relations between relative conductivity and pressure head in unsaturated soils as correlated random fields, and to cast the unsaturated flow equations in a stochastic framework.

Once statistical properties of randomly heterogeneous parameters and forcing terms (sources, initial and boundary conditions) have been determined, one can solve the corresponding stochastic flow equations analytically or numerically. The most common and widely applicable approach is to solve the stochastic

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flow equations numerically by conditional Monte Carlo simulations, and then to analyze the results statistically. The sample statistics most commonly computed from such simulations include (conditional) mean hydraulic heads and gradients, volumetric water fluxes and seepage velocities. The theoretical (ensemble) counterparts of these (conditional) sample means constitute unbiased predictors of system behavior and/or performance under uncertainty. The conditional predictors are optimal in that the sum of squared prediction errors is minimum (Mood and Graybill, 1963). Another statistic commonly computed from Monte Carlo simulations is the sample variance–covariance of the prediction errors. Its theoretical (ensemble) counterpart constitutes a measure of predictive uncertainty. It is common for Monte Carlo simulations to assume that the input variables (parameters and forcing terms) are multivariate Gaussian or log-Gaussian. When this assumption holds, the conditional predictors are optimal in that the prediction errors have minimum variance (Mood and Graybill, 1963).

Although the Monte Carlo method is conceptually straightforward and has the indisputable advantage of applying to a very broad range of both linear and nonlinear flow and transport problems, its major drawbacks have led to development of alternative methods. A conceptual disadvantage of the Monte Carlo approach is that it provides no theoretical insight into physical phenomena. Especially for highly nonlinear stochastic differential equations, there is no guarantee that Monte Carlo simulations have converged to the exact (ensemble) solution after some large number of realizations. To the best of our knowledge, there are no well-established computational criteria to predict the number of realizations required to achieve the desired accuracy. This becomes especially critical in assessing higher order moments or the probability distribution of state variables of interest.

An alternative to Monte Carlo simulations is provided by moment equations, which yield the corresponding predictions of flow and transport deterministically. Moment equations have been applied successfully to describe steady (Neuman and Orr, 1993; Neuman et al., 1996; Guadagnini and Neuman, 1999a,b) and transient (Tartakovsky and Neuman, 1998, 1999) saturated flows and transport (Neuman, 1993; Guadagnini and Neuman, 2001) in heterogeneous formations. The unifying feature of these and similar physical phenomena is that they are described by linear stochastic partial differential equations. Recently, we extended our moment equations approach to model a certain class of nonlinear processes. These include steady-state unsaturated flow (Tartakovsky et al., 1999; Lu et al., 2000) and gas flow in heterogeneous porous media (Tartakovsky, 1999, 2000; Tartakovsky and Guadagnini, 2001).

In this paper, we consider steady-state flow in partially saturated media with spatially distributed uncertain hydraulic parameters. We carry out our analysis along the lines of Tartakovsky et al. (1999) and Lu et al. (2000) by relying upon the Kirchhoff transform to derive the moment equations for pressure head, ψ, in soils with relative hydraulic conductivity described by the Gardner exponential model, $K_r = \exp(\alpha\psi)$. Applying the Kirchhoff transform to the original stochastic unsaturated flow equations has the unique advantage of fully preserving the nonlinearity of $K_r(\psi)$. This is in contrast to the recent analyses by Indelman et al. (1993), Li and Yeh (1998), and Foussereau et al. (2000), which found it necessary to linearize $K_r(\psi)$. The downside of the analyses in Tartakovsky et al. (1999) and Lu et al. (2000), as well as of the linearization-free Gaussian approximation of Amir and Neuman (2001), is that they require the pore-size distribution parameter, α, to be a random variable, which reduces the range of their practical applicability. Our main goal is to eliminate this limitation by allowing $\alpha = \alpha(\mathbf{x})$ to vary randomly in space.

We formulate the stochastic boundary-value problem in Section 2. Then we derive in Section 3 deterministic boundary-value problems for (ensemble) mean pressure head, $\langle\psi\rangle$, and for the corresponding pressure head variance, σ_ψ^2. The robustness of the approximations used in our derivations is analyzed in Section 4 for a two-dimensional example. In the absence of a better yardstick, we demonstrate, and comment upon, the accuracy of the moment equations approach by comparing our results with those obtained from a set of Monte Carlo simulations of the original nonlinear problem.

2. Problem formulation

Consider steady-state flow in a variably saturated soil (flow domain Ω) that is described by the Darcy law

$$\mathbf{q}(\mathbf{x}) = -K(\mathbf{x}, \psi)\nabla[\psi(\mathbf{x}) + x_3], \qquad (1)$$

where $\mathbf{q}(\mathbf{x})$ is the flux, $K(\mathbf{x}, \psi)$ is the unsaturated hydraulic conductivity, and x_3 is the vertical coordinate (taken to be positive upward). The Darcy law is supplemented with the continuity equation

$$-\nabla\cdot\mathbf{q}(\mathbf{x}) + f(\mathbf{x}) = 0 \qquad \mathbf{x} \in \Omega \qquad (2)$$

where $f(\mathbf{x})$ is a randomly prescribed source function. These equations are subject to the boundary conditions

$$\psi(\mathbf{x}) = \Psi(\mathbf{x}) \qquad \mathbf{x} \in \Gamma_D \qquad (3)$$

$$-\mathbf{q}(\mathbf{x})\cdot\mathbf{n}(\mathbf{x}) = Q(\mathbf{x}) \qquad \mathbf{x} \in \Gamma_N. \qquad (4)$$

Here $\Psi(\mathbf{x})$ is the random pressure head on Dirichlet boundary segments Γ_D, $Q(\mathbf{x})$ is the random flux across Neumann boundary segments Γ_N, $\mathbf{n} = (n_1, n_2, n_3)^T$ is a unit outward normal to the boundary Γ, and $\Gamma = \Gamma_D \cup \Gamma_N$. Though it is not strictly necessary, we assume for simplicity that the source and boundary functions $f(\mathbf{x})$, $\Psi(\mathbf{x})$ and $Q(\mathbf{x})$ are prescribed in a statistically independent manner. The physical quantities, such as K, ψ, and $\mathbf{q}$, are representative of a measurement volume centered about location $\mathbf{x}$, which is small relative to the flow domain size, but is large enough to guarantee that Darcy's law applies on its scale.

A choice of the functional dependence of unsaturated hydraulic conductivity on pressure head, $K(\psi)$, completes the description. In this study we employ the Gardner exponential model,

$$K(\mathbf{x}, \psi) = K_s(\mathbf{x})K_r(\mathbf{x}, \psi), \qquad K_r(\mathbf{x}, \psi) = e^{\alpha(\mathbf{x})\psi}, \qquad (5)$$

where K_s and K_r are the saturated and relative hydraulic conductivities, respectively, and α is the reciprocal of the macroscopic capillary length scale (Raats, 1976). Since the constitutive parameters K_s and α are highly variable in space and are typically under-determined by data, we describe them as random fields. This and the uncertain forcing terms render Eqs. (1)–(4) stochastic.

Choosing the Gardner model allows us to use the Kirchhoff transformation to derive the moment equations for boundary-value problem (1)–(4), which preserve the constitutive nonlinearity. By applying the Kirchhoff transform, Tartakovsky et al. (1999) have developed the first and second moment equations for stochastic steady-state unsaturated flow with the random variable α. These recursive moment equations have then been solved analytically in one dimension. More recently, Lu et al. (2000) have developed a finite elements algorithm to solve our moment equations in two dimensions. They have analyzed both mean uniform and divergent flows, with and without conditioning on discrete measurements of K_s. Using Monte Carlo simulations as a yardstick, the authors have demonstrated that their numerical solutions are remarkably accurate (more so in the conditional than in the unconditional case) for strongly heterogeneous soils with log-conductivity variance as large as 2.

These results are based on the formalism that treated α as a space-independent random variable. The authors justified this assumption on the basis of published data concerning the spatial variability α, which are quite ambiguous. Whereas the spatial variability of K_s has been studied extensively, there have been relatively few studies on the spatial statistics of α. While measuring K_s is relatively straightforward, the soil parameter α can be determined only by indirect methods. These include least square analyses of the data for relative conductivity (Russo, 1983, 1984; Ünlü et al., 1990), for water retention (Wierenga et al., 1991), and for sorptivity (White and Sully, 1992), as well as an inversion of the infiltration experiments (Russo and Bouton, 1992).

Fig. 1 summarizes the experimentally determined values of α for several soil types. While most of these studies suggest that both K_s and α are log-normal, others found α to be approximately normal. Variance of $\ln \alpha$ was found to be either larger or smaller than that of $\ln K_s$. Similarly, no general consensus exists about the relative magnitude of correlation scales for $\ln K_s$ and $\ln \alpha$ and about cross-correlation between the two. For example, Russo and Bouton (1992) reported the vertical and horizontal correlation scales of $\ln \alpha$ to be approximately three times smaller than the respective correlation scales of K_s, while Ünlü et al. (1990) found them to be larger. Analyzing three different soil types, Raga and Cooper (1993a,b) pointed out the lack of cross-correlation between

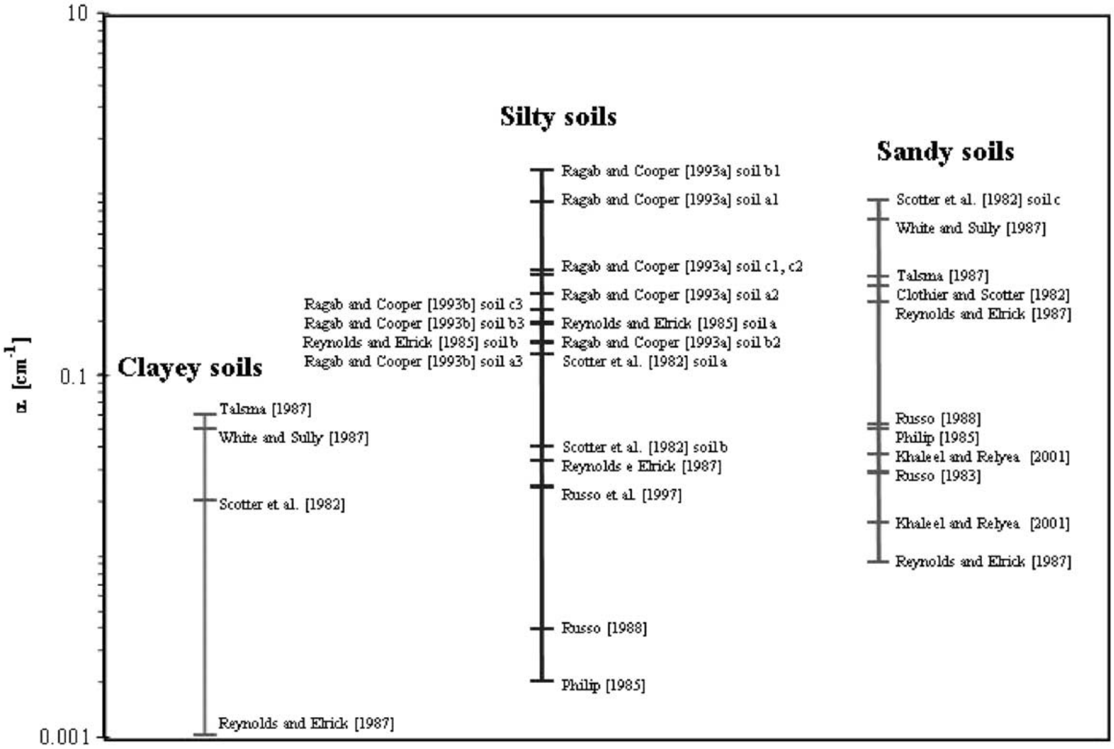

Fig. 1. Values of the Gardner parameter α for several soil types.

ln α and ln K_s. Russo and Bouton (1992) reported a weak cross-correlation. Similar findings can be found in Russo (1983, 1984) and Wierenga et al. (1991). At the same time, other studies resulted in moderate (Russo et al., 1997) or high (Ünlü et al., 1990) cross-correlations. It therefore comes as no surprise that stochastic models of unsaturated flow range from those disregarding such cross-correlations (Tartakovsky et al., 1999; Lu et al., 2000) to those incorporating them fully (Yeh, 1989).

Our approach places the following restrictions on statistics of the random fields K_s and α. In general, we do not require the random field K_s to be statistically homogeneous and allow for an arbitrary (reasonable) spatial correlation structure. Thus the moment equations in Section 3 are written for the second-order statistically homogeneous K_s field, and our numerical example in Section 4 additionally assumes K_s to be log-normal with anisotropic exponential correlation function for $Y = \ln K_s$. Although our theory requires the random field α to be first-order statistically homogeneous and arbitrary otherwise, in the example in Section 4 we take α to be the second-order statistically homogeneous Gaussian random field with an anisotropic exponential correlation function. The derivations in Section 3 do not make any assumptions regarding the cross-correlation between K_s and α, or the lack thereof. The numerical example in Section 4 assumes no correlation between the two. Of course, such a relative generality of our moment equations does not guarantee their robustness for all possible permutations of the K_s and α statistics. In fact, we expect the solutions of these equations to be excellent approximations of the 'true' distributions of $\langle\psi\rangle$ and σ_ψ^2 in some cases, while failing to do so in others. Our example in Section 4 only begins to explore this issue.

3. General theory

Our goal is to take full advantage of the Kirchhoff-transform-based approach of Tartakovsky et al. (1999), since it possesses the unique ability to fully preserve the nonlinearity of the constitutive relationship between relative hydraulic conductivity and pressure head. A spatially distributed random field α requires an extra step for the Kirchhoff transform to be useful.

3.1. Partial mean-field approximation

We employ a partial mean-field approximation,

$$K_r(\mathbf{x}, \psi) \approx \tilde{K}_r(\psi) \equiv e^{\langle\alpha\rangle\psi}, \tag{6}$$

to eliminate the explicit spatial dependence of $K_r(\mathbf{x}, \psi)$. Of course, this requires the random field $\alpha(\mathbf{x})$ to be first-order statistically homogeneous.

Quite often mean-field approximations, wherein the random system parameters are replaced with their ensemble means, proved to be at least as accurate as linearizations of the underlying stochastic differential equations (Tartakovsky and Guadagnini, 2001; Tartakovsky and Winter, 2001). Here we employ the partial mean-field approximation by keeping saturated hydraulic conductivity, K_s, random. In what follows, we explore the conditions under which this approximation remains accurate.

Clearly, the partial mean-field approximation (6) is analogous to retaining the leading term in the Taylor expansion of $\exp(\alpha\psi)$ about the mean $\langle\alpha\rangle$. Hence its accuracy requires the random fluctuations of α around $\langle\alpha\rangle$ to be small. We explore this requirement in Section 4.

3.2. Moment equations

Combining Eqs. (1) and (2), while employing approximation (6), yields

$$\nabla\cdot\{K_s(\mathbf{x})\tilde{K}_r(\psi)\nabla[\psi(\mathbf{x}) + x_3]\} + f(\mathbf{x}) = 0 \quad \mathbf{x} \in \Omega. \tag{7}$$

Upon applying the Kirchhoff transform,

$$\Phi(\mathbf{x}) = \int_{-\infty}^{\psi(\mathbf{x})} \tilde{K}_r(t)\mathrm{d}t = \langle\alpha\rangle^{-1} e^{\langle\alpha\rangle\psi}, \tag{8}$$

Eq. (7) takes the form of a linear partial differential equation,

$$\nabla\cdot[K_s(\mathbf{x})\nabla\Phi(\mathbf{x})] + \langle\alpha\rangle\frac{\partial}{\partial x_3}[K_s(\mathbf{x})\Phi(\mathbf{x})] + f(\mathbf{x}) = 0 \qquad \mathbf{x} \in \Omega. \tag{9}$$

Transforming boundary conditions (3) and (4) yields

$$\Phi(\mathbf{x}) = H(\mathbf{x}), \qquad H(\mathbf{x}) = \langle\alpha\rangle^{-1} e^{\langle\alpha\rangle\Psi} \quad \mathbf{x} \in \Gamma_D \tag{10}$$

$$K_s(\mathbf{x})\nabla\Phi(\mathbf{x})\cdot\mathbf{n}(\mathbf{x}) + \langle\alpha\rangle n_3(\mathbf{x})K_s(\mathbf{x})\Phi(\mathbf{x}) = Q(\mathbf{x}) \quad \mathbf{x} \in \Gamma_N. \tag{11}$$

Stochastic averaging of Eqs. (9)–(11), combined with the perturbation expansion of the mean pressure head into an asymptotic expansion $\langle\Phi\rangle = \langle\Phi^{(0)}\rangle + \langle\Phi^{(1)}\rangle + \cdots$ in the variance σ_Y^2 of $Y = \ln K_s$, yields for $i = 0, 1$ (see Tartakovsky et al. (1999) for more details)

$$\langle\Phi^{(i)}(\mathbf{x})\rangle = \int_\Omega f_i(\mathbf{y})G(\mathbf{y},\mathbf{x})\mathrm{d}\mathbf{y} - \langle\alpha\rangle\int_{\Gamma_D} H_i(\mathbf{y})\mathbf{n}\cdot \nabla_{\mathbf{y}}G(\mathbf{y},\mathbf{x})\mathrm{d}\mathbf{y} + \int_{\Gamma_N} Q_i(\mathbf{y})G(\mathbf{y},\mathbf{x})\mathrm{d}\mathbf{y}. \tag{12}$$

Here the kernel $G(\mathbf{y},\mathbf{x})$ is given by the zeroth-order mean Green's function for Eqs. (9)–(11) and

$$f_0 \equiv \langle f\rangle, \qquad H_0 \equiv \langle\alpha\rangle^{-1} e^{\langle\alpha\rangle\langle\Psi\rangle}, \qquad Q_0 \equiv \langle Q\rangle, \tag{13a}$$

$$f_1 \equiv \nabla\cdot\left[\frac{\sigma_Y^2}{2}\nabla\langle\Phi^{(0)}\rangle - \mathbf{r}^{(1)}\right] - \langle\alpha\rangle\frac{\partial}{\partial x_3} \times \left[\frac{\sigma_Y^2}{2}\langle\Phi^{(0)}\rangle + C_{K_s\Phi}^{(1)}\right], \tag{13b}$$

$$H_1 \equiv 0, \tag{13c}$$

$$Q_1 \equiv -\mathbf{n}\cdot\left[\frac{\sigma_Y^2}{2}\nabla\langle\Phi^{(0)}\rangle - \mathbf{r}^{(1)}\right] + \langle\alpha\rangle^{-1}n_3\left[\frac{\sigma_Y^2}{2}\langle\Phi^{(0)}\rangle + C_{K_s\Phi}^{(1)}\right]. \tag{13d}$$

The first-order approximations of the mixed moments $\mathbf{r}(\mathbf{x}) = -\langle K_s'(\mathbf{x})\nabla\Phi'(\mathbf{x})\rangle$ and $C_{K_s\Phi}(\mathbf{x}) = \langle K_s'(\mathbf{x})\Phi'(\mathbf{x})\rangle$ are given by

$$\mathbf{r}^{(1)}(\mathbf{x}) = \int_\Omega C_Y\nabla_{\mathbf{x}}\nabla_{\mathbf{y}}^{\mathrm{T}}G\nabla_{\mathbf{y}}\langle\Phi^{(0)}\rangle\mathrm{d}\mathbf{y} - \langle\alpha\rangle^{-1}\int_\Omega C_Y\langle\Phi^{(0)}\rangle\nabla_{\mathbf{x}}\frac{\partial G}{\partial y_3}\mathrm{d}\mathbf{y} \tag{14a}$$

$$C^{(1)}_{K_s\Phi}(\mathbf{x}) = -\int_\Omega C_Y \nabla_\mathbf{y} G \cdot \nabla_\mathbf{y} \langle \Phi^{(0)} \rangle \mathrm{d}\mathbf{y} + \langle \alpha \rangle^{-1} \int_\Omega C_Y \langle \Phi^{(0)} \rangle \frac{\partial G}{\partial y_3} \mathrm{d}\mathbf{y} \quad (14b)$$

where $C_Y(\mathbf{x},\mathbf{y}) = \langle Y'(\mathbf{x}) Y'(\mathbf{y}) \rangle$ is the covariance of Y'. The variance of the Kirchhoff transform, σ^2_Φ, is found in a similar manner (Tartakovsky et al., 1999).

Once Eq. (12) have been solved, one can continue by developing first-order approximations $\langle \psi^{[1]} \rangle = \langle \psi^{(0)} \rangle + \langle \psi^{(1)} \rangle$ for the mean pressure head and associated second moments,

$$\langle \psi^{(0)} \rangle = \ln \langle \Phi^{(0)} \rangle, \qquad \langle \psi^{(1)} \rangle = \frac{\langle \Phi^{(1)} \rangle}{\langle \Phi^{(0)} \rangle} - \frac{[\sigma^2_\Phi]^{(1)}}{2 \langle \Phi^{(0)} \rangle^2} \quad (15)$$

and

$$[\sigma^2_\psi]^{(1)} = \frac{[\sigma^2_\Phi]^{(1)}}{\langle \Phi^{(0)} \rangle^2}. \quad (16)$$

4. Computational example

To illustrate our computational approach and to ascertain the accuracy of our partial mean-field approximation, we consider infiltration in a vertical cross-section of a heterogeneous soil. The flow domain is taken to be a rectangle of width 40 and height 3. (Here and below some suitable consistent units are used for all physical parameters.) The top boundary, $x_3 = 3$, represents the soil surface, and the water table defines the bottom boundary, $x_3 = 0$. Infiltration at a constant rate $Q_i = 0.01$ occurs at the left portion of the top boundary, $x_1 \in [0, 5]$. The rest of the soil surface, $x_1 \in [5, 40]$, experiences the outflow at a constant rate $Q_0 = 0.0002$. At the water table, $x_3 = 0$, the pressure head $\psi = 0$. Both vertical boundaries are assumed no-flow. Loosely, such an example might represent evaporation taking place due to localized irrigation.

The soil is assumed to be statistically homogeneous and anisotropic with correlation lengths, l_Y and l_α, for both $Y = \ln K_s$ and α equal to 4.0 and 2.0 along the horizontal and vertical directions, respectively. The anisotropic exponential correlation function is selected for both random fields. We use the following values for the soil parameters and their statistics: $\langle Y \rangle = 1.0$ (or $K_G = 2.718$), $\sigma^2_Y = 0.25$, $\langle \alpha \rangle = 1.0$. Variance σ^2_α takes on several values as described below. The flow domain is subdivided into 40×30 rectangular elements 1.0 by 0.1 each.

We use the nonlocal finite element code of Lu et al. (2000) to solve our partial mean-field moment equations. Here we focus on the numerical solutions of our equations and their comparison with Monte Carlo simulations. A detailed description of both methods can be found in Lu et al. (2000). To insure the compatibility between the two computational methods, we used in the moment equations the parameter values approximated from sample realizations of Monte Carlo runs. In practical applications, one would normally infer these values geostatistically from measurements by methods such as kriging.

The Monte Carlo simulations commence with generating, by means of a Gaussian sequential simulator GCOSIM (Gómez-Hernández, 1991), 2000 unconditional realizations of a random field with zero mean and unit variance. These random fields (1000 realizations each) are then scaled as realizations of Y and α with mean and variances as specified above. We allow the variance of α to vary from case to case, to reproduce the coefficient of variation, CV_α, ranging from 0.0 to 0.5. Again, we assume that the Y and α fields have the same correlation scales and that Y and α are uncorrelated. To analyze the flow statistics, we assign to each element constant Y and α values corresponding to the point value generated at its center by GCOSIM. The unconditional auto-covariance obtained from Monte Carlo simulations compared favorably with that given theoretically.

Fig. 2 compares the mean pressure head distribution and its variance computed with the two methods for $\mathrm{CV}_\alpha = 0.0$, i.e. for deterministic α. The streamlines are drawn based on the results of the moment equations. It should be noted that the equipotential lines are not perpendicular to the streamlines, partially due to the medium anisotropy and partially due to the fact that the scale in the vertical direction has been exaggerated for graphical reasons. Since α is deterministic, our partial mean field approximation becomes exact, and the only approximation is due to a closure of

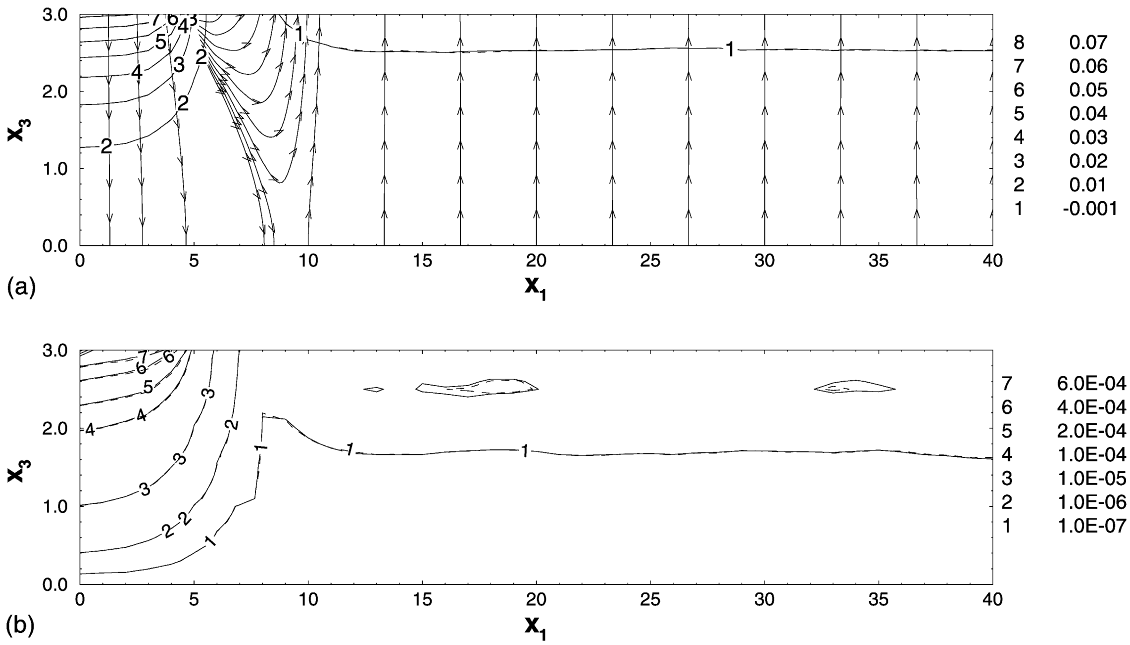

Fig. 2. Mean hydraulic head (a) and head variance (b) for $CV_\alpha = 0.0$. Dashed curves are results from Monte Carlo simulations, and solid curves are the results of the moment equations with partial mean-field approximation.

the moment equations by perturbation expansion in σ_Y^2. As expected from our previous experience, such a closure leads to the head statistics which is in almost perfect agreement with the head statistics obtained from Monte Carlo simulations.

Figs. 3–5 provide the same comparisons for $CV_\alpha = 0.1, 0.2, 0.5$, respectively. It is clear that, at least in our example problem where the medium is weakly correlated along the horizontal direction (Y and α correlation lengths are 1/10 of the horizontal domain size) and strongly correlated along the vertical direction (the characteristic vertical dimension of the domain spans 1.5 vertical correlation lengths of Y), our partial mean-field approximation captures satisfactorily the essential features of the spatial pattern of the statistical moments of pressure head. While discrepancies are almost absent for moderate CV_α, they tend to increase with CV_α. This is especially so for the head variance σ_ψ^2. However, it is worthwhile to remember that our approximation of σ_ψ^2 represents but the leading term in the perturbation expansion of the exact solution, while the more accurate approximation of the mean head consists of the two leading terms in a similar expansion.

On the basis of the few field measurements reported in the literature and reviewed in Section 2, the range of spatial variability of α is still not clear and may depend on several field factors. However, in many soils α exhibits spatial variability which is much smaller than that of saturated hydraulic conductivity K_s. Since our partial mean-field approximation remains robust for moderate values of CV_α, we expect our theory to provide accurate estimates of the pressure head statistics for many practical applications.

Figs. 5–7 demonstrate the influence of a correlation structure of the $\alpha(\mathbf{x})$ field on the accuracy of our predictions. In these simulations, we used the same parameters as in Fig. 5 except for the correlation length l_α. Fig. 6 is obtained by setting $l_\alpha = 40$ and 4 in the horizontal and vertical directions, respectively; while Fig. 7 is for infinite l_α in both directions, i.e. for the random constant α. This comparison demonstrates that the robustness of our partial mean-field approximation increases with the correlation length of α.

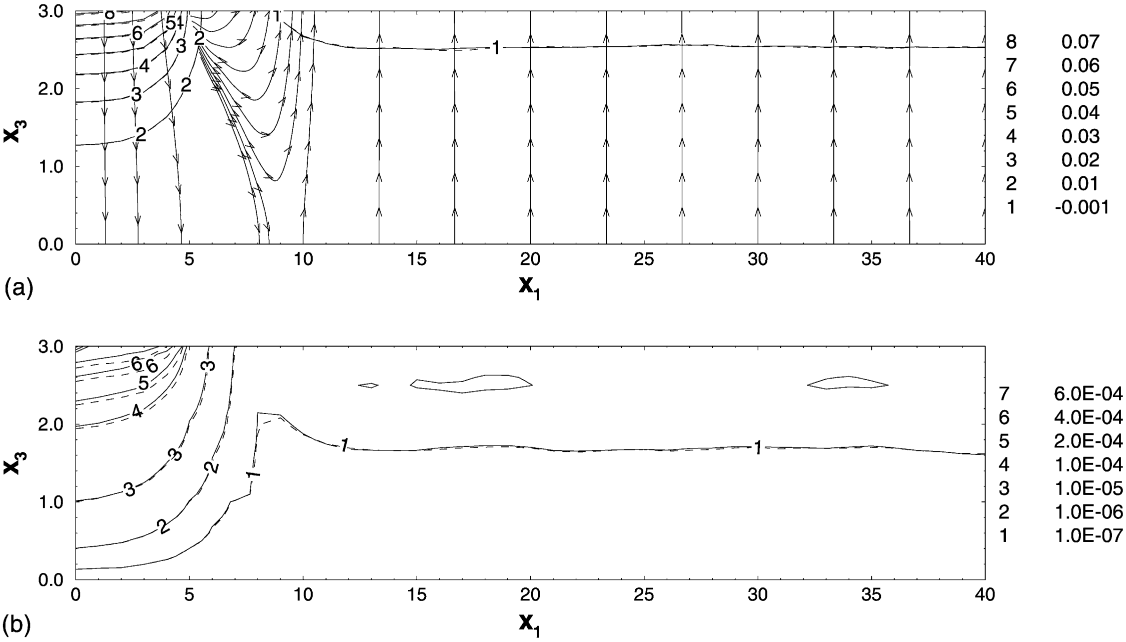

Fig. 3. Mean hydraulic head (a) and head variance (b) for $CV_\alpha = 0.1$. Dashed curves are results from Monte Carlo simulations, and solid curves are the results of the moment equations with partial mean-field approximation.

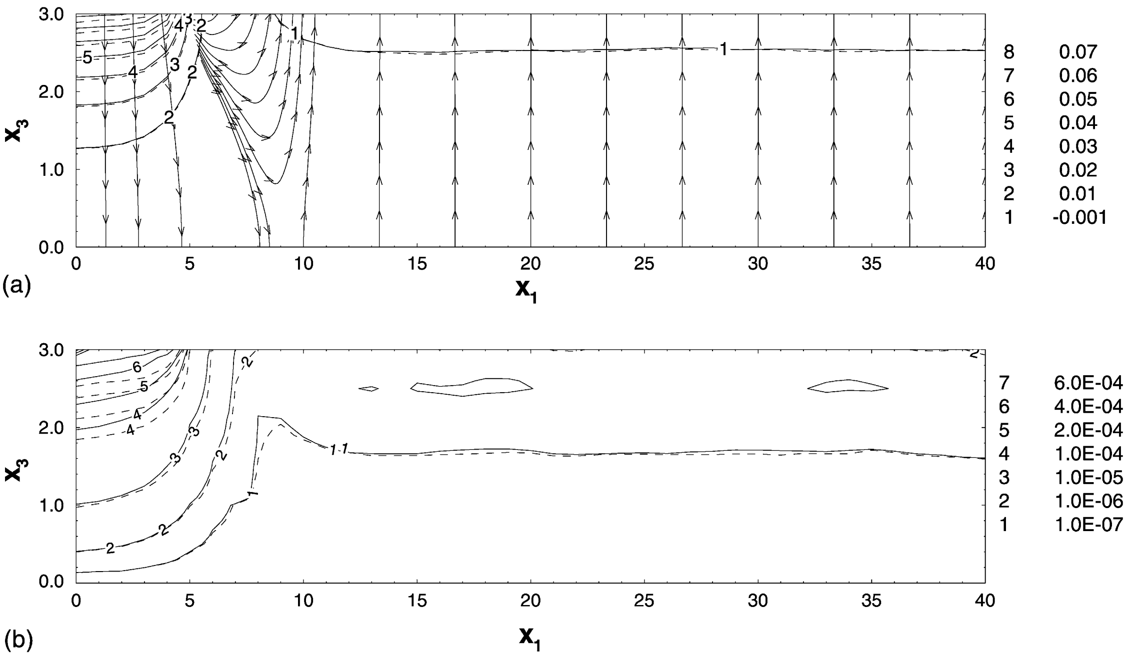

Fig. 4. Mean hydraulic head (a) and head variance (b) for $CV_\alpha = 0.2$. Dashed curves are results from Monte Carlo simulations, and solid curves are the results of the moment equations with partial mean-field approximation.

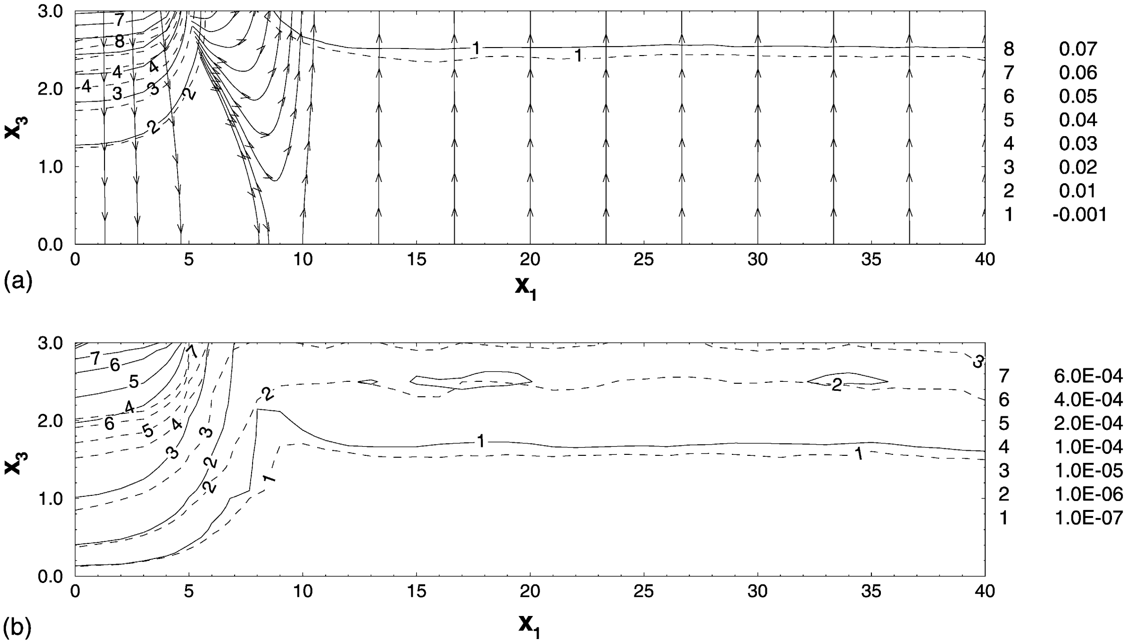

Fig. 5. Mean hydraulic head (a) and head variance (b) for $CV_\alpha = 0.5$. Dashed curves are results from Monte Carlo simulations, and solid curves are the results of the moment equations with partial mean-field approximation.

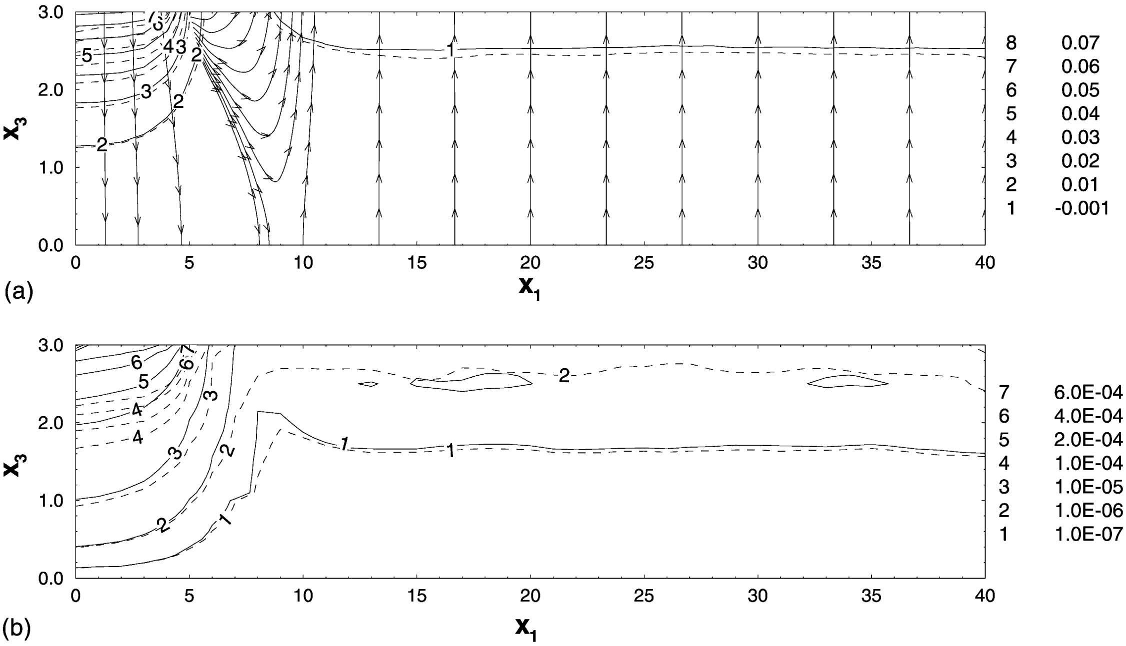

Fig. 6. Mean hydraulic head (a) and head variance (b) for $CV_\alpha = 0.5$ and $l_\alpha = 40$ and 4 in the horizontal and vertical directions, respectively. Dashed curves are results from Monte Carlo simulations, and solid curves are the results of the moment equations with partial mean-field approximation.

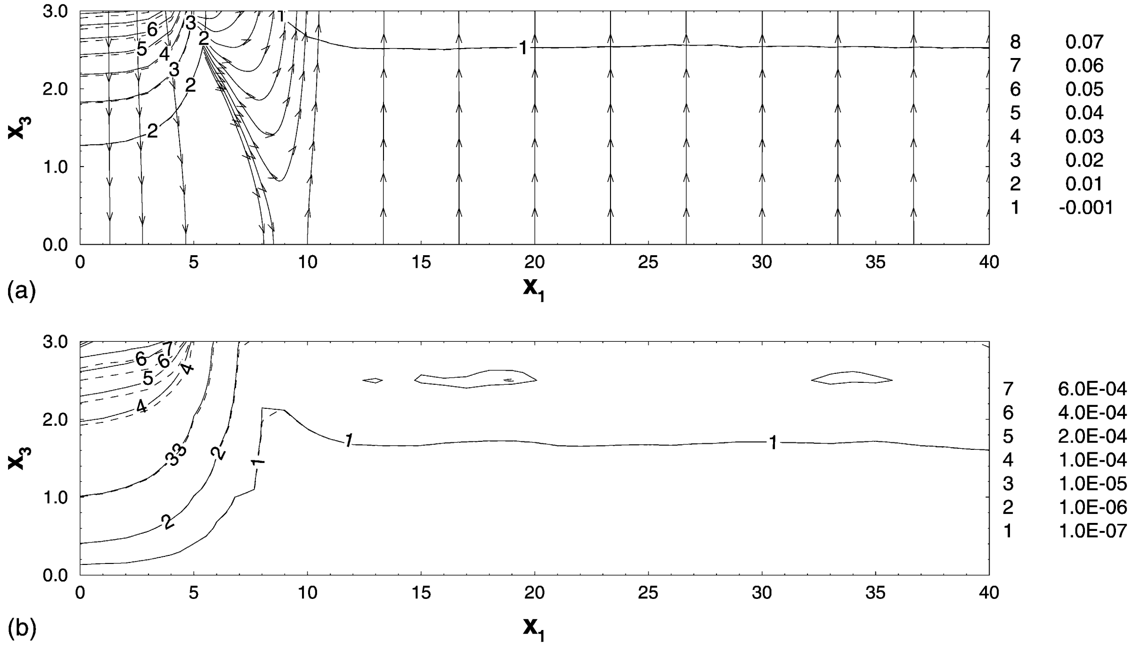

Fig. 7. Mean hydraulic head (a) and head variance (b) for $\mathrm{CV}_\alpha = 0.5$ and $l_\alpha = \infty$. Dashed curves are results from Monte Carlo simulations, and solid curves are the results of the moment equations with partial mean-field approximation.

5. Conclusions

Prediction of unsaturated flow in heterogeneous soils with uncertain hydraulic properties is complicated by the nonlinearity of the governing Richards equation. The Kirchhoff transform provides a unique advantage of preserving this constitutive nonlinearity while deriving the moment equations for pressure head, ψ. The earlier applications of this approach (Tartakovsky et al., 1999; Lu et al., 2000) assumed the parameter α in the Gardner model of relative hydraulic conductivity, $K_r = \exp(\alpha\psi)$, to be a random variable. This had an effect of limiting the range of applicability of the otherwise general theory. Our main goal was to generalize the approach presented in Tartakovsky et al. (1999) by allowing α to vary randomly in space. We accomplished this through the partial mean-field approximation that replaces the random field $\alpha(\mathbf{x})$ with its constant ensemble mean, $\langle\alpha\rangle$, in the stochastic Richards equation.

We tested the accuracy and robustness of this approximation by comparing the numerical solutions of our moment equations with those obtained from the Monte Carlo simulations of a two-dimensional infiltration problem. One expects the quality of our prediction to depend strongly on a complex interplay between the two parameters characterizing the random field $\alpha(\mathbf{x})$, its coefficient of variation, CV_α, and its correlation structure.

For moderately correlated α (when its correlation length coincides with that of saturated hydraulic conductivity), our results show that the spatial pattern of the pressure head predictions and the associated uncertainties is well reproduced by the partial mean-field approximation for the coefficient of variation CV_α as large as 0.2. The accuracy of our solution decreases with CV_α. As expected, the accuracy of the pressure head prediction is much higher than the accuracy of the predictive uncertainty.

The accuracy of our solutions increases with the correlation length of the random Gardner parameter $\alpha(\mathbf{x})$. A choice of the functional form of the corresponding correlation function might be important, but we expect it to have a second-order effect on the quality of our predictions.

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