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The impact of finite flow domain size on the statistical properties of the flow system in heterogeneous porous media

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Abstract

A modified covariance function of log hydraulic conductivity, accounting for a finite field size, is derived to assess the impact of finite flow domain size on the statistical properties of the flow system in heterogeneous porous media, such as the uncertainty of a large-scale mean model (variation of head fluctuations) and the effective hydraulic conductivity. This development is accomplished by applying a perturbation approximation and spectral representation to a steady three-dimensional flow field of finite extent. Closed-form expressions in a statistically isotropic porous medium indicate that the finite flow domain size has a direct effect on the head variance and the effective hydraulic conductivity. The comparison of the presented formulation for effective hydraulic conductivity is made with existing results.

Keywords: Stochastic analysis; Covariance function; Head variance; Effective hydraulic conductivity

1. Introduction

It is well known that most of the existing stochastic analyses of spatial variability in groundwater (e.g. Gelhar, 1986; Dagan, 1989) rely on the ergodic assumption. These analyses are valid only if the overall scale of observation is much larger than the correlation scale of the log hydraulic conductivity, i.e. for an ‘infinite domain assumption’. In such cases, a unique effective parameter (effective hydraulic conductivity), defined as the ratio of the mean specific discharge to the mean hydraulic gradient, prevails. The effective flow parameter derived from stochastic theory can then be used to simulate the mean behavior of flow system. Furthermore, the variance of the output processes is used to characterize the spatial variability about the mean model, rather than the uncertainty (or the reliability) of the mean model. Consequently, the existing stochastic theories in characterizing spatially variable flow do not reflect the influence of the overall scale variability.

In many practical applications, the information about mean behavior of flow system and variability of the output processes are required to make good management decision. However, in some cases, a bounded flow system is involved or the model parameters, such as the hydraulic conductivities or fluxes, are observed at some finite scale. In addition, some field studies (e.g. Hoeksema and Kitanidis, 1985; Gelhar, 1993) have suggested that the correlation scale

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of hydraulic conductivity is on the order of overall scale. Consequently, using theoretical results appropriate for ergodic conditions may result in significant errors in the predicted statistical properties of the flow system in finite domains. Therefore, there arises a need to incorporate the effects of finite flow domain size into a larger-scale mean model and into the reliability of the mean model.

The purpose of this paper is to quantitatively assess the impact of finite flow domain size on the statistical properties of the flow system, such as the uncertainty of a large-scale mean model (variation of head fluctuations) and the effective hydraulic conductivity. The analysis in this study uses the stochastic methodology similar to that described by Ababou and Gelhar (1990) for self-similar porous media, who extended the spectral theory to include finite-size effects by using a band-pass spectrum, with a low wavenumber cut-off proportional to the inverse size of the domain. However, in our analysis finite-size effects are largely reflected in the evaluation of the expected uncertainty of the mean hydraulic conductivity of the single realization.

2. Estimation of covariance function of log-conductivity and corresponding spectrum for a finite field size

Since the uncertainty in the resultant heads and fluxes is caused by the uncertainty in the hydraulic conductivity field, the stochastic derivation of variances rely on the determination of the spatial structure of the hydraulic conductivity variations. Therefore, the key step in evaluating the variance of output processes is to modify the covariance function of log-conductivity to account for the effects of finite field size.

Suppose that $X(t)$ is a continuous second-order stationary process over a finite interval of length T with a covariance function $R_{XX}(\tau)$ which describes the correlation between any pair of points t and $t+\tau$. An expression relating the ensemble average of covariance function estimator to the covariance function was presented in the work by Jenkins and Watts (1968)

$$E[\widehat{R_{XX}}(\tau)] = \left(1 - \frac{|\tau|}{T}\right)\{R_{XX}(\tau) - \mathrm{Var}[\overline{X}]\} \qquad (1)$$

where $E[-]$ denotes the ensemble average, and $\widehat{R_{XX}}(\tau)$ is the estimator of $R_{XX}(\tau)$. The local average of the observation over this interval takes the form

$$\overline{X} = \frac{1}{T}\int_0^T X(t)\mathrm{d}t$$

and the variance of the averaged process is given by

$$\mathrm{Var}[\overline{X}] = \frac{1}{T}\int_{-T}^{T}(1 - |\tau|/T)R_{XX}(\tau)\mathrm{d}\tau \qquad (2)$$

The reader is referred to Jenkins and Watts (1968) for a detailed derivation of Eq. (1). $\widehat{R_{XX}}(\tau)$ is a random process. Its ensemble average is the best estimate and we shall adopt it as the definition of the covariance function over a specific domain.

Our starting point is to extend Eq. (1) to the multidimensional case as follows (Wen, 1993)

$$E[\widehat{R_{XX}}(\tau)] = \frac{1}{\forall}\int_{\Omega}\{E\{[X(t+|\tau|) - \mu][X(t) - \mu]\} - \mathrm{Var}[\overline{X}]\}\mathrm{d}\Omega \qquad (3)$$

where $\mu = E[X(t)]$, $\forall$ is the sampling volume in which the data is observed, and the domain of integration is the intersection of $\forall$ with its translate by the vector $-\tau$, denoted Ω. Note that $\mathrm{Var}[\overline{X}]$ is function of $R_{XX}(\tau)$, namely

$$\mathrm{Var}[\overline{X}] = \frac{1}{\forall^2}\int_{\forall}\int_{\forall}R_{XX}(\tau_1 - \tau_2)\mathrm{d}\tau_1\,\mathrm{d}\tau_2 \qquad (4)$$

In practice, a covariance function estimator is evaluated from the data points of a single realization over a bounded domain, rather than the ensemble average across a large number of realizations. However, we are only trying to characterize the spatial correlations of a stochastic process and not those of a particular realization. Therefore, we take the ensemble average of the covariance function estimator as the definition of the covariance function of a random process over a specific domain size, which can then be calculated by Eq. (3). From the mathematical point of view, the theoretical covariance function, the first term inside the integrand of the Eq. (3), may be interpreted as the covariance function of a single realization of a random process over an infinite domain. The second term inside the integrand of the Eq. (3) is a measure of predictive uncertainty to be anticipated in applying the single realization mean.

2.1. Statistically anisotropic media

An 'infinite domain' exponential covariance function of log hydraulic conductivity for an anisotropic random field as a function of the separation vector $\xi = (\xi_1, \xi_2, \xi_3)$ is assumed to be given by

$$R_{ff}(\xi) = \sigma_f^2 \exp[-(\xi_1^2/\lambda_1^2 + \xi_2^2/\lambda_2^2 + \xi_3^2/\lambda_3^2)^{0.5}] \quad (5)$$

where the λ_i are the correlation scales of log-conductivity in the principal coordinate directions, and σ_f^2 is the total variance of the process. Here, the integral domain of total volume is considered to be an ellipsoid. Substituting Eq. (5) into Eqs. (3) and (4), and applying the Cauchy algorithm (Dagan, 1989), we obtain

$$E[\widehat{R_{ff}}(\xi)] = \left\{1 - \frac{3}{2}\left(\frac{\xi_1^2}{T_1^2} + \frac{\xi_2^2}{T_2^2} + \frac{\xi_3^2}{T_3^2}\right)^{0.5} + \frac{1}{2}\left(\frac{\xi_1^2}{T_1^2} + \frac{\xi_2^2}{T_2^2} + \frac{\xi_3^2}{T_3^2}\right)^{1.5}\right\}(R_{ff}(\xi) - \mathrm{Var}[F]) \quad (6)$$

where T_i are the length of axes of an ellipsoid, F is the ensemble average of log-conductivity

$$\mathrm{Var}[F] = \frac{6\sigma_f^2}{\pi}\int_{\phi=0}^{\pi}\int_{\theta=0}^{2\pi}\left\{\frac{2}{\omega^2} - \frac{9}{\omega^4} + \frac{60}{\omega^6} - 3\,\mathrm{e}^{-\omega}\left(\frac{1}{\omega^3} + \frac{7}{\omega^4} + \frac{20}{\omega^5} + \frac{20}{\omega^6}\right)\right\}\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta \quad (7)$$

and

$$\omega = [(T_1/\lambda_1)^2\sin^2\phi\sin^2\theta + (T_2/\lambda_2)^2\sin^2\phi\cos^2\theta + (T_3/\lambda_3)^2\cos^2\phi]^{0.5}$$

Applying the Wiener–Khintchine relation to Eq. (6), the corresponding finite-size spectrum is then

$$\widehat{S_{ff}}(K) = \frac{\sigma_f^2 T_1 T_2 T_3}{8\pi^3}\int_{\phi=0}^{\pi}\int_{\theta=0}^{2\pi} \times \frac{\mathrm{e}^{-\omega}[G_1\cos(K_aY) + G_2K_aY\sin(K_aY)] - G_3}{(\omega^2 + K_a^2Y^2)^6} \times \sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta + \mathrm{Var}[F]\frac{3T_1T_2T_3}{2\pi^2K_a^6} \times [4 + K_a^2 + (K_a^2 - 4)\cos(K_a) - 4K_a\sin(K_a)] \quad (8)$$

where $Y = \cos\phi$, $K_a^2 = (K_1T_1)^2 + (K_2T_2)^2 + (K_3T_3)^2$, and

$$G_1 = -3\omega^6(20 + 20\omega + 7\omega^2 + \omega^3) + 12K_a^2Y^2\omega^4(75 + 45\omega + 7\omega^2) + 6K_a^4Y^4\omega^2(-150 + 50\omega + 35\omega^2 + 3\omega^3) + 12K_a^6Y^6(5 - 25\omega + 7\omega^2 + 2\omega^3) + 3K_a^8Y^8(-7 + 3\omega)$$

$$G_2 = 3\omega^5(120 + 100\omega + 28\omega^2 + 3\omega^3) + 12K_a^2Y^2\omega^3(-100 - 25\omega + 7\omega^2 + 2\omega^3) + 6K_a^4Y^4\omega(60 - 90\omega - 14\omega^2 + 3\omega^3) + 12K_a^6Y^6(5 - 7\omega) - 3K_a^8Y^8$$

$$G_3 = \omega^6(-60 + 9\omega^2 - 2\omega^3) + 12K_a^2Y^2\omega^4(75 - 3\omega^2) + 6K_a^4Y^4\omega^2(-150 - 15\omega^2 + 2\omega^3) + 2K_a^6Y^6(30 - 18\omega^2 + 8\omega^3) + 3K_a^8Y^8(3 + 2\omega)$$

2.2. Statistically isotropic media

2.2.1. One-dimensional case

Consider an exponential form of the covariance function of log hydraulic conductivity associated with the infinite field size, namely

$$R_{ff}(\xi) = \sigma_f^2 \exp[-|\xi|/\lambda] \quad (9)$$

where λ is the correlation scale of log-conductivity, ξ is the separation lag, and σ_f^2 is the total variance of the process. Substituting Eq. (9) into Eq. (2) leads to

$$\mathrm{Var}[F] = 2\frac{\sigma_f^2}{\rho}\left[1 + \frac{1}{\rho}\mathrm{e}^{-\rho} - \frac{1}{\rho}\right] \quad (10)$$

where F is the average log-conductivity, $\rho = T/\lambda$, and T represents the finite field length over which the data is collected. Fig. 1 shows the scale effect on the variance of mean log-conductivity. It shows that as the sampling domain increases, the variance of mean log-conductivity decreases, and approaches zero for infinite domain. The covariance function associated

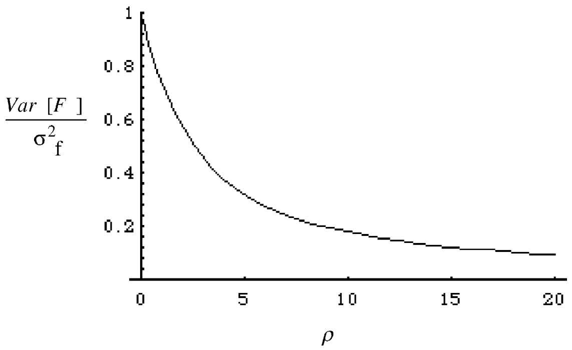

Fig. 1. Normalized variance of mean log hydraulic conductivity as a function of dimensionless field size (one-dimensional case).

with the finite field size is obtained substituting Eqs. (9) and (10) into Eq. (1)

$$E[\widehat{R_{ff}}(\xi)] = \left(1 - \frac{|\xi|}{T}\right)\{R_{ff}(\xi) - \mathrm{Var}[F]\} = \sigma_f^2\left(1 - \frac{|\xi|}{\lambda\rho}\right)\left\{e^{-|\xi|/\lambda} - \frac{2}{\rho}\left[1 + \frac{1}{\rho}e^{-\rho} - \frac{1}{\rho}\right]\right\} \quad (11)$$

It is clear that Eq. (11) converges to Eq. (9) as $\rho \to \infty$. Applying the Wiener–Khintchine relation to Eq. (11), the spectrum for a finite-length record can be derived as

$$\widehat{S_{ff}}(K) = \frac{\sigma_f^2\lambda}{\pi(1+\beta^2)}\left\{1 - e^{-\rho}\cos(\rho\beta)\left[1 - \frac{1}{\rho}\left(\frac{\beta^2-1}{\beta^2+1} - \rho\right)\right] + e^{-\rho}\sin(\rho\beta)\left[\beta - \frac{1}{\rho}\left(\rho\beta + \frac{2\beta}{\beta^2+1}\right)\right] + \frac{\beta^2-1}{\rho(\beta^2+1)}\right\} - 2\sigma_f^2\frac{\lambda}{\pi}\frac{1-\cos(\rho\beta)}{\rho^2\beta^2}\left[1 + \frac{e^{-\rho}}{\rho} - \frac{1}{\rho}\right] \quad (12)$$

where K is the wave number, and $\beta = K\lambda$. As $\rho \to \infty$, the finite-size spectrum given by Eq. (12) converges to

$$\widehat{S_{ff}}(K) = \frac{\sigma_f^2\lambda}{\pi(1+\lambda^2K^2)} \quad (13)$$

Finite scale spectra for different values of T/λ are present in Fig. 2. It seems that the scale effect plays an important role in filtering out the spectrum around the origin.

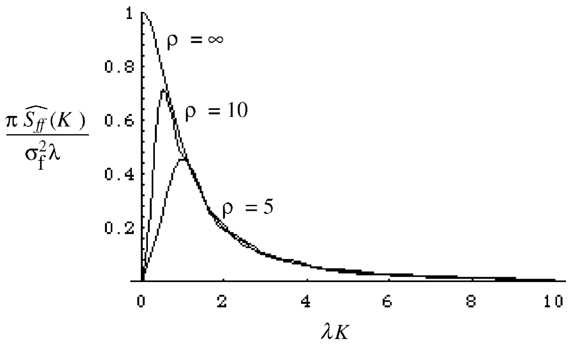

Fig. 2. Normalized spectrum as a function of dimensionless wave number (one-dimensional case).

2.2.2. Three-dimensional case

The estimation procedure for the three-dimensional case is similar to that used for the one-dimensional case, with the exception that Eqs. (1) and (2) are replaced by Eqs. (3) and (4), respectively. Since the covariance function of log-conductivity is rotationally invariant (due to statistical isotropy assumption), the integral domain of total volume is considered to be a sphere of diameter T in space.

Again, we consider the exponential form of covariance function for log-conductivity ($\lambda_1 = \lambda_2 = \lambda_3 = \lambda$ in Eq. (5))

$$R_{ff}(\xi) = \sigma_f^2 \exp[-\xi/\lambda] \quad (14)$$

Substituting Eq. (14) into Eq. (4) and applying the Cauchy algorithm (Dagan, 1989), we obtain

$$\mathrm{Var}[F] = \sigma_f^2\frac{24}{\rho^3}\left\{\left[2 - \frac{9}{\rho} + \frac{60}{\rho^3}\right] - 3e^{-\rho}\left[1 + \frac{7}{\rho} + \frac{20}{\rho^2} + \frac{20}{\rho^3}\right]\right\} \quad (15)$$

Fig. 3 depicts the normalized variance of observed mean as a function of the field scale. Finally

$$E[\widehat{R_{ff}}(\xi)] = \left(1 - \frac{3}{2}\frac{|\xi|}{T} + \frac{1}{2}\frac{|\xi|^3}{T^3}\right)\{R_{ff}(\xi) - \mathrm{Var}[F]\} \quad (16)$$

The observed variance of log-conductivity within a finite volume can be defined as

$$E\left[\widehat{\sigma_f^2}\left(\frac{T}{\lambda}\right)\right] = E[\hat{R}(0)] = \sigma_f^2 - \mathrm{Var}[F] = \sigma_f^2\left\{1 - \frac{24}{\rho^3}\left[\left(2 - \frac{9}{\rho} + \frac{60}{\rho^3}\right) - 3e^{-\rho}\left(1 + \frac{7}{\rho} + \frac{20}{\rho^2} + \frac{20}{\rho^3}\right)\right]\right\} \quad (17)$$

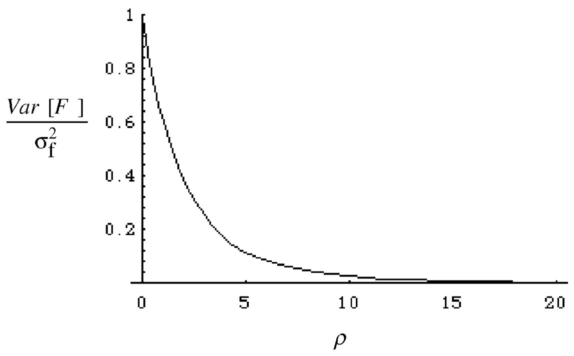

Fig. 3. Normalized variance of mean log hydraulic conductivity as a function of dimensionless field size (three-dimensional case).

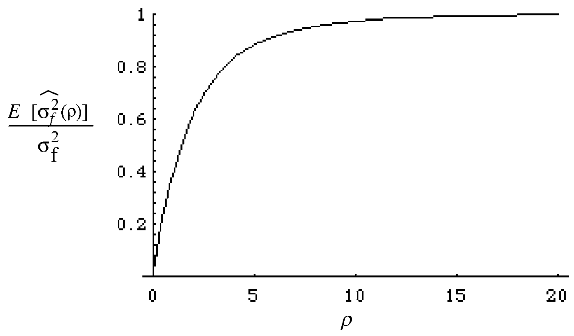

Fig. 4. Normalized observed variance of log hydraulic conductivity as a function of dimensionless field size (three-dimensional case).

It can be seen from Fig. 4 that the observed variance of a finite volume process increases monotonically from zero (for infinitesimal field size) to an asymptotic value (the total variance, σ_f^2, for $\rho \to \infty$). This behavior is related to the fact that the inclusion of larger heterogeneities results in an increased variability of hydraulic conductivity as the field scale increases.

Finally, the spectrum of the log-conductivity for a finite volume process is derived by applying the Wiener–Khintchine theorem to Eq. (16)

$$\begin{aligned}\widehat{S_{ff}}(K) &= \frac{\sigma_f^2\lambda^3}{\pi^2}\left\{\left[\frac{1}{(1+\beta^2)^2}+\frac{1.5}{\rho}\frac{\beta^2-3}{(1+\beta^2)^3}+\frac{6}{\rho^3}\frac{\beta^4-10\beta^2+5}{(1+\beta^2)^5}\right]\right.\\ &+\frac{\cos(\rho\beta)}{e^\rho}\left[\frac{1.5}{\rho}\frac{\beta^2-3}{(1+\beta^2)^3}+\frac{24}{\rho^2}\frac{\beta^2-1}{(1+\beta^2)^4}-\frac{6}{\rho^3}\frac{\beta^4-10\beta^2+5}{(1+\beta^2)^5}\right]\\ &\left.+\frac{\sin(\rho\beta)}{e^\rho\rho\beta}\left[\frac{3}{2}\frac{3\beta^2-1}{(1+\beta^2)^3}-\frac{6}{\rho}\frac{\beta^4-6\beta^2+1}{(1+\beta^2)^4}-\frac{6}{\rho^2}\frac{5\beta^4-10\beta^2+1}{(1+\beta^2)^5}\right]\right\}\\ &-\frac{\mathrm{Var}[F]}{\pi^2}\frac{3}{2}\frac{\lambda^3\rho^2}{\beta}\left[\frac{1}{\rho^3\beta^3}+\frac{4}{\rho^5\beta^5}+\cos(\rho\beta)\left(\frac{1}{\rho^3\beta^3}-\frac{4}{\rho^5\beta^5}\right)-4\frac{\sin(\rho\beta)}{\rho^4\beta^4}\right]\end{aligned} \tag{18}$$

The finite-size spectrum converges to

$$\widehat{S_{ff}}(K) = \frac{\sigma_f^2\lambda^3}{\pi^2(1+\lambda^2K^2)^2} \tag{19}$$

when $\rho = T/\lambda \to \infty$. As can be inferred from Fig. 5, the ergodic approach ($T/\lambda \to \infty$) does not reproduce the behavior of the finite-size spectrum in the low wavenumber range. It actually misses it completely. The normalized finite-size spectrum in Fig. 5 goes to zero as the wavenumber approaches zero, while the ergodic approach gives a normalized spectrum of 1.0 at zero wavenumber.

3. Effects of domain size in statistically isotropic media

In order to focus on the impact of finite flow domain size on statistical properties of the flow system and take advantage of the closed-form

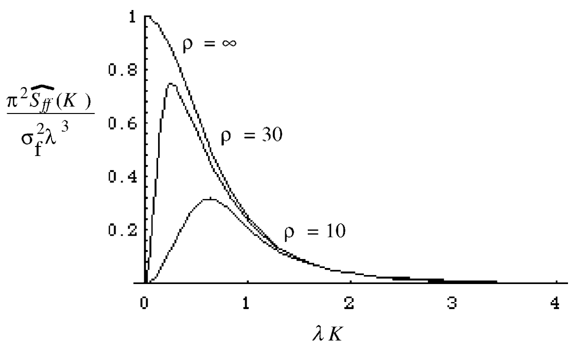

Fig. 5. Normalized spectrum as a function of dimensionless wave number (three-dimensional case).

expressions, a steady-state unidirectional mean flow in a three-dimensional statistically isotropic aquifer is considered. The mean hydraulic gradient is assumed to exist in the direction X_1 only. The behavior of statistically anisotropic system can be evaluated numerically. It is expected that this behavior will be qualitatively similar to that of isotropic ones.

Furthermore, we neglect the boundary effects on the head covariances, i.e. the hydraulic head process is assumed to be statistically homogeneous, or stationary. It is recognized that imposed boundary conditions destroy the stationarity of head process. However, theoretical investigations of flow in bounded domains by Naff and Vecchia (1986) and Rubin and Dagan (1988, 1989) through the Green's function approach imply that the assumption of a stationary head process in three-dimensional flow is not restrictive. Naff and Vecchia (1986) studied the impervious boundary effects on the head covariances for a steady three-dimensional flow in a formation of infinite horizontal extent, bounded above and below by impervious horizontal boundaries, and demonstrated that the boundary effect (namely the nonstationary effect) is largely limited to a zone near the medium boundary. Similar results were obtained by Rubin and Dagan (1988, 1989), who analyzed the effects of constant head and impervious boundary conditions on the head variation in semi-infinite aquifers.

It is of interest to note that the type of boundary conditions affects the nonstationarity of head process differently. Effects of different boundary conditions and spatially varying head gradients on both head and velocity covariance function for a two-dimensional bounded domain (rectangle) in heterogeneous media were investigated by Bonilla and Cushman (2000) using a recursive perturbation scheme. They found that for a flow of constant mean head gradient, under the Dirichlet boundary conditions (prescribed head), the head covariance function is stationary at distances larger than three to four integral scales from the boundaries. However, under the same circumstances, effects of nonstationarity caused by the Neumann boundary conditions (prescribed flux) may persist as far as three to eight integral scales from the boundary.

3.1. Variance of head fluctuations

The covariance function equation (28) of Bakr et al. (1978) is the starting point to derive a scale-dependent variance of head fluctuations.

$$R_{hh}(\xi,\chi)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\,\mathbf{K}\cdot\xi}\frac{J^2K_1^2}{K^4}\widehat{S_{ff}}(\mathbf{K})\mathrm{d}\mathbf{K} \tag{20}$$

where χ is the angle between the separation vector ξ and the direction of mean flow, $h=H-E[H]$, H is the hydraulic head, $K^2=K_1^2+K_2^2+K_3^2$, and $J=-E[\partial H/\partial X_1]$. Ababou and Gelhar (1990) have used the same equation along with the band-pass self-similar spectrum to demonstrate the effect of finite flow domain size on the head variance. Substituting Eq. (18) into Eq. (20), taking the limit of Eq. (20) as $\xi\to 0$, and integrating using polar coordinates, gives the following result for scale-dependent variance of head fluctuations

$$\sigma_h^2\left(\frac{T}{\lambda}\right)=\frac{\sigma_f^2\lambda^2J^2}{3}\left\{1-\frac{39}{5}\frac{1}{\rho}+\frac{108}{5}\frac{1}{\rho^2}+\frac{12}{\rho^3}-\frac{144}{\rho^4}+\frac{1}{\mathrm{e}^{\rho}}\left[\frac{21}{5}\frac{1}{\rho}+\frac{192}{5}\frac{1}{\rho^2}+\frac{132}{\rho^3}+\frac{144}{\rho^4}\right]\right\} \tag{21}$$

The asymptotic value of head variance ($\rho=T/\lambda\to\infty$) is

$$\sigma_h^2(\infty)=\frac{\sigma_f^2\lambda^2J^2}{3} \tag{22}$$

Eq. (22) is equivalent to that of Bakr et al. (1978). It is important to recognize that the analysis leading to Eq. (21) is restricted to relatively, in some sense, small hydraulic conductivity variations so that second terms in the flow perturbation equation can be neglected. In addition, the validity of the assumption of stationarity for the distribution of flow properties requires that the standard deviation of the random log hydraulic conductivity fluctuations, σ_f, should be small. Gutjahr and Gelhar (1981) showed that if $\sigma_f<1$, then the head field is stationary in the case of the three-dimensional isotropic log hydraulic conductivity fields but may not be stationary as σ_f increases.

The increase in head variance (Fig. 6) with the flow domain size is due to a wide range of heterogeneities

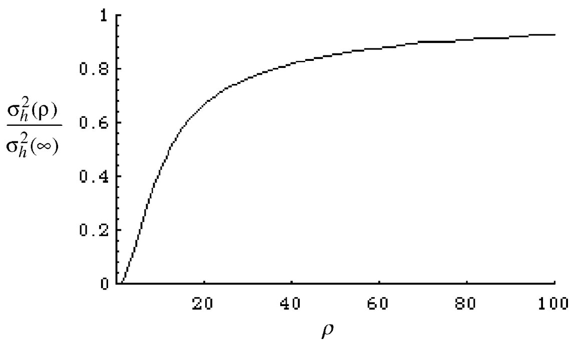

Fig. 6. Normalized head variance as a function of dimensionless flow domain size.

that contribute to the flow processes as the flow domain size increases (Fig. 4). It can be inferred from Fig. 6 that applying the classical stochastic theories to finite-size flow domains can result in inaccurate representations of variability of flow properties because the flow properties depend on the scale of observation.

3.2. Effective hydraulic conductivity

In the following, the effective hydraulic conductivity, defined as the ratio of the mean specific discharge to the mean hydraulic gradient, is estimated from the mean specific discharge equation.

Taking the expected value of Darcy equation, assuming small perturbations, and dropping the products of the perturbed quantities leads to the mean specific discharge equation (Eq. (12) of Gelhar, 1986)

$$q = e^F\left\{J\left(1+\frac{1}{2}E[\widehat{\sigma_f^2}]\right) - E\left[f\frac{\partial h}{\partial X_1}\right]\right\} \quad (23)$$

where f is the perturbation of log-conductivity. Using the spectral representation theorem, the last term in Eq. (23) can be expressed by (Eq. (13) of Gelhar, 1986)

$$E\left[f\frac{\partial h}{\partial X_1}\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{K_1^2}{K^2}\widehat{S_{ff}}(\mathbf{K})\mathrm{d}\mathbf{K} \quad (24)$$

Substituting Eq. (18) into Eqs. (23) and (24) gives the mean specific discharge

$$q = e^F J\left\{1+\frac{1}{6}(\sigma_f^2 - \mathrm{Var}[F])\right\} = e^F J\left\{1+\frac{1}{6}E[\widehat{\sigma_f^2}]\right\} \quad (25)$$

Thus, the effective hydraulic conductivity can be expressed as

$$K_e(\rho) = \frac{q}{J} = e^F\left\{1+\frac{1}{6}E[\widehat{\sigma_f^2}]\right\}$$
$$= e^F\left[1+\frac{1}{6}\sigma_f^2\left\{1-\frac{24}{\rho^3}\left[\left(2-\frac{9}{\rho}+\frac{60}{\rho^3}\right)\right.\right.\right.$$
$$\left.\left.\left.-3e^{-\rho}\left(1+\frac{7}{\rho}+\frac{20}{\rho^2}+\frac{20}{\rho^3}\right)\right]\right\}\right] \quad (26)$$

Recall from Eq. (17) that the observed variance of log-conductivity $\widehat{\sigma_f^2}$ is scale-dependent. The corresponding asymptotic values as $\rho = T/\lambda \to \infty$ is

$$K_e(\infty) = e^F\left[1+\frac{1}{6}\sigma_f^2\right] \quad (27)$$

Note that the total variance of log-conductivity σ_f^2 does not depend on the flow domain size.

The increase in the effective hydraulic conductivity as a function of the flow domain size is displayed in Fig. 7. Larger heterogeneities are included in the flow processes as the flow domain size grows. A wide range of heterogeneities in a three-dimensional flow result in patches of higher conductivity, which can contribute to increase the effective hydraulic conductivity. However, the rate of growth of the effective hydraulic conductivity with the domain size is not great, and therefore, the effective hydraulic conductivity is practically the same as its value in an unbounded domain.

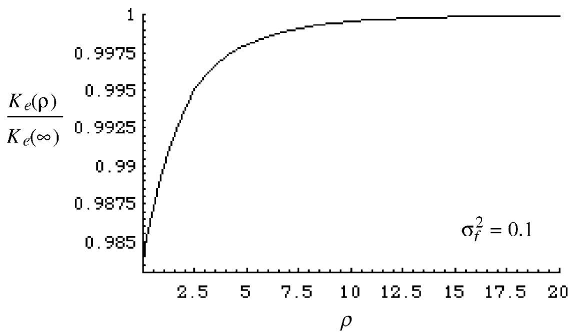

Fig. 7. Normalized effective hydraulic conductivity as a function of dimensionless flow domain size.

4. Comparison of the effective hydraulic conductivity with existing results

It is of interest to assess the validity of the presented formulation for effective hydraulic conductivity by comparing with existing theoretical results.

Expressions for effective hydraulic conductivity in bounded flow domains may alternatively be derived by using the upscaling technique (e.g. Rubin and Gomez-Hernandez, 1990; Desbarats, 1992; Indelman and Dagan, 1993a,b). The process of transferring information on scales smaller than those of interest to the resultant larger scale system is known as upscaling. There has been much effort expended in finding theoretical expressions for the statistical moments of the upscaled conductivity. For example, Indelman and Dagan (1993a,b) used a methodology based on preservation of the energy dissipation for the upscaled conductivity field to develop theoretical expressions, which relate the upscaled conductivity statistical moments to the given moments of the continuously distributed conductivity and to the size of the numerical blocks. It is important to note that in the work of Indelman and Dagan (1993a,b), the various assumptions they used in solving the upscaling problem are similar to those underlying the derivation of the effective conductivity.

According to the work of Indelman and Dagan (1993b), the explicit expressions for the upscaled conductivity statistical moments for the case of a statistically isotropic conductivity field with an exponential covariance function are given by (their Eqs. (37) and (44))

$$K_e = \frac{1+\frac{1}{6}\sigma_f^2}{1+\frac{1}{6}\tilde{\sigma}_f^2}e^F \qquad (28)$$

$$\tilde{\sigma}_f^2 = \frac{24}{\rho^3}\sigma_f^2\left[2-\frac{9}{\rho}+\frac{60}{\rho^3}-\left(3+\frac{21}{\rho}+\frac{60}{\rho^2}+\frac{60}{\rho^3}\right)e^{-\rho}\right] \qquad (29)$$

The results of Eqs. (26) and (28) are compared graphically in Fig. 8. It is evident that the presented formulation for effective hydraulic conductivity is in good agreement with the theoretical result of Indelman and Dagan (1993b).

We should note that under small perturbations, our effective hydraulic conductivity in Eq. (26) has a form

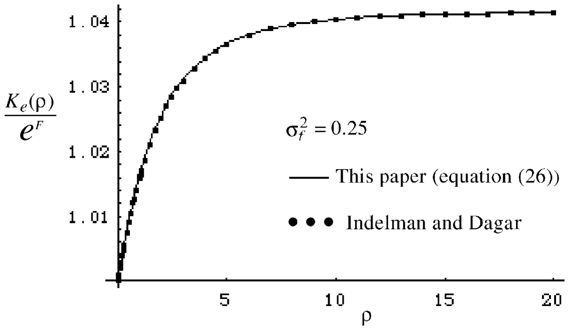

Fig. 8. Comparison of the effective hydraulic conductivity with the result of Indelman and Dagan (1993b).

similar to the Eq. (43) of Sanchez-Vila et al. (1995), developed from the definition by Rubin and Gomez--Hernandez (1990), and the Eq. (11) of Desbarats (1992), developed from an empirical power-averaging rule. Sanchez-Vila et al. (1995) analyzed three different practical approaches (namely the approaches of Rubin and Gomez-Hernandez (1990), Desbarats (1992), and Sanchez-Vila et al. (1995)) for upscaling of hydraulic conductivity numerically in isotropic heterogeneous media and found that these approaches yield very similar results in terms of actual computed values.

The other comparison for effective hydraulic conductivity is done with the results of Paleologos et al. (1996). They developed linearized analytical expressions for effective hydraulic conductivity of three-dimensional, bounded, strongly heterogeneous porous media based on the so-called Landau-Lifshitz conjecture. It is worthwhile mentioning that the boundary effects on steady state flow are included in their derivation of the effective hydraulic conductivity. Upon applying to a statistically isotropic conductivity field with an exponential covariance function, they found that for a given variance of log-conductivity, the effective conductivity decreases rapidly from the arithmetic mean toward the asymptotic value ($K_e = e^F(1+\sigma_f^2/6)$) as the characteristic length of the domain increases from zero toward about eight integral scales of log-conductivity. This is in contrast to our conclusion that for a given variance of log-conductivity, the effective conductivity increases with the domain size from the geometric mean toward the asymptotic value ($K_e = e^F(1+\sigma_f^2/6)$). The decrease in effective hydraulic conductivity of

Paleologos et al. (1996) with the domain size may be explained by the fact that in their calculation, the first two statistical moments of the log-conductivity are assumed to remain fixed as the domain size changes. In other words, the arithmetic mean of the log hydraulic conductivity is kept fixed as the domain size changes. In contrast to their assumptions, our log-conductivity covariance function for a finite domain size used in Eq. (26) essentially changes with the domain size and is obtained based on calibration to the ergodic case (a very large domain size), for which the mean and covariance function are kept fixed. This scale-dependent covariance function in Eq. (17) is basically predicated on the observation that the apparent statistical properties of parameters inferred from field data are significantly affected by the scale of the observation (e.g. Kemblowski, 1988).

The effective hydraulic conductivity in Eq. (26) appears as a function of the domain size through the term $E[\sigma_f^2]$, the observed variance of log-conductivity. In the limit as the domain size is reduced to a point, this term vanishes, and the effective hydraulic conductivity tends to the geometric mean. However, the effective hydraulic conductivity of Paleologos et al. (1996) tends to its maximal value, the arithmetic mean, as the domain size is very small. This discrepancy is attributed to the assumption of stationarity for the hydraulic head process, which has been introduced through our procedure of evaluating the effective hydraulic conductivity. As mentioned earlier, this study assumes that the boundary has negligible effect on the head variation so that the nonstationarity restricted to a thin layer near the boundary is eliminated. Therefore, the analyses in this study are limited to a finite-size domain away from the boundary. The comparison with the expressions of Indelman (1993) for effective hydraulic conductivity led Paleologos et al. (1996) to make a similar comment on the discrepancy that the effective hydraulic conductivity obtained by Indelman (1993) tends to the geometric mean, rather than to its arithmetic mean as required by the theory of Neuman and Orr (1993), when the domain size approaches zero. They stated that "…their upscaling is carried out in a region sufficiently far from the boundary" (Indelman and Dagan (1993a), p. 920), so that its effect is not felt.

5. Summary

A stochastic approach was used to assess the impact of finite flow domain size on the head variance and the effective hydraulic conductivity. Closed-form expressions are derived for the case of statistically isotropic three-dimensional porous media. The major findings from our analysis may be summarized as follows:

1. The characteristic scale of the finite field space is important in filtering out the spectrum of a finite process around the origin. Fig. 5 showed that there is an inaccurate representation of the spectrum in the low wavenumber range for a finite volume process by using the ergodic assumption.
2. A larger flow domain size means larger heterogeneities included in the flow process (Fig. 4), which results in higher variability of head fluctuations (Fig. 6).
3. The increase of effective hydraulic conductivity with the domain size (Fig. 7) is related to the fact that in the three-dimensional flow, a wider range heterogeneities caused by an increased domain size results in a easier flow-around process. However, the rate of growth of the effective hydraulic conductivity with the domain size is small, and, therefore, the effective hydraulic conductivity can be approximated by the infinite domain result. Our presented formulation for effective hydraulic conductivity compares well with the upscaling solution obtained by Indelman and Dagan (1993b).

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