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Diffusion in porous layers with memory

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SUMMARY

The process of diffusion of fluid in porous media and biological membranes has usually been modelled with Darcy's constitutive equation, which states that the flux is proportional to the pressure gradient. However, when the permeability of the matrix changes during the process, solution of the equations governing the diffusion presents severe analytical difficulties because the variation of permeability is not known *a priori*.

A diverse formulation of the constitutive law of diffusion is therefore needed and many authors have studied this problem using various methods and solutions. In this paper Darcy's constitutive equation is modified with the introduction of a memory formalism. We have also modified the second constitutive equation of diffusion which relates the density variations in the fluid to the pressure, introducing rheology in the fluid represented by memory formalisms operating on pressure variations as well as on density variations. The memory formalisms are then specified as derivatives of fractional order, solving the problem in the case of a porous layer when constant pressures are applied to its sides.

For technical reasons many studies of diffusion are devoted to the flux rather than to the pressure; in this work we shall devote our attention to studying the pressure and compute the Green's function of the pressure in the layer when a constant pressure is applied to the boundary (Case A) for which we have found closed-form formulae. The described problem has already been considered for a half space (Caputo 2000); however, the results for a half space are mostly qualitative since in most practical problems the diffusion occurs in layers.

The solution is also readily extended to the case when a periodic pressure is applied to one of the boundary planes while on the other the pressure is constant (Case B) which mimics the effect of the tides on sea coasts. In this case we have found a skin effect for the flux which limits the flux to a surface layer whose thickness decreases with increasing frequency. Regarding the effect of pressure due to tidal waters on the coast, it has been observed that when the medium is sand and the fluid is water, for a sinusoidal pressure of 2×10^4 Pa and a period of 24 hr at one of the boundaries and zero pressure at the other boundary, the flux is sinusoidal with the same period and amplitude decaying exponentially with distance to become negligible at a distance of a few hundred metres.

A brief discussion is given concerning the mode of determination of the parameters of memory formalisms governing the diffusion using the observed pressure at several frequencies. We shall also see that, as in the classic case of pure Darcy's law behaviour, the equation governing the flux resulting in the diffusion through porous media with memory is the same as that governing the pressure.

Key words: Darcy, diffusion, filtering, flux, memory, porous media.

Nomenclature

$m(x, t)$ (kg m^{-3})	Mass of fluid per unit volume in the porous medium.
$m(x, 0)$ (kg m^{-3})	Mass of fluid per unit volume in the initial condition.
m_0 (kg m^{-3})	Mass of fluid per unit volume in the undisturbed condition.
$p(x, t)$ (kg m^{-1} s^{-2})	Pressure of the fluid.
$q(x, t)$ (kg m^{-2} s^{-1})	Fluid mass flow rate in the porous medium.
R, r	Radius of the inner and outer circle, respectively, of the path of integration, to find the inverse Laplace transform of eq. (B3) shown in Fig. 6.
t (s)	Time.
x	Distance from the boundary plane.
h	Thickness of the layer.
κ (kg^{-1} m^3 s)	Ratio of the permeability of the medium to the viscosity of the fluid.
ρ_0 (x) (kg m^{-3})	Density of the fluid in the undisturbed condition.
η (m^2)	Permeability.
μ (kg m^{-1} s^{-1})	Viscosity.
a (m^{-2} s^2)	Ratio of density variation to pressure variation.
d (s^{1+n})	Memory factor of pressure gradient.
ω (s^{-1})	Frequency.
z (dimensionless)	Porosity.
k (m^{-2} s^{1-n}) $= (a/d)$	Diffusivity (Bear 1972).
k' (m^{-1} s$^{1.5+n/2}$) $= (ad)^{1/2}$	Pseudodiffusivity (see text).

1 FOREWORD

Understanding the diffusion of fluids in porous media has become increasingly important, mostly because of problems arising in the fields of petroleum extraction, agricultural and civil engineering, soil mechanics, chemical, food and pharmaceutical industries, ground water hydrology and biological research.

For some decades water shortages have occurred in various parts of the world, increasing the need for studies connected with the diffusion of water in porous media during the hydrological cycle as water travels from the ground surface into agricultural and civil use and eventually to the oceans. Also, because of frequent droughts and overexploitation in some regions, we may no longer consider water to be a renewable resource over long time periods. Since one of the causes of water shortages is the increasing need of the human population, the United Nations has established World Water Day on March 22 of each year to attract the attention of the public to the gravity of the problem. In civil and agricultural water usage, the problem often lies in loss of water in the distribution network; in other emblematic cases such as in Silicon Valley, California and in Israel, where the rainfall is below the continental average, techniques have been developed to improve the efficiency of the diffusion of water through the soil to a very high standard. The diffusion of fluid through porous media is also of great importance in monitoring the diffusion of contaminants in the ground which, aided by ground water flux, may travel large distances and poison water reservoirs intended for civil use.

It is worth recalling that the basic constitutive law of fluid dynamics, stating that the flux is proportional to the gradient of the pressure, was first established by Fick (1855), and subsequently, but independently, by Darcy (1856), and is generally known in geology and geophysics as Darcy's law; however, as noted also by Bear (1972), the law is embedded in the classic Fourier equation on the diffusion of heat. Many authors have contributed in various ways to the study of the diffusion of fluids in porous media extending Darcy's law in different modes; in general these authors set equations rigorously representing the interaction between the porous medium and the flow of fluid through it and have obtained solutions in many interesting cases (e.g. Bear 1972; Sposito 1980; Steefel & Lasaga 1994; Dewers & Ortoleva 1994; Indelman & Abramovich 1994; Mainardi 1996; Cushman & Moroni 2001; Moroni & Cushman 2001).

Because of the inadequacy of current theories in taking into account memory, some authors have also developed non-local flow theories (e.g. Hu & Cushman 1994) using general principles of statistical mechanics under appropriate limiting conditions from which the classical Darcy's law is derived for saturated flow. We should also mention here the work of Sposito (1980) who used memory for water movements in saturated soils based on Mori's approach in non-equilibrium statistical physics. In spite of this, some data on the flow of fluids in rocks exhibit properties which cannot be interpreted using the classical theory of propagation of pressure and fluids in porous media (Robinson 1939; Bell & Nur 1978; Roeloffs 1988) nor adequately with many of the new theories; therefore more studies are needed.

With regard to the correlation between seismogenesis and fluid migration and/or injection in the ground the Rangeley Colorado experiment (Raileigh *et al.* 1976) and the interpretation of the phases between the increase in water level in reservoirs and seismicity (Bell & Nur 1978) are well known: these still present unsolved problems which may find better interpretation with the use of a more complex diffusivity model than one based strictly on Darcy's law.

2 INTRODUCTION

Darcy's law allows us to obtain the flux directly from the pressure gradient. Variable permeability caused by differing amounts of fluid passing through the pores implies the presence of 'memory' in the matrix or in the fluid. In practice, this variation in permeability may occur in geothermal areas where crystals formed during expansion of the fluid obstruct the path of incoming fluid, chemical reaction of the fluid with the matrix may enlarge the pores or, in biological systems, the filtering properties of membranes may become saturated. It is clear that in these cases the variation in permeability depends on the amount of fluid passed through the matrix, which implies a system 'memory'. To represent the memory we shall introduce in the constitutive equation the derivative of fractional order which weighs the 'past' of the function.

The above-mentioned types of fluid flow imply that the permeability of the medium varies with time, and this phenomenon should be taken into account when writing Darcy's law which states proportionality between the fluid mass flow rate q per unit area in the direction x_i and the gradient of the pore pressure p:

$$q_i = -\rho_0 \kappa \, \partial p/\partial x_i. \tag{1}$$

In the present paper we try to model mathematically the possible changes in the physical properties of the matrix due to variations in its temperature and physical or chemical interactions with the fluid, introducing a memory mechanism in the constitutive eq. (1). The elastic reaction of the matrix is neglected by uncoupling the equations of diffusion from those of elasticity, which are not considered here where we focus on the pressure of the fluid with occasional attention to the flux as in the case of periodic boundary conditions. At first it may seem that the introduction of mathematical memory into the formulation of the constitutive equations complicates the problem, but as we shall see the use of the Laplace transform (LT) allows us to obtain the solution directly when the boundary conditions are given.

The scope of this paper is also to produce diffusion models potentially capable of describing the above-mentioned phenomena by rewriting the constitutive equations with a memory formalism and transferring the mathematical formalisms to the frequency domain—we consider a medium filling a layer limited by two parallel planes at a distance h; we denote by x the distance from the boundary plane positive toward the medium, t is time, and we compute the pressure in the case when the fluid pressure in the layer is initially zero and different constant pressures are applied on the boundary planes (Case A).

We will also indicate how to obtain the solution in the important case when a sinusoidal pressure is applied to one of the boundaries while the other is kept at zero pressure (Case B). The spectra of the amplitude of the filtering implied by the Green's functions will be discussed in a practical example giving an original point of view of diffusion.

3 THE MODELS

In this paper we will focus our attention on physical cases of diffusion in which the permeability varies with time depending on the previous pressure gradient and flow which mimic the above-mentioned cases when the permeability is locally variable in time. These phenomena, which we will represent mathematically with memory formalisms, have often been observed qualitatively in oil extraction, in geothermal areas and in the laboratory (Walder & Nur 1984; Elias & Hajash 1992). Very recently Iaffaldano (2003) produced quantitative data from laboratory experiments which have validated some of the formulae obtained here.

To simulate the memory formalism the following relations substitute for the classical Darcy's law (1):

$$(a + b\partial^{m_1}/\partial t^{m_1})p = (\alpha + \beta\partial^{m_2}/\partial t^{m_2})(m(x,t) - m_0) \tag{2}$$

$$(\gamma + \varepsilon\partial^{n_1}/\partial t^{n_1})q = -(c + d\partial^{n_2}/\partial t^{n_2}) \,\mathrm{grad}\, p \tag{3}$$

where $0 \le n_1 < 1$, $0 \le n_2 < 1$ and $0 \le m_1$, $0 \le m_2 < 1$,

$$\partial^n f(t)/\partial t^n = [1/\Gamma(1-n)] \int_0^t (t-\tau)^{-n} (df(\tau)/d\tau)\, d\tau \tag{4}$$

with $0 \le n < 1$ (e.g. Caputo 1969; Kiryakova 1994; Luchko & Gorenflo 1998; Podlubny 1999) and Γ is the gamma function. In the definition (4) there is convergence at $t = \tau$ for any value of t since it is assumed that $0 \le n < 1$. The continuity equation will also be enforced:

$$\mathrm{div}\, q + \partial m/\partial t = 0. \tag{5}$$

In practice the derivative of fractional order $f(t)$ is constructed with a weighted mean of the first-order derivative $df(\tau)/d\tau_\nu$ in the time interval [0, t], which is a sort of feedback system. That is, the values of $df(\tau)/d\tau_\nu$ at times ν far from t are given smaller weight than those at times ν closer to t. Hence, the weights are increasingly smaller with increasing time separation from the time t to imply that the effect of the past fades with increasing time. When $n = 0$ and $f(0) = 0$, the fractional order derivative reduces to the functions themselves. Importantly, the weights multiplying the first-order derivative of $f(t)$ inside the integral appearing in eq. (4) can be chosen in many ways. The definition adopted in eq. (4) is appropriate because it is algebraically simple, allows easy solutions and has commonly been applied in the previously cited studies.

The memory acts on a separate part of the function: namely the fraction represented by b, β, ε and d; the presence of all parameters implies the possibility that only a proportional part of the variables enter with memory in the equation. It is worth noting how the memory functions capture the 'past' (Caputo & Kolari 2001). What the fractional derivative memory functions are remembering is their past values as defined by eq. (4), which implies that the function is constructed by adding to the initial value the successive weighted increments over time. The increments per unit time are represented by the first-order derivative under the integral sign, and the weights are represented by the factor of the first-order derivative in eq. (4) which are decreasing with increasing time separation from time t. Thus, the value of a variable is a weighted mean of its past values.

It is clear that the 'memory' formalisms introduced here to describe the flux of the fluid imply the use of more than two parameters instead of only the one parameter $\kappa\rho_0$ as in Darcy's law. These cases, specified by eqs (2) and (3), are of interest in practical applications because the values of the parameters defining them may be estimated with laboratory experiments.

The introduction of fractional derivatives in the constitutive equations of the phenomena studied in geophysics is not new. In rheology they were used by Bagley & Torvik (1986) to model the rheological properties of solids, by Caputo (1967) to model the frequency-independent quality factor, by Körnig & Müller (1989) to successfully model the Fennoscandian uplift and by Caputo & Plastino (1998) to show that the constitutive equation of polarizable media, in the time domain, is represented by a relation containing these derivatives. The derivatives of fractional order have also been successfully used in other fields of research (e.g. Le Mehaute & Crépy 1983; Jacquelin 1984; Mainardi 1996; Caputo & Kolari 2001).

The equations resulting from our procedure are phenomenological; however, these types of equation, as mentioned in a recent citation for the Nobel Prize for Physics, have been rehabilitated due to their contribution, in various forms, to the rapid development of superconducting materials. These phenomenological equations, when adequately verified with experimental data, in some way represent a step forward with respect to the usual empirical equations which are still very useful in many branches of applied science and technology.

In this work we introduce the memory formalism into the constitutive equation for diffusion. The way in which memory acts is clarified in the examples given and in the following. As we will see this will also lead to substantial results.

4 THE DIFFUSION OF THE PRESSURE IN THE LAYER AND THE DISCUSSION OF THE PARAMETERS

We begin recalling (see Appendix B) the solution given by eq. (B4) for the case when the pressures on both boundaries of the layer are constant (Case A); that is

$$p(0, t) = A,\ p(h, t) = B,\ p(x, 0) = C \tag{6}$$

with A, B and C constant. The solution of this problem may readily be extended to the case when the conditions are time dependent.

Since experimental data are scarce it is advisable to reduce the number of parameters, and we will consider the simple case, which may be of interest in applications, when $n_1 = n_2 = m_1 = m_2 = n$. In our case, as was done previously when considering the half space (Caputo 2000) instead of the layer, we tentatively assume that $F_1/F_2 = k/s^n$ in which case eq. (B2) of Appendix B is

$$\begin{aligned} P = {} & (1/s)\Big((B - C)\big\{\exp\big[x(ks^{1-n})^{1/2}\big] - \exp\big[-x(ks^{1-n})^{1/2}\big]\big\} + (A - C)\big\{\exp\big[(h - x)(ks^{1-n})^{1/2}\big] \\ & - \exp\big[-(h - x)(ks^{1-n})\big]\big\}\Big)\Big(\exp\big[h(ks^{1-n})^{1/2}\big] - \exp\big[-h(ks^{1-n})^{1/2}\big]\Big)^{-1} + C/s. \end{aligned} \tag{7}$$

With the extreme value theorem it is seen from eq. (7) that $p(x, 0) = C$ and $p(x, \infty) = (B - A)x/h + A$. The latter implies that $p(0, \infty) = A$ and $p(h, \infty) = B$, which verify that the initial conditions are satisfied and also that, at sufficiently large time, the pressure varies linearly with x or its gradient is constant in the layer.

The simplification $F_1/F_2 = k/s^n$ may be obtained in many ways, the most significant of which are represented by the following two models:

(1) $\varepsilon = \beta = b = c = 0$ which implies that the memory is in eq. (3) only (the relation between the flux and the pressure gradient);
(2) $\varepsilon = b = d = \alpha = 0$ which implies that the memory is in eq. (2) only (the relation between the pressure and the mass of fluid per unit volume).

Model 1 is the most practical and interesting and will be used in this paper.

The equation of the flux will be easily obtained using the LT of eq. (3) and then, after some manipulations needed to imply the boundary conditions, taking its inverse Laplace transform. The constant pressure gradient at infinite time, obtained by means of the extreme value theorem, has different implications for the flux depending on the choice between Models 1 and 2 and on the definition of k. In fact the LT of the flux is

$$(\gamma + \varepsilon s^n)Q - \varepsilon s^n q(x, 0) = (c + ds^n)p_x - ds^{n-1}p_x(x, 0)$$

which gives, when $p(x, 0) =$ constant and then $q(x, 0) = 0$,

$$Q = -(c + ds^n)P_x/(\gamma + \varepsilon s^n). \tag{8}$$

Then, at infinite time we obtain using Model 2 that $Q = -cP_x/s\gamma = c(B - A)/hs\gamma$ or $q(x, 0) = c(B - A)/h\gamma$, which implies that at infinite time the flux is a non-zero constant. With Model 1 the flux is $Q = -ds^nP_x/s\gamma$ which implies zero flux at infinite time.

In this paper we will use Model 1 and solve the following cases:

(1) Case A: The fluid pressure is initially zero and constant pressures are applied on the boundaries.
(2) Case B: The fluid pressure is initially zero and a sinusoidal pressure is applied to one of the boundaries while the other is kept at zero pressure.

5 DISCUSSION OF CASE A

In Case A there are two parameters defining the memory: n and $k = a\gamma/\alpha d$. The boundary conditions are that the pressure at the boundary $x = h$ is constant ($B \neq 0$), and zero at $x = 0$ ($A = 0$) while it is initially zero inside the layer ($C = 0$); the solution is given by eq. (B4), reproduced here as eq. (9) for convenience:

$$p(x,t) = (B/\pi)\int_0^\infty (\{[(\exp(-rt)]/r\}dr\{\sin[b'(h+x)]\sinh[a'(h-x)] - \sin[b'(h-x)]\sinh[a'(h+x)]\})(\cosh(2ha') - \cos(2hb'))^{-1} + Bx/h \quad (9)$$

where $b' = k^{1/2}r^{\nu}\sin(\nu\pi)$, $a' = k^{1/2}r^{\nu}\cos(\nu\pi)$, $\nu = (1-n)/2$ and $k = a\gamma/\alpha d$.

When the pressure is initially zero in the medium and constant pressures are applied at the boundaries, at any given point in the medium, as shown in Fig. 1, the pressure is an increasing function of time and tends asymptotically to Bx/h, which implies a constant gradient and, according to eq. (3), a zero flux since the fractional order derivative of a constant is zero. The pressure, at any given time, is a decreasing function of the distance from the boundary $x = h$. However, there is a memory effect relative to the case of strict application of Darcy's law ($n = 0$): at any given time and point the pressure is smaller when the memory is present and the difference is an increasing function of n.

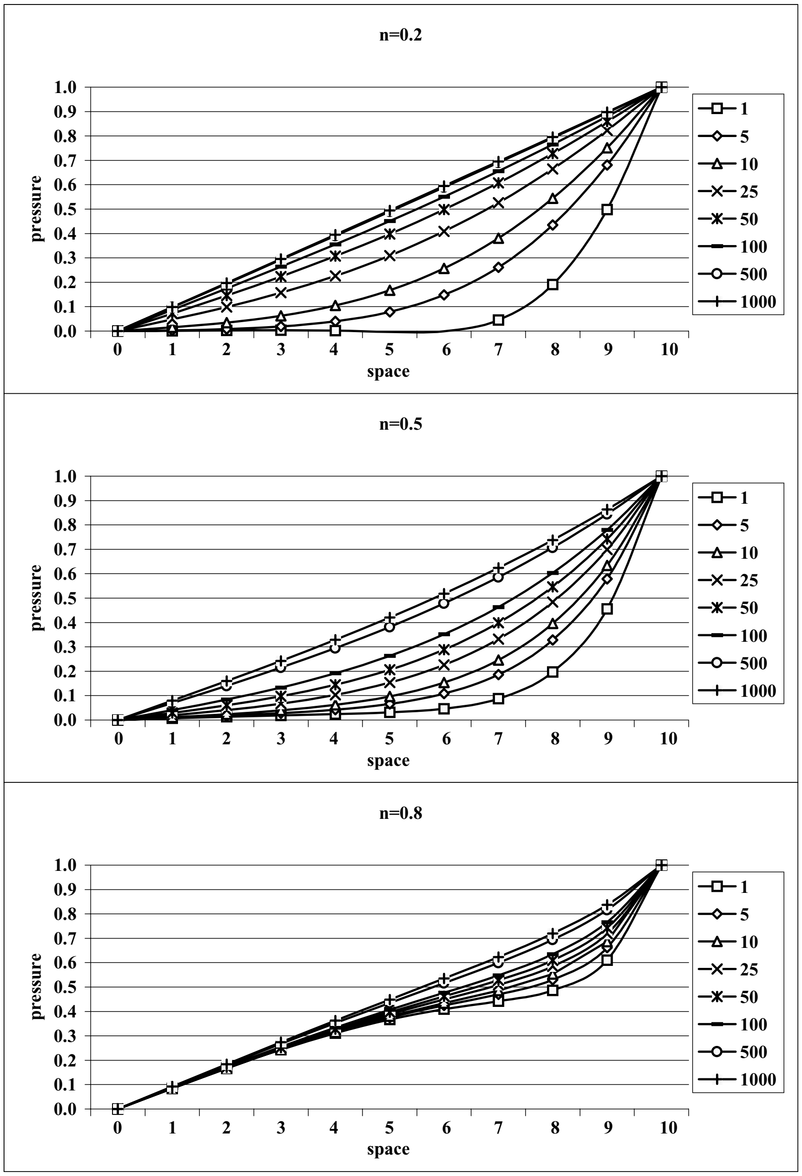

Figure 1. Pressure in a layer for zero initial pressure in the layer, a constant pressure on the boundary $x = h$ and zero pressure on the boundary $x = 0$. The abscissa x as well as h are measured in units of k. The ordinate is in units of the amplitude of the pressure applied on the boundary $x = h$. The panel on the right indicates the time in seconds.

With the extreme value theorem applied to eq. (8) it is verified that the flux at infinite time is zero; it is numerically verified that the flux is initially increasing, it reaches a maximum and then decreases to zero asymptotically.

Iaffaldano (2003) did an interesting experiment to measure the flux of water through a sample of sand contained in a cylinder with a constant hydraulic pressure difference between the boundary surfaces limiting it. The cylinder was 0.1 m long and the size of the sand grains was less than 6×10^{-4} m with an average of 3.5×10^{-4} m. The pressure on one side of the sample was obtained with a water column generating a pressure of 2×10^4 Pa, the flux was then measured at the other side of the sample. The flux at the beginning of the experiment was about 40 g s^{-1}, and after 10 hr decreased to about 27 g s^{-1}, that is about 70 per cent of the initial flux, implying that the flux generated by a constant pressure difference was not constant as Darcy's law would require. After the experiment a compaction of the sand was observed, implying a reduction of porosity.

We recall that the diffusion is governed by the parameter a/d, with dimensions m^{-2} s^{1-n}, defined in eq. (9), where for convenience we have assumed $\varepsilon = c = 0$ and $\gamma = \alpha = 1$, which simplifies the procedure without altering the meaning of the results. The parameter $(ad)^{1/2}$ substitutes for the diffusivity in the classic equation and we call it 'pseudodiffusivity'; the parameter's combination $d\mu/\rho_0$, with dimensions m^2 s^n may be called the 'pseudopermeability' of the medium. In conclusion, a/d and n define the diffusion properties of the medium coupled with the fluid.

The work of Iaffaldano (2003) shows that for some porous materials, in his case sand, two parameters define the permeability of the medium, in the 1-D case, and not one as in classic diffusion. The data observed by Iaffaldano (2003) in five laboratory experiments are well fitted with the average values $n \approx 0.5$, $d \approx 1.7 \times 10^{-2}$ s^{1+n} and give a 'pseudopermeability' $d\mu/\rho = 10^{-8}$ m^2 s^n, affected by experimental errors of a few per cent. It is not surprising that the value of the 'pseudopermeability' implied by the result of these laboratory experiments, i.e. 10^{-8} m^2 s^n, is out of the range of the classic permeability of sand without a 'memory' effect, which is 10^{-13} to 10^{-9} m^2 (Bear 1972; Bear *et al.* 1968) since the two parameters are formally different and have different dimensions.

6 DISCUSSION OF CASE B

A case leading to interesting conclusions is when one of the boundaries is subject to a sinusoidal pressure while the other is kept at constant pressure, that is when $A = 0$, $C =$ constant and $B = D \sin\omega t$ which simulates the effect of tides and sea waves against porous walls in harbours or on the water level of wells near the coastline (Robinson 1939), as shown in Fig. 2. The solution is obtained using formula (B5) of Appendix B

$$\begin{aligned} p(x,t) = (D\omega/\pi) \int_0^\infty & dr \exp(-rt)\{\sin[b'(h+x)] \sinh[a'(h-x)] \\ & - \sin[b'(h-x)] \sinh[a'(h+x)]\}/(\omega^2 r^2)[\cosh(2ha') - \cos(2hb')] \\ & + D\{\exp[J(L+M)] \sin[S(L-M)+\omega t] - \exp[J(L-M)] \sin[S(L+M)+\omega t] \\ & - \exp[J(-L+M)] \sin[S(-L-M)+\omega t] \\ & + \exp[J(-L-M)] \sin[S(-L+M)+\omega t]\}/2[\cosh(2MJ) - \cos(2MS)] \end{aligned} \tag{10}$$

where

$J = \cos[(1-n)\pi/4]$, $S = \sin[(1-n)\pi/4]$, $L = kx\omega^{(1-n)/2}$, $M = kh\omega^{(1-n)/2}$,
$b' = kr^\nu \sin(\upsilon\pi)$, $a' = kr^\nu \cos(\nu\pi)$, $\nu = (1-n)/2$ and $k = (a\gamma/\alpha d)^{1/2}$.

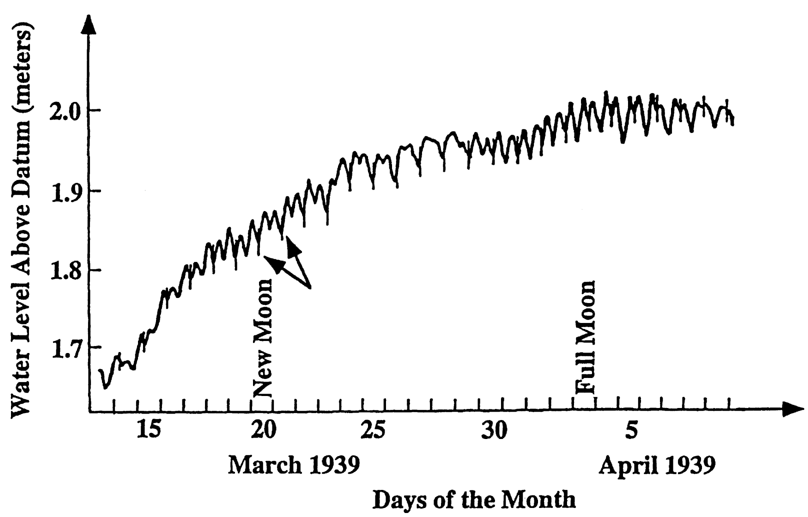

Figure 2. Recording of the water level in a well caused by ocean tides. The levels are corrected for the effect of atmospheric pressure. The arrows point to the segments indicating the time of the Moon's transit at upper culmination (Robinson 1939).

Fig. 3 shows that at any given point the pressure varies almost sinusoidally. There is only a perturbation due to the presence of the integral term in eq. (10) which, however, is rapidly vanishing with increasing time. In Fig. 3 it is also seen that the amplitude of these sinusoidal variations decreases with decreasing distance from the boundary $x = h$. However, it is seen in Fig. 4 that, at any given point the amplitude of the sinusoid decreases with increasing frequency, which indicates that most of the energy is confined in the boundary layer near $x = h$. In the case of strict application of Darcy's law ($n = 0$) the decrease in amplitude with increasing distance from the boundary $x = h$ would be linear; the departure from linearity due to the 'memory' effect is an increasing function of n. The fast decrease of the pressure is similar to the skin effect observed with electrical current and also in rheology (Caputo 1999).

The flux is obtained from eq. (B7), reproduced here for convenience:

$$\begin{aligned} q(x,t) = {} & -k'D\big(LT^{-1}[\omega/(\omega^2+s^2)](sF_1F_2)^{1/2}\big\{\exp\big[x(sF_1/F_2)^{1/2}\big] + \exp\big[-x(sF_1/F_2)^{1/2}\big]\big\}\big/\big[\exp\big[h(sF_1/F_2)^{1/2}\big] \\ & - \exp\big[-h(sF_1/F_2)^{1/2}\big]\big]\big) + \omega^{(1+n)/2}k'D\{\exp[J'(L'+M')]\cos[S'(L'-M')+\omega t-\nu\pi/2] \\ & - \exp[J'(L'-M')]\cos[S'(L'+M')+\omega t-\pi/2] + \exp[J'(-L'+M')]\cos[S'(-L'-M')+\omega t-\nu\pi/2] \\ & - \exp[J'(-L'-M')]\cos[S'(-L'+M')+\omega t-\pi\nu/2]\}/2[\cosh(2M'J') - \cos(2M'S')] \end{aligned} \tag{11}$$

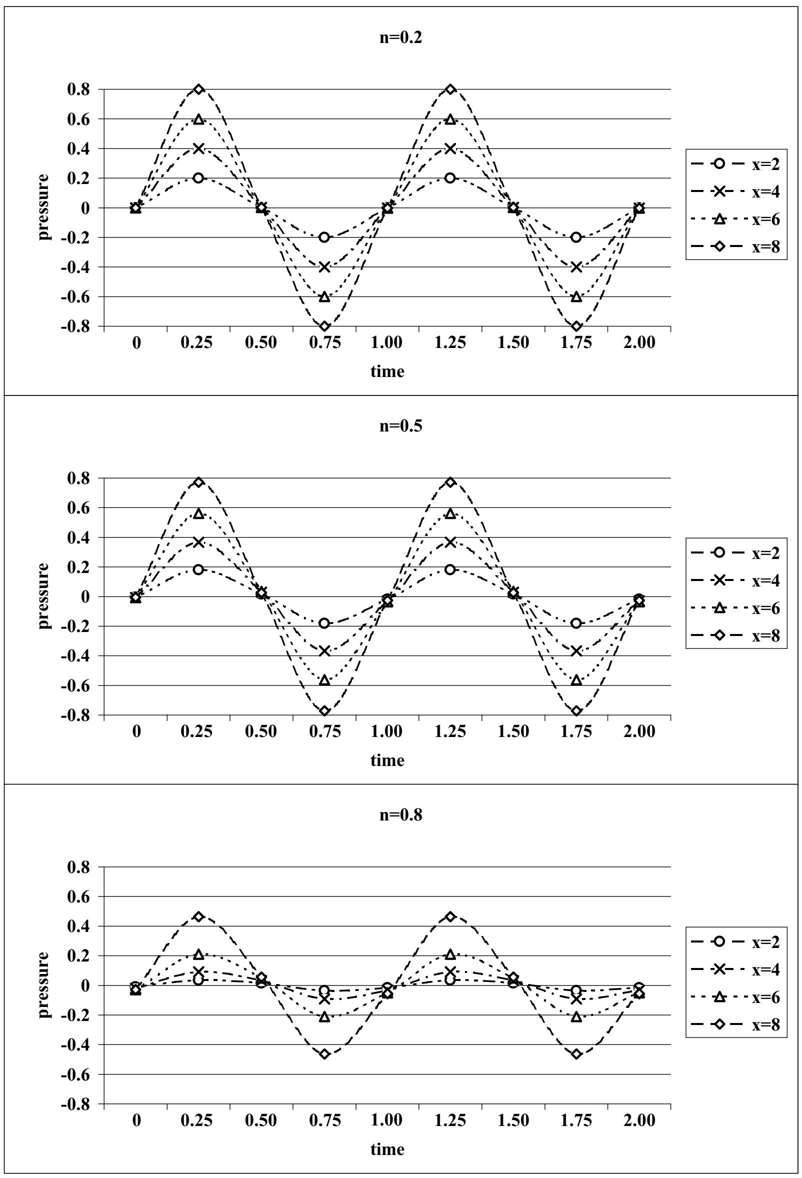

Figure 3. Pressure in a layer for a sinusoidal pressure applied on the boundary $x = h$, zero pressure on the other boundary $x = 0$ and zero initial pressure in the layer. The abscissa is time in units of the period of the applied pressure and x as well as h are measured in units of $k^{1/2}$. The ordinate is in units of the amplitude of the sinusoidal pressure applied on the boundary $x = h$. The panel on the right indicates the distance inside the layer of thickness 10 m.

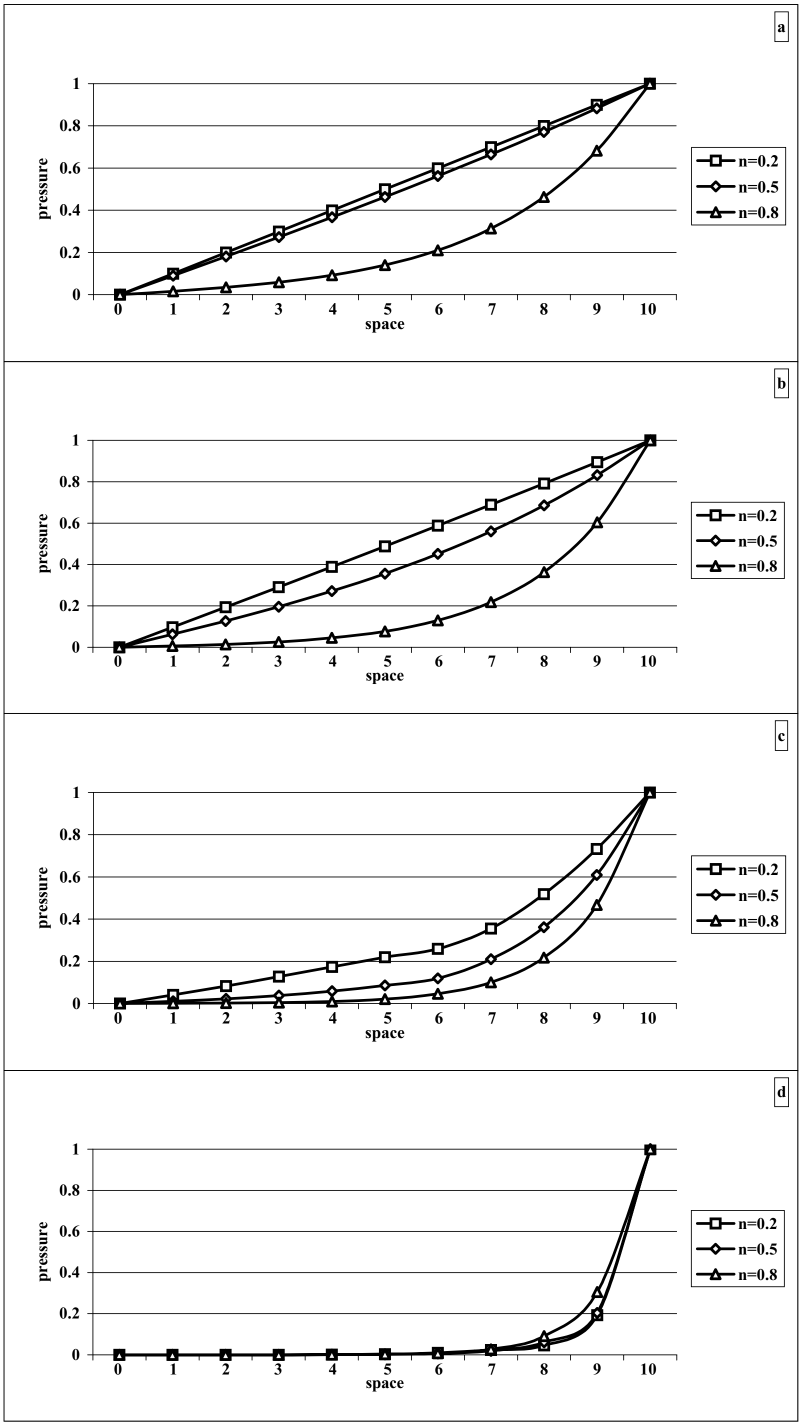

Figure 4. Variation of the amplitude of the sinusoids recorded at the distance x measured in units of $k^{1/2}$ (abscissa), for various values of the frequency ($a = 1.2 \times 10^{-5}\ \mathrm{s}^{-1}$; $b = 2.8 \times 10^{-4}\ \mathrm{s}^{-1}$; $c = 1.7 \times 10^{-2}\ \mathrm{s}^{-1}$; $d = 1\ \mathrm{s}^{-1}$). The ordinate is in units of the amplitude of the sinusoidal pressure applied on the boundary $x = h$. Note that at any given x the amplitude decreases with increasing frequency. The abscissa is measured in units of the thickness of the layer. The panel on the right indicates the order of fractional differentiation.

where

$$J' = \cos[(1-n)\pi/4], \quad S' = \sin[(1-n)\pi/4], \quad L' = kx\omega^{(1-n)/2}, \quad M' = kh\omega^{(1-n)/2}, \quad k = (a\gamma/d\alpha)^{1/2},$$

$$k' = (ad/\gamma\alpha)^{1/2} \quad \text{and} \quad \nu = (1-n)/2.$$

which is proportional to k'. The transient represented by the inverse Laplace transform (LT) term is rapidly vanishing and then the flux is stationary and represented by the sum of the residues:

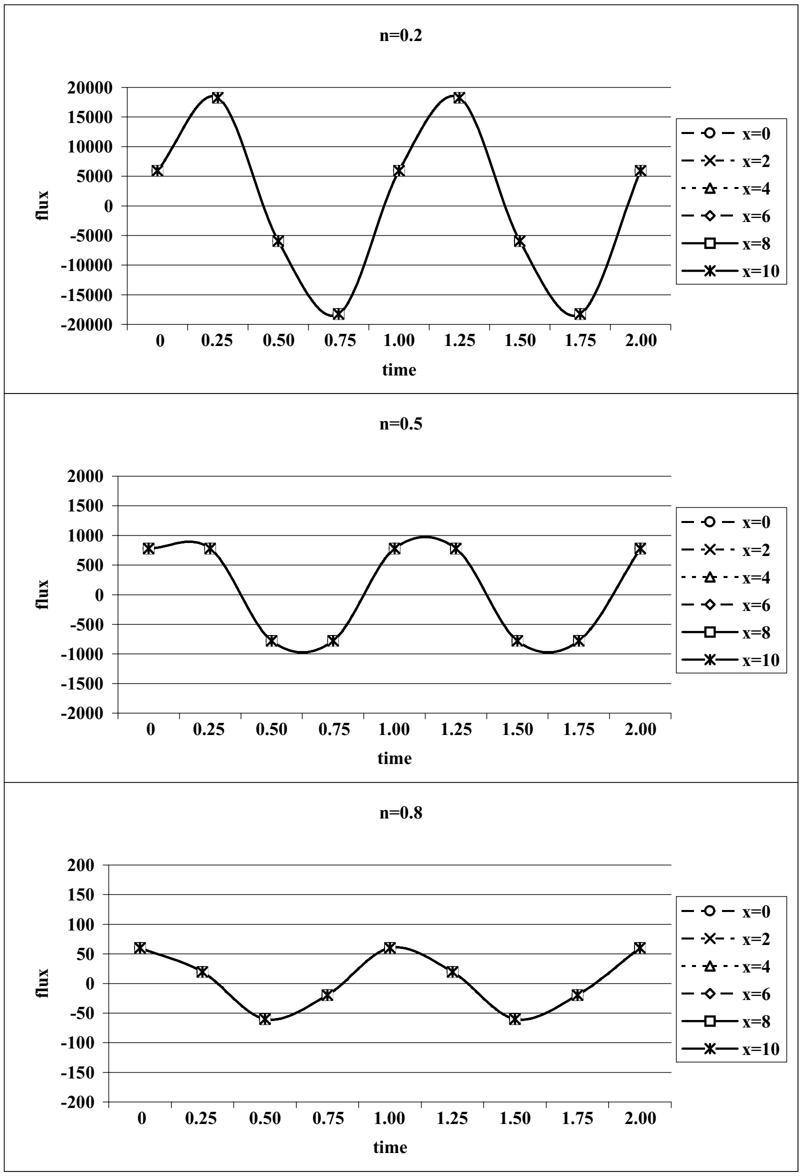

Figure 5. Flux across a layer caused by a sinusoidal pressure with period 24 hr applied at the boundary $x = h$ while the boundary $x = 0$ is kept at zero pressure. The abscissa is time in units of the period of the applied pressure. It is assumed that $k^{1/2} = 0.0155\ \mathrm{m}^{-1}\ \mathrm{s}^{(1-n)/2}$. The ordinate is measured in units of k'. Note that the curves are for the different values of x, including $x = 0$ and $x = h$; they are graphically indistinguishable. The panel on the right indicates the distance inside the layer of thickness 10 m.

$$\begin{aligned} q(x, t) = {} & \omega^{(1+n)/2} k' D (\exp[J'(L' + M')] \cos[S'(L' - M') + \omega t - \nu\pi/2] \\ & - \exp[J'(L' - M')] \cos[S'(L' + M') + \omega t - \pi/2] + \exp[J'(-L' + M')] \cos[S'(-L' - M') + \omega t - \nu\pi/2] \\ & - \exp[J'(-L' - M')] \cos[S'(-L' + M') + \omega t - \nu\pi/2]) \, (2[\cosh(2M'J') - \cos(2M'S')])^{-1}. \end{aligned} \quad (12)$$

When the product $kx\omega^{(1-n)/2} \ll 1$ then, for x in a limited range of few kilometres, with n limited in the range 0.2–0.8 which is almost always sufficient when fitting the data, L' and M' are much smaller than unity and, as is seen with a series expansion in terms of L' and M', the terms of first order in L' cancel out and the sum of the residues is practically independent of x: this is the case for low frequencies and for values of k of geophysical interest. The flux is then given by eq. (12) and is a sum of cosines with practically the same phase.

Concerning the flux, Fig. 5 shows the flux caused by a periodic pressure with period of 24 hr and amplitude 2×10^4 Pa applied at $x = h = 10$ m with $k = 0.0155\ \mathrm{m}^{-1}\ \mathrm{s}^{(1-n)/2}$ at several points in the layer and for several values of n. We have also verified that with this k value the flux at a distance of few hundred metres would be much lower than reported in the figure, and it could not explain the variation of water level in the wells observed by Robinson (1939), which implies that the medium between the beach and the wells is not common sand but perhaps gravel or that it has small underground communicating channels.

Observation of the flux in laboratory experiments made at several different frequencies would allow us to obtain reliable estimates of the parameters of models more complex than that considered in this paper which is characterized by just the two parameters k and n.

7 CONCLUSIONS

The experimental estimate of the parameters defining the pressure in a layer is not as simple as in the case of the classic Darcy's law, not only because there are two parameters defining the diffusion ($k = (a\gamma/\alpha \mathrm{d})^{1/2}$ and n) instead of one (the diffusivity), but also because of the more complicated form of the formulae required to fit the observed data which are all of an integral form instead of the simple exponential form in the Darcy's law case.

To estimate of the parameters k and n identifying the flux in the medium the simplest procedure is to observe the flux at $x = 0$ (Iaffaldano 2003) at given times when constant pressures are applied to the boundary, and compare them with those of the model curves of the flux.

Using the classic values of the diffusion reported above it may be assumed that an ocean tide with amplitude D m at the border of the water table induces, at a given distance a flux proportional to D. Assuming that the pressure at the boundary is periodic with a period of 24 hr and amplitude 2×10^4 Pa, we estimate that, at several tens of metres from the ocean, according to the results of Iaffaldano (2003) and the theory presented in this paper, the flux would hardly be observable.

Seven parameters should be used here in the complete mathematical formulation of the memory formalism, but due to the structure of formula (A11) this number may be readily reduced by simple grouping of the parameters, for instance c/γ, d/γ and ε/γ in eq. (3).

A more efficient laboratory method for estimating the parameters describing the diffusion is the observation of the pressure and/or the flux, and their amplitude spectra, at several distances from the source when a periodic pressure is applied at the boundary. The theoretical spectrum of the flux, when a sinusoidal pressure with frequency σ is applied to the boundary, is obtained by multiplying the spectrum of the Green's function by $\sigma/(\sigma^2 + \omega^2)$, with ω the frequency variable, which implies low-pass frequency filtering; note that the spectrum also depends on x. In m experiments made at different frequencies and observed at p different distances from the source, we have pm observed data and spectra and therefore pm experimental curves to fit to the theoretical ones in order to infer the parameters describing the diffusion.

In summary: the constitutive equations with 'memory' implied by the fractional order derivative introduced into Darcy's law represent media in which the flux decays in time more rapidly and delays the effect of the pressure at the boundary relative to the effects of the classic Darcy formula. Using fractal language, if it might be useful: the fractional differentiation will change the fractal dimension of the flow and the change will be larger for larger values of n. We also tentatively conclude that Darcy's is more general then previously believed upon the introduction of memory into its parameters.

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APPENDIX A: THE GREEN'S FUNCTION OF THE DIFFUSION IN THE LAYER

The Green's function of the pressure problem will be found with the Laplace transform (LT) method. The LTs of eqs (2), (3) and (5), assuming the initial pressure throughout the medium is given and also zero initial variation of mass per unit volume from the undisturbed condition, are

$$(\gamma + \varepsilon s^{n_1})Q = -(c + ds^{n_2})P_x + ds^{n_2-1}p_x(x,0) \tag{A1}$$

$$Q_x + sM - m(x,0) = 0 \tag{A2}$$

$$M = \left[\beta s^{m_2} m(x,0) - bs^{m_1} p(x,0) + \alpha m_0\right]/s(\alpha + \beta s^{m_2}) + (a + bs^{m_1})P/(\alpha + \beta s^{m_2}). \tag{A3}$$

Substituting eq. (A3) in eq. (A2) we find

$$Q_x = -\left[\beta s^{m_2} m(x,0) - bs^{m_1} p(x,0) + \alpha m_0\right]/(\alpha + \beta s^{m_2}) - s(a + bs^{m_1})P/(\alpha + \beta s^{m_2}) + m(x,0). \tag{A4}$$

Differentiating eq. (A1) with respect to x we get

$$Q_x = -(c + ds^{n_2})P_{xx}/(\gamma + \varepsilon s^{n_1}) + ds^{n_2-1}p_{xx}(x,0)/(\gamma + \varepsilon s^{n_1}) \tag{A5}$$

and equating to eq. (A4) we obtain

$$\begin{aligned} &-\left[\beta s^{m_2} m(x,0) - bs^{m_1} p(x,0) + \alpha m_0\right]/(\alpha + \beta s^{m_2}) - s(a + bs^{m_1})P/(\alpha + \beta s^{m_2}) + m(x,0) \\ &\quad = -(c + ds^{n_2})P_{xx}/(\gamma + \varepsilon s^{n_1}) + ds^{n_2-1}p_{xx}(x,0)/(\gamma + \varepsilon s^{n_1}). \end{aligned} \tag{A6}$$

Assume now that the initial pressure in the layer is a linear function of x, then we may write

$$\begin{aligned} P_{xx} = &\left[\beta s^{m_2} m(x,0) - bs^{m_1} p(x,0) + \alpha m_0\right](\gamma + \varepsilon s^{n_1})/(c + ds^{n_2})(\alpha + \beta s^{m_2}) \\ &+ s(a + bs^{m_1})(\gamma + \varepsilon s^{n_1})P/(\alpha + \beta s^{m_2})(c + ds^{n_2}) - m(x,0)(\gamma + \varepsilon s^{n_1})/(c + ds^{n_2}) \end{aligned} \tag{A7}$$

or, assuming

$$F_1 = (a + bs^{m_1})/(\alpha + \beta s^{m_2}) \quad \text{and} \quad F_2 = (c + ds^{n1})/(\gamma + \varepsilon s^{n_2}), \tag{A8}$$

we obtain

$$\left[\beta s^{m_2} m(x,0) - bs^{m_1} p(x,0) + \alpha m_0\right]/F_2(\alpha + \beta s^{m_2}) + s(F_1/F_2)P - m(x,0)/F_2 = P_{xx}. \tag{A9}$$

For our purpose is sufficient to simplify the solution assuming that there is no memory in the fluid, then $b = 0$, $\beta = 0$, as in Model 1), and $m(x,0) - m_0 = ap(x,0)/\alpha$.

The solution of eq. (A9), when $p(x,0)$ is constant or a linear function of x, is

$$P = -[m_0 - m(x,0)]\alpha/a + C_1(s)\exp\left\{x[s(F_1/F_2)]^{1/2}\right\} + C_2(s)\exp\left\{-x[s(F_1/F_2)]^{1/2}\right\}. \tag{A10}$$

For our purpose it is sufficient to simplify the solution assuming $F_1 = a/\alpha$ and also that $m_2 = m_1 = n_2 = n_1 = n$. With these assumptions, eq. (A10) finally simplifies to

$$P = p(x,0)/s + C_1(s)\exp\left\{x[s(F_1/F_2)]^{1/2}\right\} + C_2(s)\exp\left\{-x[s(F_1/F_2)]^{1/2}\right\} \tag{A11}$$

where $F_1 = a/\alpha$ and $F_2 = (c + ds^n)/(\gamma + \varepsilon s^n)$.

APPENDIX B: COMPUTATION OF THE GREEN'S FUNCTION OF THE LAYER

A case of interest is when the pressure at the boundaries and the initial pressure are constant:

$$p(0,t) = A,\ p(h,t) = B,\ p(x,0) = p(0) = C \tag{B1}$$

with A, B and C constant. To satisfy the conditions one must assume, as is seen in eq. (A11),

$$P(x,s) = C_1\exp[x(sF_1/F_2)]^{1/2} + C_2\exp[-x(sF_1/F_2)^{1/2}] + p(0)/s \tag{B2}$$

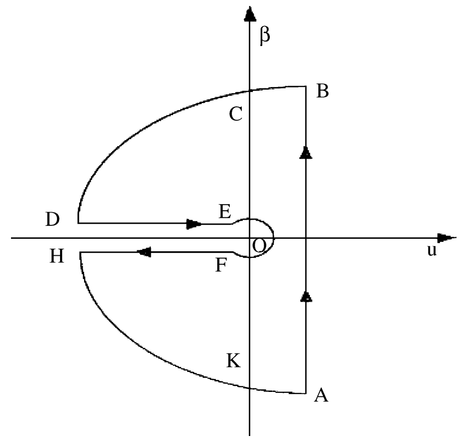

Figure 6. Path of integration of formulae (B4), (B8) and (B10) in the complex plane. The path begins at A, follows the direction of the arrows and returns to A.

where

$$C_2 = (1/s)\big(B - C + (C - A)\big\{\exp\big[-h(sF_1/F_2)^{1/2}\big]\big\}\big)\big(\exp\big[h(sF_1/F_2)^{1/2}\big] - \exp\big[h(sF_1/F_2)^{1/2}\big]\big)^{-1}$$

$$C_1 = (1/s)\big(C - B + (A - C)\big\{\exp\big[h(sF_1/F_2)^{1/2}\big]\big\}\big)\big(\exp\big[h(sF_1/F_2)^{1/2}\big] - \exp\big[-h(sF_1/F_2)^{1/2}\big]\big)^{-1}$$

which gives with Model 1

$$P = (1/s)([B - C + (C - A)\exp(-hs^\nu\sqrt{k})]\exp(xs^\nu\sqrt{k}) + [C - B + (A - C)\exp(hs^\nu\sqrt{k})]\exp(-xs^\nu\sqrt{k}))(\exp(hs^\nu\sqrt{k}) - \exp(-hs^\nu\sqrt{k}))^{-1} + C/s \tag{B3}$$

where $b' = k^{1/2}r^\nu\sin(\nu\pi)$, $a' = k^{1/2}r^\nu\cos(\nu\pi)$, $\nu = (1-n)/2$ and $k = a\gamma/\alpha\, d$. Finding the inverse Laplace transform of formula (B3) we obtain the solution for Case A when $(A = C = 0,\ B \neq 0)$:

$$p(x,t) = (B/\pi)\int_0^\infty (\{[\exp(-rt)]/r\}dr\{\sin[b'(h+x)]\sinh[a'(h-x)] - \sin[b'(h-x)]\sinh[a'(h+x)]\})(\cosh(2ha') - \cos(2hb'))^{-1} + Bx/h \tag{B4}$$

where the last term in eq (B4) results from the integration on the circle around the origin of the path shown in Fig. 6.

An interesting case arises when $A = C = 0$ and $B = D\sin\ \omega t$ which is of interest in laboratory experiments for the determination of the parameters $ca/\alpha\gamma$, $\alpha d/a\gamma$ and ε/γ and simulates the effect of tides and sea waves against porous walls in harbours or on the water level of wells near the coastline (Robinson 1939) (shown in Fig. 2); in this case eq. (B2), taking into account the residues resulting from the zeros of $\omega^2 + s^2 = 0$, gives

$$\begin{aligned} p(x,t) = {} & LT^{-1}[\omega D/(\omega^2+s^2)]\big(\big\{\exp\big[x(sF_1/F_2)^{1/2}\big] - \exp\big[-x(sF_1/F_2)^{1/2})\big]\big\}\big/\big\{\exp\big[h(sF_1/F_2)^{1/2}\big] - \exp\big[-h(sF_1/F_2)^{1/2}\big]\big\}\big) \\ & + D\big\{\exp[J(L+M)]\sin[S(L-M)+\omega t] - \exp[J(L-M)]\sin[S(L+M)+\omega t] - \exp[J(-L+M)]\sin[S(-L-M)+\omega t] \\ & + \exp[J(-L-M)]\sin[S(-L+M)+\omega t]\big\}/2[\cosh(2MJ) - \cos(2MS)] \end{aligned} \tag{B5}$$

where the terms in the second and third lines represent the sum of the residues at the points $s = \pm i\omega$ which are the poles of the LT of sin ωt,

$$J = \cos[(1-n)\pi/4], \quad S = \sin[(1-n)\pi/4], L = Kx\omega^{(1-n)/2}, \quad M = Kh\omega^{(1-n)/2}, \quad K = (sF_1/F_2)^{1/2} \tag{B6}$$

and the inverse Laplace transform is obtained by integrating along the circuit of Fig. 6, where now zero is the contribution from the integral around the origin.

Since we are interested in the effect of ocean tides on the beaches it is interest to find the flux caused by a sinusoidal pressure on the boundary $x = h$. The flux is obtained from eq. (8) which we rewrite here for Model 1 with the assumption that $p(x, 0) = 0$ which implies $q(x, 0) = 0$

$$Q = -ds^n P_x/\gamma.$$

Substituting P_x obtained from eq. (B5) it is seen that the inverse Laplace transform of Q is found by integrating along the path of Fig. 6 and adding the residues due to the presence of the zeros of $\omega^2 + s^2 = 0$; the integral is a transient which is negligible in the case of the frequencies of ocean tides and the sum of the two residues is

$$\begin{aligned} q(x,t) = {} & -k'D\{LT^{-1}[\omega/(\omega^2+s^2)]s^{(1-\nu)/2}k'[\exp(xs^\nu k) + \exp(-xs^\nu k)]/[\exp(hs^\nu k) - \exp(-hs^\nu)]\} \\ & + \omega^{(1+n)/2}k'D\{\exp[J'(L'+M')]\cos[S'(L'-M') + \omega t - \nu\pi/2] \\ & - \exp[J'(L'-M')]\cos[S'(L'+M') + \omega t - \pi/2] + \exp[J'(-L'+M')]\cos[S'(-L'-M') + \omega t - \nu\pi/2] \\ & - \exp[J'(-L'-M')]\cos[S'(-L'+M') + \omega t - \pi\nu/2]\}/2[\cosh(2M'J') - \cos(2M'S')] \end{aligned} \tag{B7}$$

where $J' = \cos(\nu\pi/2)$, $S' = \sin(\nu\pi/2)$, $L' = kx\omega^\nu$, $M' = kh\omega^\nu$, $k = (a\gamma/d\alpha)^{1/2}$ and $k' = (ad/\gamma\alpha)^{1/2}$.