# Decadal free polar motion of triaxial Earth

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Accepted 2003 October 7. Received 2003 September 22; in original form 2003 May 6

## SUMMARY

By using the elliptic function solution, the exact rotation solution of triaxial Earth is provided as motion of couple periods with an image decadal period longitudinally and Chandler wobble as transverse wave. To obtain the evident function result, we had to treat with Euler rotation equations with three times separated polar motions. The image period means transverse wave modulation that appears as phase modulation and the phase time series of the free wobble show decadal variation. The decadal wobble causes frequency modulation for free polar motion. The decadal wobble has 14.6 yr period that is almost 73/6 times or about 12 times of 1.191 yr period for Chandler wobble so that subharmonic resonance takes place. Applying subharmonic resonance theory in non-linear oscillations, the amplitude of polar motions excited by the decadal additional wobble is provided as about 8.2 times magnifying the excitation amplitude. The decadal wobble of amplitude about 13 mas excites the polar motion on the average amplitude of 132 mas in addition of an about 108 mas amplitude modulation, range in (24, 240). This amplitude variation is almost identical to the observed data series.

**Key words:** elliptic function solution of rotation, polar motion, subharmonic resonance, triaxially asymmetric Earth.

## 1 INTRODUCTION

Polar motion of Earth's rotation is a very interesting problem of science. Many mechanisms of motion have not been worked out in the studies of scientists of geodesy, geophysics, and astronomy. Especially, the long term polar motion is unproven, appearing as amplitude modulation. Early in the 18th century, Euler (1765) provided rigid body rotation equations to model polar motion and provided the free sway term of 305 d period. In the last century, scientists treated these equations with elastic bodies, to determine the free sway period as 396 d. Considering ocean, liquid core, and other effects, the theoretical free sway period is conformed at 435.2 d, approximating to the observed average period discovered by Chandler (1891) and determined as precisely 433.5 d by recent scientific works, say, in Vondrák (1999).

In the late half of the last century, each author retained a personal view in discussion of the strange natural phenomena of variations for the free wobbles. Carter (1981) supported the viewpoint of frequency modulation. Later, Vondrák (1987) pointed out that the Chandler frequency could not be determined as an invariant constant and provided an exponent curve to fit the frequency–amplitude relation. Gao (1994) treated with amplitude dependence and obtained a similar logarithm curve fitting the frequency and amplitude. Wang (1998, 1999a) introduced a non-linear dynamic model to explain the excitation of the Chandler wobble and concluded that it is the weak resonance that relates the amplitude and frequency by a bell curve other than a monotone curve. Wang (1999b, 2000a) introduced a parameter time-dependent model to modify the Euler rotation equations adapting the deformation Earth. Wang (1999c, 2000a, 2000b, 2001, 2002) obtained a series of results identical to the observed reality by the model. Wang (1999a, 2000a, 2002) introduced bifurcation solution excited by the annual function and provided sub-period bifurcation of 0.594 yr with amplitude *ca* 12 mas. Later in another study, bi-period bifurcation of 2.3 yr with amplitude *ca* 8 mas is obtained. All these discussions verify that the model of polar motion must be non-linear and non-linear dynamics may work for the kinetic Earth's rotation.

Almost all the investigations for excitation source of polar motion are focused on the Earth's physical amounts. Almost all the Earth's environmental physical amounts are considered in decades. Few studies become suspicious of the linearized physics model, describing polar motion that could cause the variation of the free wobbles. Indeed, the rotation equations of rigid body and non-rigid body due to Euler–Liouville may not be suspicious of for the model is succinct and perfect to describe the rotation. Furthermore, the linearized form of the model and its solution for polar motion are not suspicious as its precision is applied over *ca* 200 more years. However, it should be pointed out that the prime model of Euler was non-linear. It was linearized and solved for the practical Earth by some symmetric feature assumptions. As the solution was very much approximate, with the calculated average parameters from observed data (except the variation of the frequency and the amplitude) Earth's rotation was regarded as linear motion without suspicion.

As known, Earth is not so triaxially symmetric with respect to the inertia momentum ellipsoid. Thus Euler's rotation model must be non-linear. The property of superposability does not work for the non-linear model, and the solutions made previously are established on the basis of the linear model possessing the property of superposability. Additionally, a more serious problem is that the non-linear model has many solutions which differ from the linear model, such as weak resonance solution, bifurcation solution, subharmonic resonance solution and super-harmonic resonance solution, etc. The exact solution of Euler's non-linear rotation equations is of the form of the Jacobi elliptic function with couple periods. These important solutions would explain many rotation phenomena that could not be seen in the linearized process. Just by this solution, many unproved problems might be made clear.

Many works investigated the effect of the second term in the solution process of Euler homogeneous equations. Smith & Dahlen (1981, P230) pointed out that the effect might be about 0.14 d for perturbing the free Chandler wobble. Kinoshita & Sasao (1989) derived that the effect of the difference between the minor equatorial inertia momentum $A$ and the major $B$ might only affect the polar motion to a $10^{-14}$ variation. Van Hoolst & Dehant (2002) treated the case for rotation of triaxiality of Earth and the effect of second terms, concluding that the effects might be less than 0.01 day. However, almost all of the former studies discussed on the effects of the second terms and of the revision of period for Chandler wobble. Abraham & Marsden (1980) stated the triaxial rigid body rotation principle, and Ratiu (1980) provided the verification and generalized it to n-axial case. In fact, the rotation model of triaxiality is changed into the non-linear case, and the linearized solution of the former approach can only provide the rotation about the principal pole. As non-linear equations, motion dominated by the system may possess complex process other than free wobble about the maximum while there exists stable motion about the axis of the minimum moment of inertia. However, revision of the classical solution for Euler–Chandler free wobble is not the right route to revealing the real motion of Earth. Theoretical mechanics indicates that the exact solution of rotation for a triaxial rigid body should be elliptic functions rather than the former trigonometric functions that have been applied for 200 more years. Even from the end of 19th century, after Chandler's discovery of the real free wobble, scientists all over the world focused on studying the effects of elastic mantle and liquid core but pay no attention to the existence of multi-solutions for the non-linear model.

The more difficult problem in polar motion is the presence and consequence of frequency modulation phenomena. There are many long-period polar motions seen in the Earth's rotation data, especially of the period 14.6 yr. If the free wobble possesses one inherent frequency, then the trajectory of the pole position should be an ellipse rather than the observed spiral. It is evident that the observed amplitude of Chandler wobble varies 90 per cent upon its mean with long quasi-periods. Markowitz (1960) pointed out that there might be an empirical term of period 21 yr in polar motion, later revised to 24 yr. Vondrák (1999) provided the Markowitz period as 30 yr and a term of period *ca* 14 yr with a series of decades periods. Schuh & Nagel (2001) obtained decadal period spectrum for polar motion data series. Poma (2000) surveyed the Markowitz's polar motion. Wang (2001) detected about 16 mas modulation with decadal period and estimated it as 18.6 yr. Sidorenkov (2000a) also discovered 18.6 yr lunar tides in atmospheric excitation series. What dynamic force causes the long periodic motion?

Formerly, the effects considered as exciting the decadal polar motions are those of magnetic effect of core–mantle coupling, load and melting of glacial ice and snow, ground water and sea level changing, etc. Eubanks (1993) provided some evidence for sea level, ground water, glacial ice affecting decadal polar motion, but the real period has not been obtained. Investigations were carried out into the effect of snow load and water in lakes and reservoirs, but the conclusion was reached that all the excitation for decadal polar motion may not reach 10 per cent of the real excitation.

In this study, we verify that there exist a decadal free polar motion term and apply subharmonic resonance to solve the amplitude modulation of polar motion. Sections 2 to 5 discuss the rotation for a triaxially asymmetric Earth, the exact solution of elliptic functions, decadal free wobble and frequency modulation model. Subharmonic resonance is a kind of non-linear oscillation in which the forcing period is many times larger than the inherent period. In Section 6, the principle of subharmonic resonance is discussed, applying the time-dependent parameter model for polar motion introduced in this study. That is to say the periods of the inherent and the forcing, should have a simple fractional ratio. In Section 7, the subharmonic resonance solution for interactions of Chandler wobble and the decadal free polar motion is provided. Conclusions are made in Section 8.

## 2 MOMENTS ON THREE POLES

The main relative difference $H_0 = (C - A)/B \approx (C - B)/A = 0.00328475$ of the principal inertia momenta for the polar motion of a triaxially symmetric Earth with $A \approx B$, is given as dynamic ellipticity $H_0$. From this dynamic ellipticity, it may be deduced that there is a free polar motion mode, of period about 304.5 d, for a rigid Earth. Smith & Dahlen (1981) provided the hybrid formula for the Earth rotation model of elastic mantle with ocean. Wahr (1981) expanded the eigenfunction for rotation mode in higher order solid tide case. Zhu (1991) considered the effect of the viscoelastic mantle on polar motion. The free wobble mode is thus determined as *ca* 1.43 times longer for the real Earth with respect to the rigid Earth. We set the coefficient at 1.43, given as the dynamic coefficient, without discussing the detailed effects of elastic mantle, liquid core and ocean.

In this study, the Earth is assumed to have heterogeneous inertia momenta in the three principal axes. As known, rotation dynamics of a rotation planet may be considered only by its inertia momentum ellipsoid in spite of its shape. By the attraction potentials of Moon and Sun, Earth's equatorial bulge emerges differently in two rectangular directions. Liu & Chao (1991) obtained that the relative difference amounts to

$(B - A)/C = 2.1946 \times 10^{-5}$, *ca* $1/150$ of $H_0$, and the principal semi-major inertia momentum $B$ lies approximately in the directions of *ca* 105°W and 15°W for the principal semi-minor inertia momentum $A$ using inertia tensor. It can be seen that the inertia momentum ellipsoid may possess triaxial asymmetry with the principal inertia momenta $C > B > A$. By this assumption, the dynamic mechanism of polar motion is deduced with important difference to the case $B \approx A$ of Earth's biaxiality.

As $(B - A)/C$ has the same form as the dynamic ellipticity $H_0$, this relative difference may have analogous effect for the dynamic ellipticity that causes free wobble. Here $H_1 = (B - A)/C$ is defined as the average equatorial dynamic ellipticity according to the assumption of a triaxial Earth. Although the defined parameter is tiny, less than 1/150 of the polar dynamic ellipticity as distinct from the equatorial, we shall deduce that the effect may not be neglected. In fact, the effect of inertia momentum for Earth rotation is intense and 1/150 of the dynamic ellipticity may be larger than the effects of atmospheric, oceanic, seismic and other angular momentum variations. Earth's seismic energy, water flow, reservoir building, even atmospheric variations have been taken into account for seeking the energy causing the rotation variations. Maybe none of these mentioned forces may be greater than the effect of difference of inertia momenta $1/150$ of the polar dynamic ellipticity. Why has the source always been neglected? An important reason is that the rotation model is seen as linear with little effect on higher order decimal amounts. Some works considered the effect of the difference but concluded that the affection of $H_1$ is too tiny to be detected. As known by non-linear dynamics, a tiny variation in non-linear model may cause intense or even qualitative changes especially in multisolutions. We consider the tiny departure of amount *ca* $1/45000$ and investigate the diversity yielded by non-linearity of Euler's model with triaxial Earth.

Moreover, if the principal inertia momenta are assumed as more triaxially asymmetric on the same axis in the different direction, as in the case of $A' > A$, $B' < B$ and $C' > C$ with the Southern Hemisphere fatter than the Northern, there may be two different equatorial dynamic ellipticities $H_1$ and $H_2$ yielding moments on two non-rotating poles for the hexad-axial Earth. The effects of the two equatorial dynamic ellipticities are illustrated in Fig. 1, but we do not go deep in this study about the effects for rotation. As seen in Fig. 1(a) and (b), the force ***C*** on point $E$ exerts moment on axis $A$ by an arm of force *OE* that is the difference of inertia momenta $B$ and $A$, so that the moment directs in ***OF*** longitudinally. ***OE*** is too tiny and was always ignored. By discussion of the additional polar motion, it is enough to consider $A < B < C$. However, for deriving the secular trend in polar motion, the case of $A' > A$, $B' < B$ and $C' > C$ had to be taken into account. It can be seen that the moments form a longitudinal speed. Liu & Chao (1991) also treated with this effect as diurnal and semi-diurnal speed. Dickey & Schutz (1989) tried to detect the speed accumulation.

For deriving wobbles of non-rotating poles the moments and total rotation should be considered as in eqs (1) and (2). The inertia moment exerted on axis $Z$ does not include the inertia moments on axes $X$ and $Y$. But because the difference between inertia momenta $A$ and $B$ should

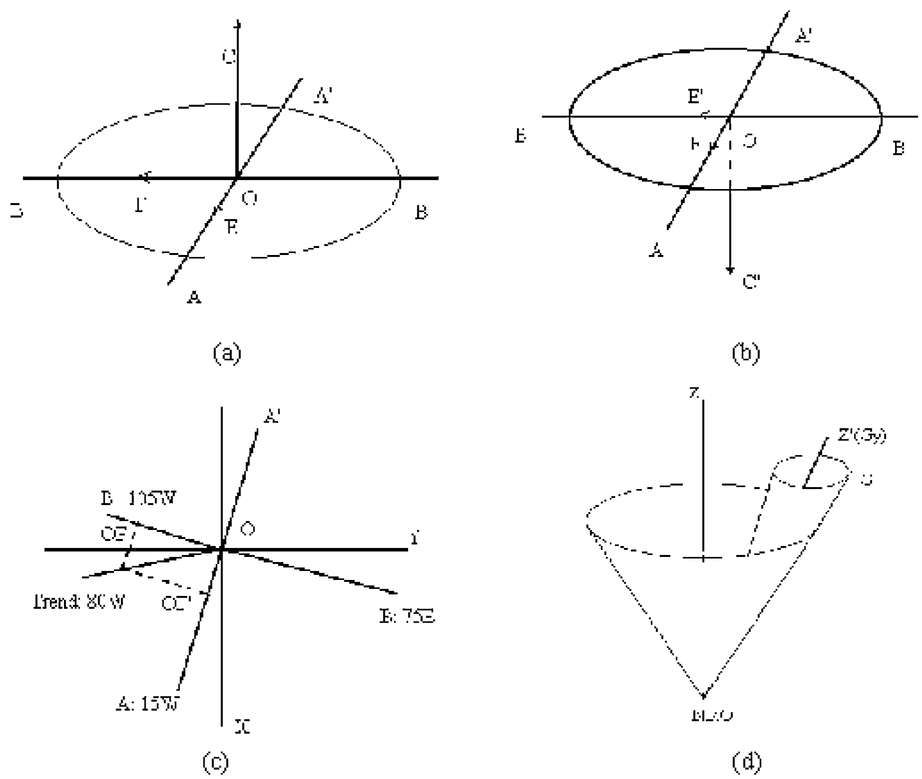



**Figure 1.** Relation for longitudinal moments on the principal inertia momentum poles *C*—*B*—*A*, where the ellipse represent the equatorial plane. (a) The difference of *B*–*A* as arm ***OE***′ of force ***OC*** the inertia force with the moment ***OF*** rectangular to both ***OE***′ and ***OC*** following the right hand spiral criterion. (b) The difference of *B*′–*A*′ as arm ***OE***′′ of force ***OC***′ the inertia force with the moment ***OF***′ rectangular to both ***OE***′′ and ***OC***′ following the right hand spiral criterion. (c) Longitudinal components ***OF*** and ***OF***′ form the speed vector of the secular trend. (d) Latitudinal contribution superposed onto the rotating pole causing the wobble cone of frequency modulation.

not be neglected, on axes $X$ and $Y$ there are inertia moments. As seen in Fig. 1(a), although the moments have the arm of force less than $1/150$ of that on axis $C$, the moments may cause the wobbling of the two non-rotating poles. The action may exert directly on axes $A$ and $B$ and indirectly exert onto axis $C$ producing tiny moments. Atmospheric variations can drive Earth's annual wobble with significant amplitude that is often taken into account. In comparison to the inertia momentum difference, though less than $1/150$ of the moment that causes the free wobble about axis $C$, atmospheric angular momentum may not be much greater in magnitude. In this sense, the moments on axes $A$ and $B$ may not be neglected.

## 3 ROTATION ABOUT THREE POLES

According to momentum theorem, the total momentum of a rotation planet remains constant in the way. Earth's total rotation momentum $T$ can be obtained by the summation of the rotation momentum components about three principal axes. Rotation component about axis $C$ is defined as $T_Z$, while that about axis $A$ as $T_X$ and that about axis $B$ as $T_Y$ respectively. There holds

$$T = T_X + T_Y + T_Z \tag{1}$$

Moritz & Mueller (1987) provided the detail for deriving rotation model by momentum criterion. Unfortunately, Earth is not so well axially symmetric according to inertia momentum. Earth's inertia momentum ellipsoid should be regarded as pear shaped by differences of principal inertia momenta. Except the rotation component about axis $C$, there are also rotation components about axis $A$ and axis $B$ caused by torques $T_X$ and $T_Y$ that are invisible because the two axes are travelling in the equator plane rectangular to the variable instantaneous rotating pole. See in Capitaine (2000).

The Earth may rotate about any of its three figure axes with respect not only to rotation about a maximum inertia axis, but also may turn somersaults all the way. It should be pointed out that total rotation of a planet noted by $\mathbf{P}$ in frame $X$—$Y$—$Z$ is formed by three-pole rotation components $\mathbf{P}_Z$, $\mathbf{P}_X$, $\mathbf{P}_Y$ respect to torques on each axis with each representing rotation component about one axis in frames $Z_Z$—$X_Z$—$Y_Z$, $X_X$—$Y_X$—$Z_X$ and $Y_Y$—$Z_Y$—$X_Y$ respectively. There is a relation

$$\mathbf{P} = \mathbf{P}_X \otimes \mathbf{P}_Y \otimes \mathbf{P}_Z \tag{2}$$

Here $\otimes$ represents direct product of rotations. Later, rotations $\mathbf{P}_X$ and $\mathbf{P}_Y$ are derived and the multiplication interaction $\otimes$ is obtained. See Section 4.

In the process, Euler's rotation equations may be the essential starting point. Euler's model for rotation was introduced for the case of $A < B < C$ by angular movement in reference to the geocentre, or fix point of rotating Earth defined in Capitaine *et al*. (1986).

$$\begin{aligned} A\dot{\omega}_1 + (C - B)\omega_2\omega_3 &= L_1 \\ B\dot{\omega}_2 + (A - C)\omega_3\omega_1 &= L_2 \\ C\dot{\omega}_3 + (B - A)\omega_1\omega_2 &= L_3 \end{aligned} \tag{3}$$

Here $\omega_1$, $\omega_2$, $\omega_3$ are components of the angular velocity vector with respect to the figure center and $A$, $B$, $C$ the inertia momenta of principal figure axes. In the right hand side, $L_1$, $L_2$, $L_3$ are additional attraction potential components exerting on respective principal figure axes that are set as zero in this study for homogeneous case of free wobbles.

For convenience to solve eq. (3) in approach to elliptic functions, Euler rotation equations may change to formulation of distinct variables with modes $k$ and $k'$ by ignoring the difference of $(C - A)/B$ and $(C - B)/A$ and preserving that of $A < B$.

$$\begin{aligned} \dot{\omega}_1^2 &= H_0H_1(a_1^2 - \omega_1^2)(k'^2a_1^2 + k^2\omega_1^2) \\ \dot{\omega}_2^2 &= H_0H_1(a_2^2 - \omega_2^2)(a_2^2 - k^2\omega_2^2) \\ \dot{\omega}_3^2 &= H_0^2(a_3^2 - \omega_3^2)(\omega_3^2 - k'^2a_3^2) \end{aligned} \tag{3$'$}$$

Here $a_i$ $(i = 1, 2, 3)$ are integral constants determined by initial conditions. In the case of $k \to 0$ while $k' \to 1$, differentiating the both sides of each equation with respect to time $t$, eq. (3′) may be simplified to the case of $A = B$ with polar motion and LOD (length-of-day) been distinct.

$$\begin{aligned} A\dot{\omega}_1 + (C - A)\omega_2\omega_3 &= 0 \\ A\dot{\omega}_2 + (A - C)\omega_3\omega_1 &= 0 \\ C\dot{\omega}_3 &= 0 \end{aligned} \tag{3$''$}$$

Evidently, eq. (3″) is the special case of eq. (3) as well as eq. (3′) by which the so-called polar motion and LOD may be separated into two problems considered.

In fact, eq. (3′) is a kind of non-linear model. The exact analytical solution for the homogeneous case may be obtained by elliptic functions, seen in Greenhill (1959).

$$\begin{aligned} \omega_1 &= a_1\mathrm{cn}(\omega t - \varphi, k) \\ \omega_2 &= a_2\mathrm{sn}(\omega t - \varphi, k) \\ \omega_3 &= a_3\mathrm{dn}(\omega t - \varphi, k) \end{aligned} \tag{4}$$

Here$a_i$ ($i = 1, 2, 3$) and others are amplitudes, frequency, initial phases and the main period $4K/\omega$ relative to mode $k$. And cn, sn and dn are elliptic cosine, elliptic sine and the third class of elliptic sine functions with double period variables $K$ and $K'$, that are generalizations of trigonometric functions. As elliptic functions, cn, sn and dn possess couple periods with $K'/K$ equal to an pure image number. If in the main frequency $\omega$, condition of the auxiliary mode $k'^2 = 1 - k^2$ tends to 1 or mode $k$ tends to zero, then cn, sn and dn tend to cos, sin and constant 1 respectively. In other words, cn and sn have graphs like sine waves with a tiny diversity, while dn is an even function with a few oscillations under the horizontal line of Earth's average rotation velocity. The latter illustrates that the solution of the rotation velocity of Earth is about a constant but with a very little free oscillation on it. Generally, for Earth of triaxially asymmetric principal inertia momenta $A < B < C$ there should be speed in semi-diurnal term of length-of-day with tiny amplitude about 0.005 ms.

We estimate the main period $K$ and $K'$ by mode $k$ with respect to principal inertia momenta $A < B < C$ using the formula of $k$ in Greenhill (1959).

$$k^2 = \left[-\frac{A^2\omega_{10}^2}{(B-C)/BC} + \frac{B^2\omega_{20}^2}{(C-A)/CA}\right] \Big/ \left[-\frac{C^2\omega_{30}^2}{(A-B)/AB} + \frac{B^2\omega_{20}^2}{(C-A)/CA}\right] \tag{5}$$

$$K = \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2\sin^2\phi}} d\phi = [1 + (1/4)k^2 + (9/64)k^4 + \ldots\ldots] \times \pi/2$$

Here $\omega_{10}^2$, $\omega_{20}^2$, $\omega_{30}^2$ are initial values of variables and $K$ the complete elliptic integration determined by $k$. Simple calculation and specifying $K$ from the expansion of $K$ in eq. (6) provide $k \approx 1.156 \times 10^{-}$ or $4K/\omega \approx (1 + 3.34 \times 10^{-15}) \times 2\pi/\omega$. This means that the main period for elliptic function solutions is a very slightly larger than that of the linear sine and cosine functions. As known the main period for Chandler wobble may be about 435 d, the diversity for the non-linear elliptic function solution is about $1.45 \times 10^{-12}$ d for rigid Earth case and $2.08 \times 10^{-12}$ d for real Earth deviating from the approximate period of 435 d in linearized case. Evidently the affection for the variation is caused by linearized solution of sine functions for the Euler's equations approximating the exact elliptic functions. But the consideration only takes into account the deviation of the elliptic function solution with respect to the sine function approximation without noticing another period in elliptic functions. The former solutions are constrained with trigonometric function so that different approaches give different approximations. But here in this study, we obtain exact solution in approach of elliptic function including couple periods.

There should, however, be an additional period in the exact solution of elliptic functions for polar motion of Euler's non-linear equations with common frequency $\omega$ in components. It can be obtained by elliptic function theory that

$$\omega = (H_1H_0)^{1/2}a_1 = (H_1H_0)^{1/2}a_2 = H_0a_3 \tag{6}$$

eq. (6) means that the main frequency $\omega$ is determined as $H_0a_3$. For $a_3$ may be very approximating the mean rotation velocity $\Omega$ of Earth, $\omega$ can be easily specified as the inherent frequency $\sigma_c$ of Chandler wobble as the square root of $(C - A)(C - B)/AB$ for the rigid and modified as $1/1.43$ times to about 0.84 cpy or of period 1.191 yr for real Earth. Here the elastic mantle and liquid core affect the frequency by the formula

$$\sigma_c = H_0\Omega\left(1 - \frac{k_2}{\kappa}\right)\frac{A}{A^m}$$

Where $k_2$ is the Love number of elastic Earth and $\kappa$ the classical secular Love number. $A$ and $A^m$ are moments of inertia for the whole Earth and the mantle. If the ocean coefficients and effects of viscoelastic deformation are considered, the modified factor may be changed a little to $1/1.43$.

Solutions for the first two components cn and sn are also coupled periodic functions, but for periods of more than 200 years the polar motion has been seen as one period elliptic trajectory. The additional period caused by triaxiality is easy to estimate from the equatorial dynamic ellipticity $H_1$ and $H_0$ similar to the main free wobble term determined by dynamic ellipticity $H_0$. When $H_1$ is taken into account, the additional free polar motion possesses frequency as the square root of $H_1H_0 = (B - A)(C - B)/AC$ for the rigid or $0.84^2/150$ or $1.191 \times 1.191 \times 150 = 14.6^2$ for the real. As the main free wobble of Chandler, the decadal frequency of the other free polar motion $\sigma_a$ is also modified by the same coefficient as effects of the mantle elasticity, liquid core boundary and ocean.

$$\sigma_a = \sqrt{H_0H_1}\,\Omega\left(1 - \frac{k_2}{\kappa}\right)\frac{A}{A^m} \tag{7}$$

So it is estimated that there exists a decadal free polar motion of period 14.6 yr or frequency *ca* 0.0685 cpy.

## 4 LINEARIZED APPROXIMATION

It is seen that the elliptic function approach for rotation of the triaxial Earth provides difficult non-evident function for inconvenient usage. Most of geodetists may not be familiar with the Jacobi elliptic functions. As mentioned above, in the Earth's rotation the parameter $k$ is very tiny, so that elliptic functions are approximated to the trigonometric functions. However, by applying trigonometric functions the other free polar motion may be rejected in the process. Can the additional free polar motion be solved from Euler rotation equations by trigonometric function approach? The answer is positive. Since Chandler wobble may be solved in a linearized process, it needs the theorem in theoretical mechanics for rotation of triaxial rigid body. It indicates that the rotation of the free rigid triaxial body about a fixed point has two stable elliptic motions about the maximum axis of inertia momentum as well as about the minimum. Here a concise verification is stated as follows.

In fact, the total momentum $H$ of the body remains constant in angular velocity form while the angular momentum also remains constant. There hold

$$H = \tfrac{1}{2}M(\omega_1^2/A + \omega_2^2/B + \omega_3^2/C)$$
$$\omega_1^2 + \omega_2^2 + \omega_3^2 = m^2 \tag{8}$$

Here $M$ is the mass of the body and $m$ as known a constant. Let $x_i = \omega_i/m$, $(i = 1, 2, 3)$ and $\lambda_1 = 1/A$, $\lambda_2 = 1/B$, $\lambda_3 = 1/C$, the first of eq. (8) changes to a quadratic polynomial of $x_i$ with $\lambda_i$ the respective eigenvalues of eigenvectors.

$$H = \frac{1}{2}m^2M(\lambda_1x_1^2 + \lambda_2x_2^2 + \lambda_3x_3^2) = \frac{1}{2}m^2M[(\lambda_1 - \lambda_3)x_1^2 + (\lambda_2 - \lambda_3)x_2^2 + \lambda_3] \tag{9}$$

Theory of quadratic polynomial shows that if, and only if, the determinant of the last formula of eq. (9) is positively definite, then it has stable elliptic solution, otherwise it has unstable saddle point solution. So it is evident for $C > A$ and $C > B$ that eq. (3) has a stable elliptic solution corresponding to the usual case of Chandler wobble. But eq. (9) has another form as

$$H = \frac{1}{2}m^2M\left[\lambda_1 + (\lambda_2 - \lambda_1)x_2^2 + (\lambda_3 - \lambda_1)x_3^2\right] \tag{10}$$

For the case of $A < B$ and $A < C$, the quadratic polynomial (10) also satisfies the positive definite condition. So there must be another stable elliptic solution for the triaxial Earth.

The verification confirms that there are two stable elliptic solutions about axes of the maximum and the minimum moments of inertia, while about the middle moment axis the solution is unstable. The process of the verification also implies that the solution for Euler rotation equations may be very complex so that one process of assumption of rotation about the maximum axis cannot finish the solution. If the rotation is assumed about the maximum moment axis, then the stable solution about the minimum moment axis might be neglected. In order to provide the total solution of Euler rotation equations for triaxial Earth, we had to ignore the rotation about the maximum moment axis to find the rotation about the minimum moment axis. It is just because of this viewpoint we discuss total rotation of a rigid body noted by $\mathbf{P}$ in frame $X$—$Y$—$Z$ formed by three-pole rotation components $\mathbf{P}_Z$, $\mathbf{P}_X$, $\mathbf{P}_Y$ with respect to moments on each axis with each representing rotation component about one axis in frames $Z_Z$—$X_Z$—$Y_Z$, $X_X$—$Y_X$—$Z_X$ and $Y_Y$—$Z_Y$—$X_Y$ respectively.

Previously, axial symmetric rotation is noticed so that the angular velocity components are set as $\omega_1 = m_1\,\Omega$, $\omega_2 = m_2\,\Omega$, $\omega_3 = (1 + m_3)\Omega$ with $m_1$, $m_2$, $1 + m_3$ as direction cosines of Earth's rotating inertia momentum axis with respect to the figure axis $C$ in frame $Z_Z$—$X_Z$—$Y_Z$. Here $m_1$, $m_2$, $m_3$ are dimensionless amounts of about $10^{-6}$. Since there is one in $\omega_3 = (1 + m_3)\Omega$ which is much more intense than the other two components, rotation variation component $m_3$ can be easily separated with the wobble components $m_1$, $m_2$. The rotation equations change into

$$\begin{aligned}
&\dot m_1 + \sigma_2(1 + m_3)m_2 = 0\\
&\dot m_2 - \sigma_1 m_1(1 + m_3) = 0\\
&\dot m_3 + \sigma_3 m_2 m_1 = 0
\end{aligned} \tag{11}$$

Ignoring second-order amounts and separating variables $m_1$, $m_2$ in $X$—$Y$ frame, eq. (11) is simplified as

$$\begin{aligned}
&\ddot m_1 + \sigma_1\sigma_2 m_1 = 0\\
&\ddot m_2 + \sigma_1\sigma_2 m_2 = 0\\
&\dot m_3 = 0
\end{aligned} \tag{12}$$

Here $\sigma_1 = \Omega(C - B)/A \approx \sigma_2 = \Omega(C - A)/B \approx \sqrt{\sigma_1\sigma_2} = \sigma_c$ and $\sigma_3 = \Omega\,(B - A)/C$ are frequency components. $\sigma_c$ is the average free wobble frequency for rigid Earth as Euler's frequency $365/304.7 = 1.1987$ cpy. By considering the effects of mantle elasticity, liquid core action and ocean excitation except the excitations out side of Earth's body, the inherent frequency for the free wobble may be reformed to decrease about 1/1.43 times. As $(B - A)/C = 2.1946 \times 10^{-5}$, $\sigma_3 = 2.1946 \times 10^{-5} = 1/45466.4\ \mathrm{day}^{-1} = 1/124.76\ \mathrm{yr}^{-1} = 0.00951\sigma_c$, *ca* $1/150$ of $\sigma_c$, so is often ignored.

If $\sigma_1$ is close enough to $\sigma_2$, then the free wobble has one frequency $\sigma_c$ and the trajectory of the pole should be approximately ellipse. If $\sigma_1$ is different enough to $\sigma_2$, then the pole position should wander according to a curve named as Lissajous' graph, seen in Marion (1970) for instance. By eq. (3), ignoring the difference between $\sigma_1$ and $\sigma_2$, rotation about axis $C$ is of average frequency as $\Omega$ and pole position of coordinates $(X, Y) = (m_1, m_2)$ circles in ellipse around the convention international origin (CIO). Therefore, the rotation about axis $C$ is described.

For Lissajous' trajectory, there are definitions of periodic and quasi-periodic motions. If the ratio of the two frequency components is a rational number, then the motion of Lissajous' trajectory is periodic in rigorous sense. If the ratio is an irrational number, then the motion is quasi-period with Lissajous' graph not closing for ever but it almost closes so that quasi-period is often described as almost period. It can be seen that in astronomy, many more rigorously periodic motions are in fact quasi-periodic as the periodicity is always broken by tiny or intense interactions. Quasi-periodic motion may not be easy to be detected by using common filtering methods, as the periodicity is partly hidden and partly visible. Generally, bandpass filtering and wavelet may work for quasi-periodic analysis, but here the difference for two periods to form Lissajous' trajectory is too intense so that the periodicity may not be evident.

In order to obtain the rotation about axis $A$, the rotation pole may be set near axis $A$ in frame $X_X$—$Y_X$—$Z_X$ and the rotation angular velocity components should be set as $\omega_1 = (i + w_1)\Omega$, $\omega_2 = w_2\ \Omega$, $\omega_3 = w_3\ \Omega$ with $w_1$, $w_2$, $w_3$ dimensionless amounts according to eq. (10). Notice that the rotation about axis $C$ has been obtained in the previous process separated from the total rotation. This time $\omega_1$ is much more intense in mode than the other two components. Then Euler's rotation equations becomes

$$\begin{aligned} &\dot{w}_1 = 0 \\ &\ddot{w}_2 + \sigma_a^2 w_2 = 0 \\ &\ddot{w}_3 + \sigma_a^2 w_3 = 0 \end{aligned} \tag{13}$$

Here rotation velocity variation about axis $A$ is separated from motion of axis $A$ determined by the second and the third equation of (13). Thus the three direction cosines in eq. (3) are alternately symmetry but the principal inertia momenta $A$, $B$, $C$ may dominate the rotation behaviour about a certain axis like the rotation about axis $C$ discussed previously. The second and the third equation of (13) are clearly oscillation with Lissajous' graph. However, $\sigma_3$ is estimated as $1/124.76$ cpy so that rotation about axis $A$ may not be comprehended. So long as Earth is considered as a potato-shape, then it rotates about axis $A$ possibly by angular velocity 0. Similar to eqs (12) and (13) it can be solved easily.

$$\begin{aligned} &w_1 = \text{const} \\ &w_2 = \beta \cos(\sigma_a t + \varphi) \\ &w_3 = \delta \sin(\sigma_a t + \varphi) \end{aligned} \tag{14}$$

Here $\beta$ and $\delta$ are amplitudes of elliptic wobble orbit, and $\varphi$ the initial phase to be specified. The constant for $w_1$ also may be determined as approximate to 0 so that the rotation angular velocity may have $\omega_1 = i\Omega$. This illustrates that axis $A$ itself may not rotate but the rotation pole is the pole cross with axis $A$ in direction $-90°$. This must be axis $C$ and is identical to the fact of Earth's rotation.

Solution for eq. (13) can be obtained as that of for eq. (12) and the inherent frequency can be specified as image stated in the previous section $\sigma_a = \sqrt{\sigma_1 \sigma_3} = \sqrt{0.832 \times 124.76} = 1/10.2$ cpy or there is an inherent wobble of period 10.2 yr for axis $A$ of rigid case. However, the frequency components may have a period of 0.832 yr latitudinal along axis $C$ and a period of *ca* 124.76 yr longitudinal along axis $B$. In other words, since $\sigma_3$ is about 1/150 of $\sigma_2$, the frequency field is a very narrow rectangular square; almost a thin line. If the parameter $\sigma_3 = (B - A)/C$ were zero or regarded as zero, then there would be no wobble for axis $A$. But now $\sigma_3$ is not zero so that there is a wobble of period about 124.76 yr for axis $A$ to tilt towards a longitudinal along the defined equator. As to the inherent period, $\sigma_3$ and $\sigma_2$ are different enough so that the Lissajous' trajectory for wobble of axis $A$ possesses no such evident eigenvalue but unfolds period band ranging from 1.19 yr to 180 yr with resonance on 14.6 yr called as quasi-period. As known, the effects of mantle elasticity, liquid core action and ocean excitation contribute the free wobble of rigid Earth as 304.5 d to 435.2 d or lengthen the period of more than about 1.43 times according to eq. (7). So the inherent quasi-period for wobble of axis $A$ may be specified as about 14.6 yr.

For the rotation of axis $B$ about axis $Y$, the solution is not similar to that of about axis $A$. If the similar process is taken as assuming $\omega_1 = v_1\ \Omega$, $\omega_2 = (i + v_2)\Omega$, $\omega_3 = v_3\ \Omega$ with $v_1$, $v_2$, $v_3$ in eq. (3), then the equations may change to

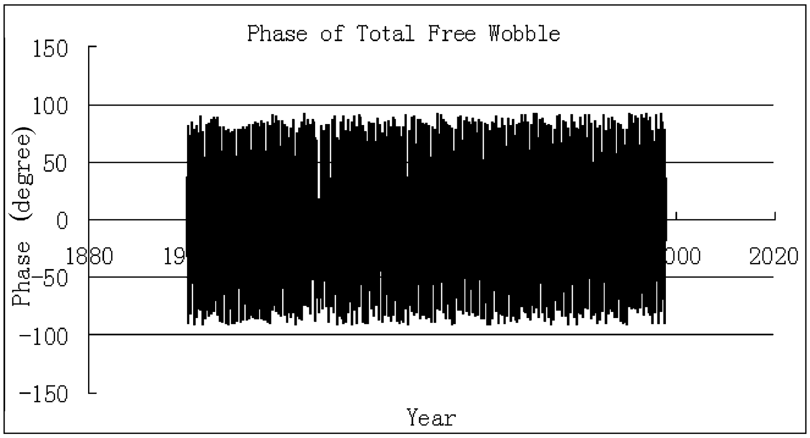
$$\begin{aligned} &\ddot{v}_1 - \sigma_b^2 v_1 = 0 \\ &\dot{v}_2 = 0 \\ &\ddot{v}_3 - \sigma_b^2 v_3 = 0 \end{aligned} \tag{15}$$

As the first and the third equations have minus signs for the variables, the solution may not provide the form similar to that about axis $A$, i.e. without oscillation about axis $B$. The solution has the form

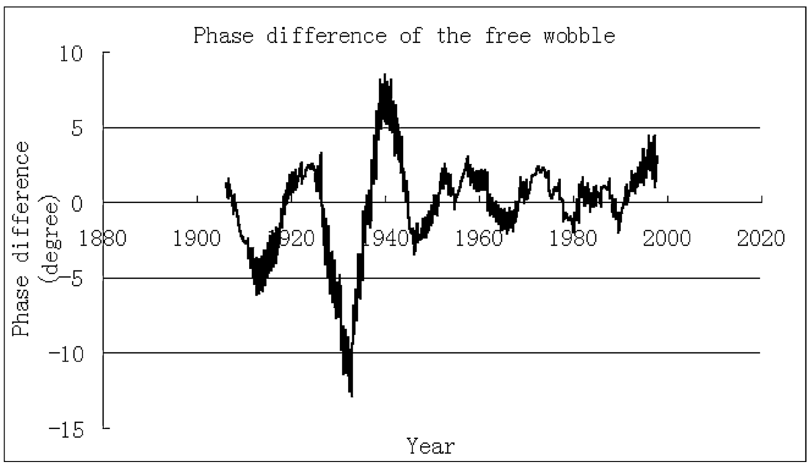
$$\begin{aligned} &v_2 = \text{const} \\ &v_1 = R_1 e^{\sigma_b t} + R_2 e^{-\sigma_b t} \\ &v_3 = R_1' e^{\sigma_b t} + R_2' e^{-\sigma_b t} \end{aligned} \tag{16}$$

Here $R_1$, $R_2$, $R_1'$, $R_2'$ are constants determined by initial conditions and $\sigma_b$ is also determined by geometric mean of $\sigma_1$ and $\sigma_3$. This solution may be instable and the instable solution may not appear to be visible. So there is neither longitudinal nor latitudinal wobble components about axis $Y$ as decadal free wobble about axis $A$.

Therefore, the total rotation of triaxially asymmetric Earth is formed by rotation of axis $C$ with rotation components of axis $A$ by considering excitations on each inertia poles. Or the total polar motion in frame $X$—$Y$—$Z$ is formed by wobbles in frames $Z_Z$—$X_Z$—$Y_Z$, $X_X$—$Y_X$—$Z_X$. The wobble in frame $X_X$—$Y_X$—$Z_X$ of period 14.6 yr may be described as additional free wobble. The linear approach is evident basically approximate to the elliptic function approach for exact solution of rotation of triaxial case. Fig. 2 shows observed amplitude additional wobbles of quasi-period about 14.6 yr in polar motion obtained from C01 data series. Annual wobble is caused by annual forcing angular momentum compounded by atmospheric variation, ocean flow and other factors so that energy cannot reserve there in long period. It can be seen that there is a term of free wobbles with period 14.6 yr in polar motion data series.




(a)




(b)

**Figure 2.** The phase time series in (a) computed from IERS C01 data series of the so-called Chandler wobble position coordinates. In fact, the series represent the total free wobbles including mainly Chandler wobble and the decadal free wobble introduced in this study. The result in (b) illustrates phase difference relating to the referred phase of Chandler wobble with roughly smooth average filtering. The phase difference shows variations with period approximately 11 to 19 yr and amplitude *ca* 3° in the last half-century.

## 5 FREQUENCY MODULATION MODEL

As seen in the Section 3, the elliptic function solution for triaxial Earth includes couple periods of a real period $4K/\omega$ and an image period $4K'i/\omega$. One may misunderstand the image period as a strange result of attenuation occurring in the homogeneous solution. In fact, this is another kind of application of complex number in the science of polar motion. Those who study the science of polar motion know that there have been three types of different applications of complex number in studying polar motion. Moreover, here we apply the fourth type.

Firstly, as the equatorial inertia momenta are set with $A = B$ in the axially symmetric case, the variation of LOD and pole displacement are separated into two variables. One is the so-called LOD and the other is described as pole position. The pole position may be expressed as a complex number with two coordinates ($X$, $Y$). For convenience, the complex number may be written as $X + iY$. This is the first type to apply complex number in polar motion.

Secondly, in order to consider Earth visco-elastic deformation affection in polar motion, the frequency of the free wobble may be expressed as a complex frequency setting attenuation index as $\lambda$. Thus the complex frequency may be written as the sum of a real frequency component and an image frequency component $\sigma = \sigma_c + i\lambda$, where $\lambda = \sigma_c/(2Q)$ with $Q$ the quality factor. Fortunately, $Q^{-1}$ can be empirically deemed as energy loss by way of oscillation. Therefore, the expression is very much adapted to the physics mechanism of polar motion.

Thirdly, by application of the complex number in frequency to describe the physics background of energy loss in visco-elastic deformation, the important physics amount Love number needs to be seen as a complex number. It can be seen in Zhu (1991) that if the Love number is treated as a complex number, then the visco-elastic effect can be quantified and the period of polar motion can be specified more accurately.

Evidently, each time the complex number is applied in studying polar motion, the physics meaning is different to the other. The complex frequency may be likely to yield the complex Love number. However, the two cases of applications are not the same at all.

Now in this study, the elliptic function solution of the Euler's rotation equations includes a complex period. The real part represents transversal wave of the latitudinal variations, while the image part reveals longitudinal wave yielded by radial moment. The component of radial moment caused by equatorial dynamic ellipticity $H_1$. So the image part of the period in elliptic function solution for polar motion does not show attenuation mechanism as image part of frequency. This is nothing but a different kind of application of complex number in the science of polar motion.

In fact, there are transverse waves and radial waves for physical motions of motive body. Especially in polar motion, the transversal moments cause latitudinal oscillations while the radial moments are considered formerly in LOD. However, if the rotating axis deviates from the figure axis, the radial moments will not cause variations at all. This assertion is not suspected except where the magnitude of the longitudinal variation is large enough. By application of the former methodology for solving Euler's rotation equations, the longitudinal moment and the longitudinal displacement of the pole were often ignored or calculated as tiny. Only by the elliptic function approach, does the longitudinal variation emerge as available. It should be pointed out that the longitudinal polar motion exists all the time. However, the former solution methodology cannot unveil the longitudinal polar motions.

As known, the variations of polar motions are expressed with trigonometric functions. Trigonometric functions can only describe one oscillation at a time; either transverse wave or radial wave. If there were transverse and radial waves occurring in one motion, then trigonometric functions would not work to express both the two waves. When the main free polar motion is solved from Euler's rotation equations, a radial wave caused by radial moment cannot be seen generally, except where elliptic functions are applied. As the elliptic functions here in polar motion are approximated sufficiently to the trigonometric function solution, the exact solution of the Euler rotation equations in the triaxial Earth case is rejected so that the longitudinal variations of tiny polar motion have never been obtained. In this sense, the main difference of the elliptic function and the trigonometric, is that the former possesses a couple periods especially the longitudinal polar motion while the latter expresses only the transverse wave or latitudinal polar motion.

Earth is a global body. So long as the rotating pole deviates from its figure axis, radial moments will cause polar motion as well as latitudinal. Thus while the latitudinal moments and latitudinal variations are considered, the longitudinal ones must be taken into account at the same time. The success of the elliptic function in polar motion may be emphasized as the consideration of the longitudinal oscillations with the main free polar motion. This causes the decadal free polar motion term to be discovered after a hundred years of Chandler's result for main free wobble and more than two centuries after Euler's prediction for the free motion of the rigid Earth.

It should be noticed that in fact the radial moment of about 180 yr period is only a component. The radial moment component causes an oscillation synthetically together with the latitudinal component different to that of the main free polar motion synthesized by two sufficiently approximate latitudinal components. The former two synthetic components possess sufficiently different frequencies of periods 1.191 yr and 180 yr. The sufficiently different frequencies synthesize the so-called Lissajous's graph of a trajectory other than an ellipse. In this case, the synthesized period may not be as rigorous than that appearing as quasi-periodic oscillation. The longitudinal and latitudinal torque components synthesize a different latitudinal oscillation such that it mixes together with the main free polar motion and is observed as total free polar motion. In fact, the data series in IERS (International Earth Rotation Service) C01 that is often seen as Chandler wobble series is nothing but the total free polar motion mainly combined by the Chandler term and the decadal free wobble term introduced in this study.

In the frequency field, the complex period means that the additional oscillation superposes radial wave onto the main frequency seen as transverse wave. The additional polar motion causes frequency modulation for the main polar motion or Chandler wobble. By observation, Chandler frequency is seen with great variation. Here the frequency modulation is verified theoretically and the modulation period of 14.6 yr is specified. It can be seen by filtering of phase time series of free wobbles in Fig. 2 that there exists about 14.6 yr period modulation in phase modulation sense.

The inherent frequency of a period *ca* 14.6 yr interacted onto the free wobble about axis $C$ modulates the inherent frequency of period 1.191 yr. The interaction of the decadal free wobble onto the free wobble makes the Chandler wobble more complex. If free polar motion is expressed by pole position of coordinates $(X, Y)$, then $X$ and $Y$ should satisfy the following parameter time-dependence equations

$$\dot{X} + \lambda X + \sigma Y = \sigma \Psi_2 \quad \dot{Y} + \lambda Y - \sigma X = -\sigma \Psi_1 \tag{17}$$

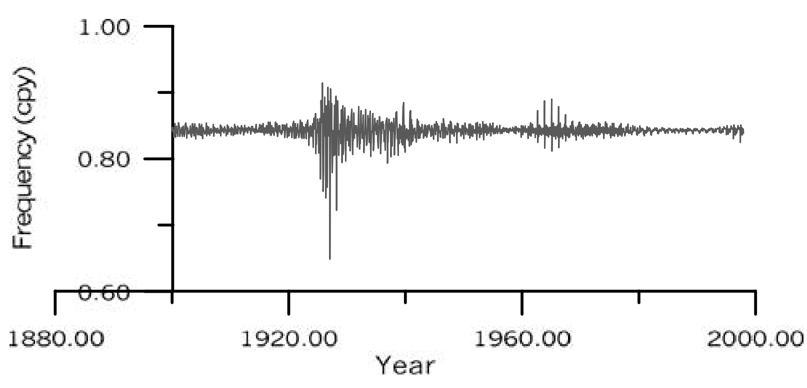



**Figure 3.** CW instantaneous frequency (1900–1997) un-filtered just transformed from the phase determined by C01 data series showing with modulation.

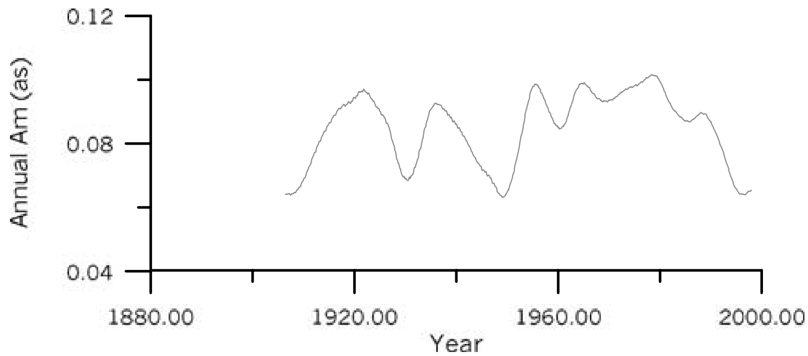



**Figure 4.** Annual polar motion in 6.5 yr smoothing shown oscillating in an *ca* 14.6 yr period.

$$\lambda(t) = \overline{\lambda} + \lambda_0 \cos(\omega_0 t) \quad \sigma(t) = \sigma_c + \Delta \cos(\omega_0 t) \tag{18}$$

Here, $\omega_0$ is the modulation frequency of range $\omega_0 = \sigma_c \pm 1/14.6$ cpy and $\Delta$ as the modulation level. $\sigma_c$ is the inherent frequency of free wobble about axis $Z$ and $\lambda = \sigma_c/(2Q)$ the attenuation index of mantle viscoelastic damping respective to quality factor $Q$ with $\overline{\lambda}$ the average attenuation index and $\lambda_0$ the perturbation level. Evidently, this model is approximate.

As free wobble, Chandler wobble should possess an invariant eigenfrequency of about 0.84 cpy. The main perturbation may only form the modulation of the decadal free wobble. The decadal free wobble has inherent frequency about $1/14.6 = 0.0685$ cpy. As this frequency superposes onto Chandler wobble, the total wobble frequency varies on a level of 0.84 cpy with a period varying by $\pm 25$ days. The total wobble period may appear as range from 408 to 458 d. Fig. 3 shows the instantaneous frequency computed according to IERS C01 data without any filtering. The real frequency of the total wobble in range from 0.815 to 0.895 cpy is identical to the discussion above.

It is necessary to point out that axis $A$, like studies in variations of polar motion and length-of-day, may be perturbed by excitations expressed in eq. (17) from angular momentum variations of atmosphere, ocean current and other excitation sources. These are constrained by inhomogeneous equations not discussed here.

In addition to the frequency modulation, the additional free wobble contributes amplitude excitation on Chandler wobble. Here the amplitude of the decadal wobble may be estimated according to method of proportion by the theorem of energy conservation as momentum equal to potential and the Chandler wobble average amplitude observed as the given inherent amplitude. The inertia momentum $H$ of a moving planet is proportional to the square of its angular velocity as $H = \frac{1}{2}m\omega^2$ and the potential is proportional to its displacement of pole deviating as $U = mgh$. Thus by equality of the two, wobble displacement $h$ is proportional to $\omega^2$ the square of its angular frequency or velocity. Suppose that the free mode of Chandler wobble has inherent frequency $\omega_c = 0.8425$ cpy and stable amplitude seen as speed $h_c = 135$ mas cycle$^{-1}$, then the additional decadal free wobble of period 14.6 yr or frequency $\omega_a = 0.0685$ cpy may have a speed of about $h_a = 0.892$ mas yr$^{-1}$ by relation of $h_a : h_c = \omega_a^2 : \omega_c^2$. The speed $h_a$ is too small to be taken into account but prolongs a period of 14.6 years so that the amplitude is estimated as 13 mas cycle$^{-1}$. Thus by the proportional method and the observed average amplitude of Chandler wobble as given, the additional decadal amplitude may be obtained as about 13 mas. This force can only be considered as excitation in the right hand sides of eq. (17).

The longitudinal wave modulation is a phenomenon of phase modulation. In the period of 14.6 yr, the main periodic motion may vary the phase slightly by transversely waving. To estimate the phase modulation magnitude, it may be seen that the 14.6 yr period possesses frequency of 0.0685 cpy. The frequency is distributed by half period of 14.6 yr. Then, there is variation of 0.00938 cpy/cycle in the period of 360°. Thus the peak-to-peak variation of the average phase modulation is about 3.38°. In other words, the observed free wobble should have average phase modulation of 3.38° with a 14.6 yr period, seen in Fig. 2. Fig. 4 shows observed decadal periodic amplitude modulation in polar motion from C01 data series. The amplitude is identical to the analysis above.

## 6 SUBHARMONIC RESONANCE PRINCIPLE

Considering the instability of the sway frequency, the modulation makes the sway frequency and attenuation index possess a time dependent feature as in eq. (17). Thus the damping and elastic parameters in eq. (17) can be reformed as a time-dependent case in form of eq. (17). By process of solving the time-dependent equation, the non-linear terms in the left hand side of eq. (17) may be translated to the right hand side as forcing or excitation inhomogeneous terms. We provide the variation for coordinate $X$ of the free polar motions including Chandler wobble and the decadal free polar motion.

By condition eq. (18), eq. (17) may be changed to a time-dependent equation for $X$

$$X'' + 2\lambda(t)X' + \sigma^2(t)X = A_X\sigma_c \cos(\varpi t + \beta) \tag{19}$$

Here $A_X$, $\varpi$ and $\beta$ are amplitude, frequency and initial phase of excitation from the decadal free wobble, respectively. To discuss the subharmonic and super-harmonic solutions of (19), it can be written as a functional form as

$$x'' + g(x) = -h(x, x')x' + F(t) \tag{20}$$

Here, $F(t)$ represents the excitation function in eq. (19) that satisfies the periodic condition $F(t+P) = F(t)$ with: $P$ the minimum period; $g(x)$ the non-linear elastic force; and, $-h(x, x')\,x'$ the non-linear positive damping force. It can be verified that, under non-linear interaction, the periodic solutions of eq. (20) may be harmonic and subharmonic.

Suppose that eq. (20) has a solution of period $P_1$ under excitation function $F(t)$, then it must satisfy periodic conditions

$$x(t+P_1) = x(t) \quad x'(t+P_1) = x'(t)$$

Evidently periodic solutions $x(t)$ and $x(t+nP_1)$ ($n$ the integer) satisfy eq. (20),

$$x''(t) + g(x(t)) = -h(x(t), x')x'(t) + F(t)$$

$$x''(t+nP_1) + g(x(t+nP_1)) = -h(x(t+nP_1), x'(t+nP_1))x'(t+nP_1) + F(t+nP_1)$$

Consider the above equations, substituting into eq. (21), it becomes

$$x''(t) + g(x(t)) = -h(x(t), x'(t))x'(t) + F(t+nP_1) \tag{21}$$

Eq. (21) subtracts eq. (20) gives

$$F(t) = F(t+nP_1) \tag{22}$$

Since $F(t)$ is periodic function of period $P$, and it gives by eq. (21) that $F(t)$ is also periodic function of period $P_1$, the period $P_1$ must satisfy the relation of $m$ integer times to period $P$.

$$mP = nP_1 \quad \text{or} \quad P_1 = \frac{m}{n}P \tag{23}$$

**Case 1:** $m/n = 1$. Or if there is an excitation of period as that of the inherent frequency, then the solution of the equation would be harmonic resonance and the energy would reach high level without bound in case of ignoring the damping.

**Case 2:** $m/n = \boldsymbol{Z}^{+}$ is an integer or with $m$, $n$ as prime numbers to each other. This is the case of subharmonic oscillation with period of $1/Z$ multiple annual. The sub-annual polar motion is a subharmonic solution. Accordingly, there may be $1/3$-annual and $1/5$-annual terms, etc. Those are determined by the noearity of polar motion and solid tides. Some of the publishers classify the case with $m$, $n$ as prime numbers to each other as super-subharmonic resonance that is a special subharmonic resonance.

**Case 3:** $n/m = Z^{+}$ is an integer. In this case the solution of the system may be super-harmonic oscillations. Considering that the forcing period is an integer multiple of the inherent period, the super-harmonic resonance may take place and finite excitation may cause great resonant solution.

Therefore, it is verified that the periodic solutions of eq. (17) may be harmonic, subharmonic and super-harmonic oscillations. Here the forcing period has *ca.* $73/6$ times over the inherent period so that subharmonic resonance occurs.

## 7 SUBHARMONIC RESONANT SOLUTION

For introducing the explicit function solution to approximate the elliptic function solution of triaxially asymmetric Earth, non-linear oscillation of subharmonic resonance should be considered. Non-linear oscillation theory has been studied including subharmonic resonance solutions, seen in Nafeh & Mook (1979). The interaction of the two free wobbles may cause composite motions other than that simply superposed and the equation of frequency modulation can be solved in subharmonic case. The forcing period should be considered as 14.6 yr, and the inherent frequency as 0.84 cpy or a period of 1.191 yr with the ratio approximate to the simple ratio $73/6$.

Consider the position coordinate equation of $X$ in an inherent frequency $\sigma_c$ the frequency of Chandler wobble. First, eq. (17) reforms to

$$\ddot{X} + \sigma_c^2 X = \mu f(X, \dot{X}) + F\cos\varpi t \tag{24}$$

Where $\mu$ is a small coefficient introduced, $f$ is a non-linear function of $X$ and $\dot{X}$, $F = A_X\,\sigma_c$. $A_X$ is the excitation magnitude of $X$ coordinates and $\varpi$ the decadal polar motion as the forced frequency. Take transformation $\varpi\, t = \tau + \varphi$, such that the initial condition $X_i'\,(0) = 0$ is satisfied, here $X'$ represents differential with respect to $\tau$. Then eq. (21) changes to

$$\varpi^2 X'' + \sigma_c^2 X = \mu f(X, \varpi X') + F\cos(\tau + \varphi) \tag{25}$$

Let solution be $X(\tau)$, then $X(\tau)$ and $\varphi$ can be expanded into power series with respect to $\mu$,

$$X(\tau) = X_0(\tau) + \mu X_1(\tau) + \mu^2 X_2(\tau) + \ldots\ldots \tag{26}$$

$$\varphi = \varphi_0 + \mu\varphi_1 + \mu^2\varphi_2 + \ldots\ldots \tag{27}$$

Here $X_i(\tau)$ are all periodic function of period $2\pi$, and satisfy the initial conditions

$$X_i(0) = 0 (i = 0, 1, 2, \ldots\ldots) \tag{28}$$

Substituting eqs (26) and (27) into (28) and expanding $f(X_0, \varpi\, X_0')$ and $\cos(\tau + \varphi)$ into power series with respect to $\mu$, comparing the same order power term of $\mu$ by two sides, it can be obtained that

$$\begin{aligned} &\varpi^2 X_0'' + \sigma_c^2 X_0 = F\cos(\tau + \varphi) \\ &\varpi^2 X_1'' + \sigma_c^2 X_1 = f(X_0, \varpi X_0') - F\varphi_1 \sin(\tau + \varphi_0) \end{aligned} \tag{29}$$

Here, $f = x + x(\Delta/2\sigma_c) - x'(\overline{\lambda} + \lambda_0)/(\sigma_c\Delta)$, $\mu = -2\sigma_c\Delta$.

In the case $\varpi \approx \sigma_c/n$, the solution of the first equation of eq. (29) can be written as

$$X_0 = M_0 \cos n\tau + N_0 \sin n\tau + \frac{F}{\varpi^2(n^2-1)}\cos(\tau + \varphi_0) \tag{30}$$

Here the third term of the right hand side of eq. (30) is the particular solution of eq. (26). Substituting eq. (30) into the second equation of (29), there holds

$$\begin{aligned} X_1'' + n^2 X_1 = {} & \frac{1}{\varpi^2} f(M_0 \cos n\tau + N_0 \sin n\tau + \Phi(\tau, \varphi_0), \\ & -\varpi M_0 n \sin n\tau + \varpi N_0 n \cos n\tau + \varpi \Phi'(\tau, \varphi_0)) - \frac{F}{\varpi^2}\varphi_1 \sin(\tau + \varphi_0) \end{aligned} \tag{31}$$

Here $\Phi\,(\tau, \varphi_0) = [F/\varpi^2(n^2-1)]\cos\,(\tau + \varphi_0)$, $\Phi'(\tau, \varphi_0) = [-F/\varpi^2(n^2-1)]\sin\,(\tau + \varphi_0)$. For getting periodic solution $X_1\,(\tau)$, the coefficients of term $\sin(n\tau)$ and $\cos(n\tau)$ should be set as zero. Then, it exists

$$\begin{aligned} P(M_0, N_0) = \int_0^{2\pi} & f(M_0 \cos n\tau + N_0 \sin n\tau + \Phi(\tau, \varphi_0), \\ & -\varpi M_0 n \sin n\tau + \varpi N_0 n \cos n\tau + \varpi \Phi'(\tau, \varphi_0)) \sin n\tau d\tau = 0 \end{aligned} \tag{32}$$

$$\begin{aligned} Q(M_0, N_0) = \int_0^{2\pi} & f(M_0 \cos n\tau + N_0 \sin n\tau + \Phi(\tau, \varphi_0), \\ & -\varpi M_0 n \sin n\tau + \varpi N_0 n \cos n\tau + \varpi \Phi'(\tau, \varphi_0)) \cos n\tau d\tau = 0 \end{aligned} \tag{33}$$

By non-evident function theorem, if function Jacobian determinant holds

$$\left|\frac{\partial(P, Q)}{\partial(M_0, N_0)}\right| \neq 0 \tag{34}$$

Then $M_0 = M_0\,(\varphi_0)$, $N_0 = N_0\,(\varphi_0)$ can be the zero point of eqs (32) and (33). By initial condition $X_i\,(0) = 0$, it holds another equation of relation

$$\sin\varphi_0 = \frac{\varpi^2(n^2-1)}{F} n N_0 \tag{35}$$

So, by eqs (32)~(35), $M_0$, $N_0$, $\varphi_0$ can be determined. Substituting into eq. (30), solution $x_0\,(\tau)$ would be the derived as a result of the eq. (24) under time-dependent parameter condition (18).

Thus eq. (32) is determined by parameters $M_0$, $N_0$, $\varphi_0$. The similar process provides the solution of $x_1$ as

$$x_1(\tau) = M_1 \cos n\tau + N_1 \sin n\tau + \Phi_1(\tau, \varphi_1) \tag{36}$$

Here $\Phi_1\,(\tau, \varphi_1)$ is the particular solution of the second equation of eq. (29). Further, parameters $M_1$, $N_1$, $\varphi_1$ can be specified and substituting into eq. (36), the second order approximate solution would provide precision of the solution $x_0\,(\tau)$. As the solution of $x_0\,(\tau)$ has enough precision in the study of polar motion excited by the 14.6 yr period additional wobble, the second order approximate solution may be derived no more here.

Solving eqs (27) and (28), the integration gives the following equations with parameter given in eq. (24) letting $\eta = (\overline{\lambda} + \lambda_0)/(\sigma_c\Delta)$,

$$\begin{aligned} & N_0\left(1 + \frac{\Delta}{2\sigma_c}\right)\pi + \frac{4F}{\varpi^2(n^2-1)^2}\left(1 + \frac{\Delta}{2\sigma_c}\right)\left[\frac{1}{n+1}\cos(n\pi + \varphi_0) - \frac{1}{n-1}\cos(n\pi - \varphi_0) + \frac{2}{n^2-1}\sin\varphi_0\right] \\ & -\eta\varpi M_0 n\pi - \frac{F\eta\varpi n}{\varpi^2(n^2-1)^2}\left[\frac{-1}{n+1}\sin(n\pi + \varphi_0) + \frac{1}{n-1}\sin(n\pi - \varphi_0) - \frac{2}{n^2-1}\sin\varphi_0\right] = 0 \end{aligned} \tag{37}$$

$$\begin{aligned} & M_0\left(1 + \frac{\Delta}{2\sigma_c}\right)\pi + \frac{Fn}{\varpi^2(n^2-1)^2}\left(1 + \frac{\Delta}{2\sigma_c}\right)\left[\frac{-1}{n+1}\sin(n\pi + \varphi_0) + \frac{1}{n-1}\sin(n\pi - \varphi_0) - \frac{2}{n^2-1}\sin\varphi_0\right] \\ & -\eta\varpi N_0 n\pi + \tfrac{4F\eta\varpi}{\varpi^2(n^2-1)^2}\left[\tfrac{1}{n+1}\cos(n\pi + \varphi_0) - \tfrac{1}{n-1}\cos(n\pi - \varphi_0) + \tfrac{2}{n^2-1}\sin\varphi_0\right] = 0 \end{aligned} \tag{38}$$

Noticing initial condition $X_i\,(0) = 0$, in condition $\varpi\ t = \tau + \varphi$, it holds $\varphi = 0$, or $\varphi_0 = 0$. So, the term of $\sin\ \varphi_0$ would be 0 and $\cos\ \varphi_0 = 1$. Eqs (37) and (38) are simplified as

$$N_0\left(1 + \frac{\Delta}{2\sigma_c}\right)\pi + \frac{2F}{\varpi^2(n^2-1)^2}\left(1 + \frac{\Delta}{2\sigma_c}\right)(1 - \cos n\pi) - \eta\varpi M_0 n\pi - \frac{2F\eta\varpi}{\varpi^2(n^2-1)^2}\sin n\pi = 0 \tag{39}$$

$$M_0\left(1 + \frac{\Delta}{2\sigma_c}\right)\pi + \left(1 + \frac{\Delta}{2\sigma_c}\right)\frac{2F}{\varpi^2(n^2-1)^2}(-\sin n\pi) + \eta\varpi N_0 n\pi + \frac{2F\eta\varpi}{\varpi^2(n^2-1)^2}(1 - \cos n\pi) = 0 \tag{40}$$

Therefore, $M_0$, $N_0$, $\varphi_0$ can be obtained and determined by eq. (35).

$$M_0 = \frac{2F[(1 + \Delta/2\sigma_c)^2 - \eta\varpi]}{n\pi\varpi^2(n^2-1)^2(1 + \Delta/2\sigma_c)}\,\frac{1 - \sin n\pi - \cos n\pi}{\pi(1 + \Delta/2\sigma_c)^2 - (\eta\varpi n)^2} \tag{41}$$

$$N_0 = \frac{4F}{\pi\varpi^2(n^2-1)^2(1+\Delta/2\sigma_c)}\left[\frac{\eta\varpi n\pi((1+\Delta/2\sigma_c)^2-\eta\varpi)(1-\sin n\pi-\cos n\pi)}{\pi(1+\Delta/2\sigma_c)^2-(\eta\varpi n)^2} + \eta\varpi\sin n\pi-(1+\Delta/2\sigma_c)(1-\cos n\pi)\right] \tag{42}$$

The amplitude of $x_0$ can be determined by $\sqrt{M_0^2+N_0^2}$. The similar process can obtain the solution of $y_0$. But, since the orbit of elliptic Chandler wobble has very tiny difference between the two coordinates, the last amplitude of the motion may approximate as that of $x_0$. Furthermore, $x_1$ and $M_1$, $N_1$, $\varphi_1$ may be calculated by the similar process but more composite. Here, it had to be omitted and the approximation may be high enough, because the consequence can provide the higher precision for the solution. Phase $\varphi_0$ is determined from eq. (37), but the phase is often given by tan $\varphi_0$.

$$\tan\varphi_0 = \varpi^2(n^2-1)nN_0/\sqrt{F^2-\varpi^4(n^2-1)^2n^2N_0^2} \tag{43}$$

The process for deduction subharmonic resonance solution is somewhat composite and the result is not very concise. But for more accurate solution of non-linear oscillation, we had to treat the calculation with more patience. The solution here provides modulation for the amplitude of Chandler wobble. At last, to determine the subharmonic resonance amplitude of the 14.6 yr period additional wobble superposed onto the free wobble of the rotating pole exciting the polar motion, parameters are given as $\varpi = 1/14.6$ cpy, $\sigma_c = 1/1.191 = 0.84$ cpy, $n = 73/6$, $\Delta = 0.025$, $\overline{\lambda} = \pi/70$, $\lambda_0 = \pi/100$, $F = A_X\sigma_c$ and $A_X$ assumed to be the maximum absolute position of $X$ coordinates in excitation function of a 14.6 yr period excitation, say *ca* 13 mas. Mean while, polar motion coordinate increment $Y$ excited by a 14.6 yr period excitation would be also the same as represented in eq. (32) as coordinate increment $X$ deduced in this section.

Hence, $M_0 = 0.245A_X$, $N_0 = -8.28A_X$. So, $A_{MX} = \sqrt{M_0^2+N_0^2}A_X = 8.3A_X$. Similarly, $A_r$ has almost the same amplitude as $A_X$. Especially, we have modulation amplitude $A_S = 108$ mas. That is to say, if there is the maximum absolute position of $X$ coordinates in excitation function of a 14.6 yr period excitation, then the polar motion of super-harmonic resonance in beating of Chandler would be 108 mas at the maximum. Considering that the mean amplitude of Chandler wobble is about 132.5 mas, the peak amplitude of modulation would be 240 mas. The minimum amplitude of the free wobble with superposed additional wobbles excitation may be estimated by substracting 108 mas from 132.5 mas equal to 24 mas. That is almost identical to the observed Chandler amplitude seen in Fig. 5. Fig. 6 shows the observed amplitude of free polar motion compared with the amplitude obtained from frequency modulation with annual excitation superseded by annual wobble including the decadal as excitation. Therefore, the amplitude modulation of Chandler wobble may be aware of that in subharmonic

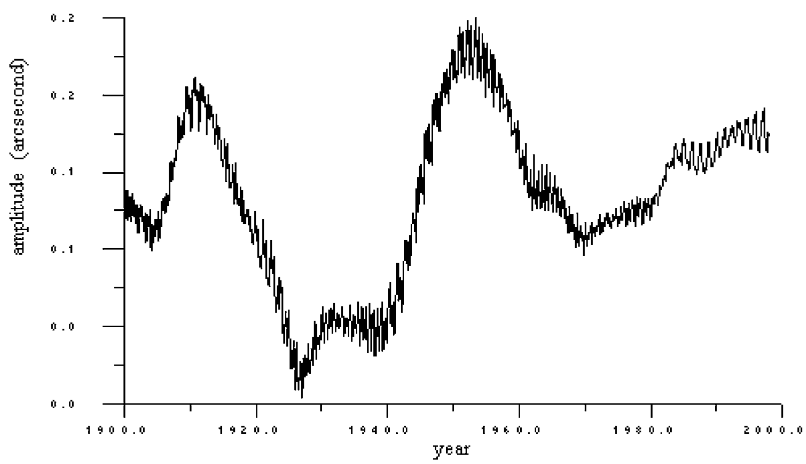



**Figure 5.** Amplitude of observed Chandler wobble with long term amplitude modulation.

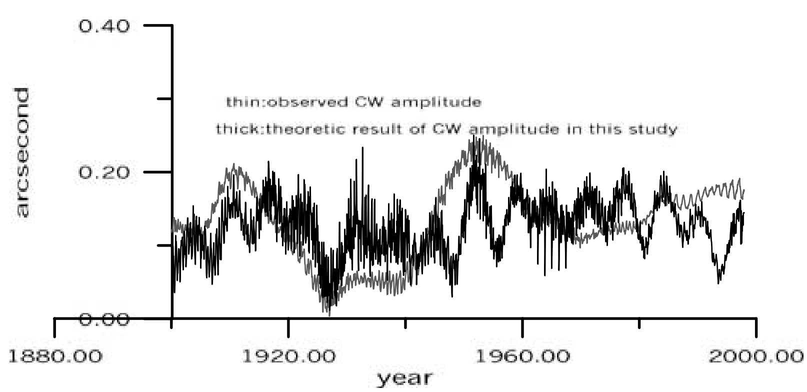



**Figure 6.** Observed amplitude of Chadler wobble with long period amplitude modulation (thin) and series calculated by parameter resonance (thick).

resonance. The total amplitude may reach the observed height seen in the theoretical analysis process. The mechanism of amplitude modulation in Chandler wobble may be explained by non-linear resonant actions.

## 8 CONCLUSION

The rotation equations for triaxial rigid body introduced by Euler in 18th century were primarily non-linear. Since the non-linear equations were not easy to solve in the former era, the equations were linearized to obtain the main period solution. Thus, Euler provided 305 days period free sway. Later, Liouville modified the rigid body rotation model into non-rigid biaxial body case in linear form. One should be suspicious of whether the non-linear model of Earth's rotation should be solved by linearization without considering non-linear affection.

The triaxial Earth rotation should be of two stable elliptic solutions by approach of elliptic functions as well as linearized approach. In fact, the observed polar motion is the compound of the free wobbles about the rotating pole and additional free wobble about the minimum moment axis. The theoretical analysis provides a quasi-periodic wobble of *ca* 14.6 yr with amplitude about *ca* 13 mas. Fortunately, filter result of phase time series provided in this study supports the theoretical result with evident long-period motion making the free wobble modulated. The average moving filter also illustrates quasi-periodic oscillation of amplitude *ca* 15 mas in observed polar motion data time series.

It may be noticed that the additional wobbles cannot simply superposed onto the free wobble of the rotating pole. The two free wobbles interact with each other in appearance of frequency modulation. The 14.6 yr periodic modulation for the inherent free wobble period possesses a ratio of $73/6$ or *ca* 12 times with respect to Chandler wobble. Non-Linear oscillation theory tells that this may cause subharmonic resonance in the sway system. Under calculation for the subharmonic resonance solution, it can be obtained that an oscillation excitation of period of *ca* 14.6 yr and amplitude 13 mas, may force the 1.191 yr free wobble oscillating of amplitude magnifying *ca* 8.3 times, so that the observed polar motion of amplitude increases by *ca* 108 mas on the average free wobble level. Therefore the observed polar motion amplitude is approximately 240 mas in maximum and 24 mas in minimum. This result is almost identical to the observed free wobble data series.

Therefore, amplitude modulation of Chandler wobble possesses long decades of period and that would have beat with longer period shown in graph of IERS data. The theory of additional torques and additional moments in this study explains clearly the reason why the observed polar motion has feature of frequency modulation and amplitude varying about 90 per cent.

## ACKNOWLEDGMENTS

This study is supported by CAS Hundred Talents Program and NSFC 40174006. The head of this program in our Institute, Dr L.T. Liu is gratefully thanked for reading through the article. One of the referees, Dr A. Poma, and the other referee are gratefully thanked for pertinent commentary on the paper. I also want to show my appreciation to the members of the Editorial Board for attaching importance to this work.

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