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On the equivalence between stratified media and oscillators

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SUMMARY

A new method of modelling scattered seismic pulses is developed, based on the deformation that suffers an initial pulse due to scattering and on a reinterpretation of the product of convolution of a Green function with a source time function. In this new interpretation, the convolution product is equivalent to the solution of an ordinary differential equation, in which the source term corresponds to the solution of the homogeneous differential equation. The medium response (identified with the Green function) corresponds to the inhomogeneous term of the differential equation. Numerical solutions show the equivalence between the proposed method and the classical one based on convolution. This method can easily be generalized to any kind of complex medium and source time function.

Key words: Green's function, layered media, scattering, wave propagation.

1 INTRODUCTION

Multiscattering is encountered in many branches of physics, and more specifically in wave propagation in heterogeneous media. A good example is provided by coda waves. Whereas a few cycles of long period *P* and *S* and surface waves are predicted well by linear wave propagation, after these few cycles a clear disagreement is being developed that grows rapidly with increasing time, accounting for the progressive complexity of the medium. The wavefield is now a scattered wavefield. According to these observations, low-magnitude events (except for *P*- and *S*-wave arrivals) and coda waves have usually been considered to be random. When dealing with coda waves, only the envelope is used in their analysis, following the pioneering work of Aki & Chouet (1975). The advent of broadband seismometers clearly showed the complexity of the wavefield, reflecting the complexity of the scattering medium, but it does not mean that they are completely random. As an example, Correig & Urquizú (1996) found strong evidence in favour of coda waves being chaotic.

A common way to approach the study of multiple scattering consists of modelling the recorded signal (in the present case the displacement ground motion) at a fixed point (the recording station) as a sum over all possible paths followed by the signal along its path from source to receiver. Analytically, this displacement field can be written (Snieder *et al.* 2002) as

$$u(t) = \sum_{P} A_P S(t - t_P), \tag{1}$$

where $u(t)$ is the resulting wavefield, t_P is the traveltime along path P, A_P is the corresponding (complex) amplitude that accounts for the effect of scattering and $S(t)$ is the source wavelet. The present work is focused on the deformation suffered by the source pulse due to scattering, and so we will not take into account the influence of geometrical spreading and the influence of (frequency depending) attenuation.

In a study on the propagation of acoustic waves obliquely reflected from a medium of higher sound velocity; Arons & Yennie (1950) showed that the reflected seismic pulse was distorted with respect to the incident one, and explained this distortion as being due to the frequency dependence of the reflection coefficient for angles of the incident pulse larger than the critical one. In this paper, in Section 2 we reformulate the results of Arons & Yennie (1950) and present a generalization to heterogeneous layered (scattering) media; the resulting expression is interpreted in terms of an ordinary differential equation. In Section 3 we present several numerical simulations to show the equivalence between the new formulation in terms of ordinary differential equations and the usual methods based on the convolution between a Green function and a source term. Finally, in Section 4 the results are discussed and some possible extensions of the method are pointed out.

2 PROPAGATION AND REFLECTION OF A SEISMIC PULSE

The propagation of seismic waves in homogeneous media is well modelled in terms of the wave equation along with appropriate boundary conditions. For heterogeneous (scattering) media, direct integration is not possible, so that some approximations such as the WKBJ, Gaussian beams, etc, are proposed, see Chapman & Orcutt (1985) for a review. The solution is usually expressed in terms of the Green function, the solution of the wave equation for a heterogeneous medium to an impulse source function in space and time, convolved with an appropriate source time function. For a heterogeneous medium, boundary conditions have to be satisfied on the displacement and stress fields at both sides of the interfaces, and the resulting reflected and transmitted wavefields are obtained in terms of reflection and transmission coefficients, usually under the hypothesis of plane waves. Let us now deduce the corresponding expressions for the reflection and transmission coefficients for the case of an incident seismic pulse.

Consider, for simplicity, a scalar plane wave, as a decoupled *SH* wave. Following Ben-Menahem & Singh (1981, Ch. 2), the motion can be represented by a wave equation for the scalar potential $\psi(\mathbf{r}, t)$, for which the general solution in terms of plane waves is

$$\psi(\mathbf{r}, t) = f\left(t - \frac{\mathbf{p}\cdot\mathbf{r}}{\beta}\right), \tag{2}$$

where $\mathbf{p}$ is a unit vector, $\mathbf{r}$ is the vector position, β the shear wave velocity and f is an arbitrary function, twice differentiable, that corresponds to progressive plane waves of constant phase propagating along the $\mathbf{p}$ direction. We assume that the shape of the pulse is preserved between any two scatters, although it may be distorted upon interaction at an interface.

Assume now that instead of plane waves we are in the presence of a pulse (a wave packet) $f(t)$ impinging an interface located at $z = 0$, see Fig. 1, and that $f'(t)$ is a causal function, i.e. for $t \le 0$, $f(t) = 0$, and $f'(t) = 0$ and $f''(0) \neq 0$ at $z = 0$ and $t = 0$.

At an interface, part of the energy of the incident pulse is transmitted and part is reflected, and just at that time the incident, reflected and transmitted pulses coexist. The boundary conditions at the interface are the continuity of the energy flux at both sides, which means continuity of displacements and stress (Achenbach 1973). Thus, the sum of the displacements (stresses) corresponding to the incident plus the reflected pulse has to be the same as the displacement (stress) of the transmitted pulse.

Displacements and stress are given, respectively, by (Ben-Menahem & Singh 1981) $\xi = -\frac{1}{\beta} f'(s)\mathbf{p}\times\mathbf{k}$ and $\tau_{zy} = \mu\frac{\partial\xi_y}{\partial z}$. Assuming that $\mathbf{p} = (\sin\theta, 0, \cos\theta)$, $\mathbf{k} = (0, 0, 1)$ and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, see Fig. 1, the following expressions are obtained.

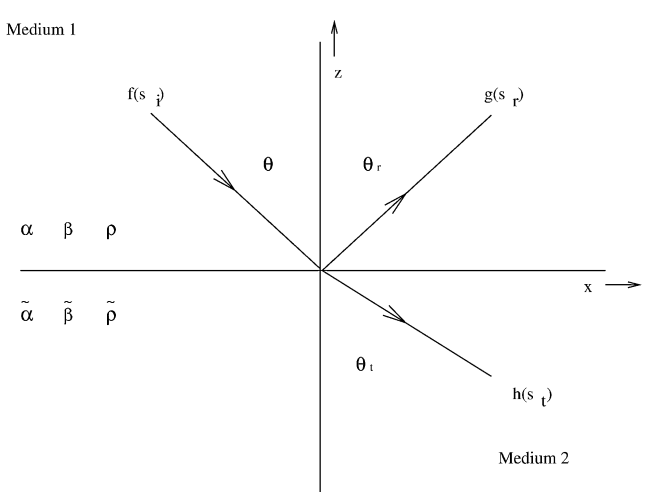

Figure 1. Scheme of the geometry of reflection and transmission at an interface. $f(s_i)$ corresponds to the incident pulse from medium 1, whereas $g(s_{\rm r})$ and $h(s_{\rm t})$ correspond to the reflected and transmitted waves. s_i is defined as $s_i = t - (x_0 \sin\theta_i + z\cos\theta_i)/\beta$, and similarly for $s_{\rm r}$ and $s_{\rm t}$. α, β, ρ are the compressional and shear velocities and density of medium 1, and $\tilde\alpha$, $\tilde\beta$, $\tilde\rho$ are the corresponding parameters of medium 2.

Displacement

$$\begin{aligned}\xi_1 &= -\frac{1}{\beta}\left[\sin\theta f'\left(t - \frac{x\sin\theta - z\cos\theta}{\beta}\right)\right.\\ &\quad\left. + \sin\theta_{\rm r} g'\left(t - \frac{x\sin\theta_{\rm r} + z\cos\theta_{\rm r}}{\beta}\right)\right]\\ \xi_2 &= -\frac{\sin\theta_{\rm t}}{\tilde\beta} h'\left(t - \frac{x\sin\theta_{\rm t} - z\cos\theta_{\rm t}}{\tilde\beta}\right)\end{aligned} \tag{3}$$

Stress

$$\begin{aligned}\tau_1 &= -\frac{\mu}{\beta^2}\left[\sin\theta\cos\theta f''\left(t - \frac{x\sin\theta - z\cos\theta}{\beta}\right)\right.\\ &\quad\left. - \sin\theta_{\rm r}\cos\theta_{\rm r} g''\left(t - \frac{x\sin\theta_{\rm r} + z\cos\theta_{\rm r}}{\beta}\right)\right]\\ \tau_2 &= -\frac{\tilde\mu}{\tilde\beta^2}\sin\theta_{\rm t}\cos\theta_{\rm t} h''\left(t - \frac{x\sin\theta_{\rm t} - z\cos\theta_{\rm t}}{\tilde\beta}\right)\end{aligned} \tag{4}$$

for media 1 and 2, respectively, where f, g and h refer to the incident, reflected and transmitted pulse, and a prime refers to the derivative with respect to the argument.

Performing the Fourier transform on both sides of eqs (3) and (4) and applying the boundary conditions at $z = 0$ we obtain

$$[F(\omega) + G(\omega)] = H(\omega) \tag{5}$$

and

$$\frac{\mu\cos\theta_i}{\beta}[F(\omega) - G(\omega)] = \frac{\tilde\mu\cos\theta_{\rm t}}{\tilde\beta}, \tag{6}$$

where $\mathcal{F}(f') = F(\omega)$, $\mathcal{F}(g') = G(\omega)$, $\mathcal{F}(h') = H(\omega)$, and $\mathcal{F}$ denotes the Fourier transform. In order that the boundary conditions are satisfied for all values of x and t Snell's law follows.

By solving the system of eqs (5) and (6) the coefficients of reflection and transmission are obtained

$$G(\omega) = C_{\rm r}F(\omega), \tag{7}$$

$$H(\omega) = C_{\rm t}F(\omega). \tag{8}$$

If the coefficients of reflection and transmission are frequency independent, the shape of the pulse will remain the same after reflection and transmission, and the only effect will be, for a purely elastic medium, a decrease of the amplitude due to geometrical spreading and to the reflection/transmission coefficients. Upon interaction of the incident pulse with the interface, the component j of the displacement field is given below.

Medium 1

$$\xi_{j\,\text{total}}(t, \mathbf{r}) = \xi_{j\,\text{incident}}(t, \mathbf{r}) + \xi_{j\,\text{reflected}}(t, \mathbf{r}). \tag{9}$$

Medium 2

$$\xi_{j\,\text{total}}(t, \mathbf{r}) = \xi_{j\,\text{transmitted}}(t, \mathbf{r}). \tag{10}$$

Let us define the normalized potential for *SH* waves as

$$\Phi_i(t) = \frac{f'(t)}{f''(0)}, \qquad \Phi_{\rm r}(t) = \frac{g'(t)}{f''(0)}. \tag{11}$$

In terms of the potential (11) and taking into account (7), eq. (9), corresponding to the wavefield at medium 1, can be rewritten as

$$\Phi_{\text{total}} = \Phi_i + \Phi_{\rm r} = \left[f' + \int_{-\infty}^{\infty} F(\omega)C(\omega)e^{i\omega t}\,d\omega\right]\frac{1}{N}, \tag{12}$$

where $\frac{1}{N} = \frac{1}{f''(0)}$ and it has been assumed that the reflection coefficient may be frequency dependent. From the above equation, the reflected wave is defined as

$$\Phi_{\mathrm{r}} = \frac{1}{N}\int_{-\infty}^{\infty} F(\omega)C_{\mathrm{r}}(\omega)\mathrm{e}^{i\omega t}\, d\omega. \tag{13}$$

A similar expression can be found for the transmitted wavefield at medium 2. Taking into account the properties of the Fourier transform on the convolution of two functions

$$\Phi_{\mathrm{r}} = \mathcal{F}^{-1}\left[\frac{F(\omega)C(\omega)}{N}\right] = \int_{t_o}^{t} \Phi_i(t-\gamma)c_{\mathrm{r}}(\gamma)\, d\gamma \tag{14}$$

we can readily see that the reflected wave is the convolution of the functions $\Phi_i(t)$ and $c_{\mathrm{r}}(t)$. For the sake of simplicity, we will refer to $c_{\mathrm{r}}(t)$ simply as $c(t)$.

From a mathematical point of view, eq. (14) can be interpreted as the solution of an inhomogeneous differential equation, for which Φ_i is the solution of the homogeneous equation and Φ_{r} is a particular solution of the inhomogeneous equation. Thus, $c(t)$ is the inhomogeneous term evaluated as the superposition of the several reflections and transmissions suffered by the source term during its travel from source to receiver. Another way to write the particular solution of an inhomogeneous differential equation is to introduce the operator $\mathcal{D}$, which denotes differentiation with respect to time, d/dt. Then, if we define a polynomial function of $\mathcal{D}$ as G, a particular solution of the differential equation can be written as

$$\Phi_{\mathrm{r}}(t) = \frac{1}{G(\mathcal{D})}c(t). \tag{15}$$

By using the one side Fourier transforms (well suited for causal functions) at both parts of the last equation, we obtain

$$\hat{\Phi}_{\mathrm{r}}(\omega) = \frac{1}{G(i\omega)}\hat{c}(\omega) = g(i\omega)\hat{c}(\omega), \tag{16}$$

where a hat denotes the Fourier transform. By comparing (16) and (14), $\hat{c}(\omega) = C_{\mathrm{r}}(\omega)/N$ and $g(i\omega) = F(\omega)/N$ can be immediately identified. Then, for example, if $g(i\omega) = \frac{1}{1-i\omega}$ it is equivalent to the differential equation $\mathcal{D}+1$, for which its homogenous solution is e^{-t}. These equivalences will provide us with a way of defining the source operator in terms of a Fourier representation.

It is now straightforward to interpret $c(t)$ as the external force that drives a dynamic system characterized by a function Φ_i, which has the meaning of the medium response satisfying $\mathcal{P}\Phi_i = c(t)$, where $\mathcal{P} \equiv G(\mathcal{D})$ is an operator that contains the source characteristics and $c(t)$ denotes the effect of scattering on the pulse. In order to interpret Φ_{r} as a particular solution of a differential equation of degree 2, the following conditions must be satisfied (Elgotz 1983): (1) for $s < 0$, $\Phi(s) = \Phi'(s) = 0$ and (2) for $s = 0$, $\Phi(s) = 0$, $\Phi'(s) = 1$, where $s = t - \gamma$.

In the particular case we are dealing with, following Arons & Yennie (1950) we can write

$$C(\omega) = C\mathrm{e}^{-i\omega/|\omega|}. \tag{17}$$

This expression can be generalized to any frequency dependence, and applying the inverse Fourier transform $\mathcal{F}^{-1}(|C(\omega)|\mathrm{e}^{i\omega t})$ we deduce (see the Appendix)

$$c(t) = |C|\left[\cos(\Theta)\delta(t-t_i) + \sin(\Theta)\frac{1}{\pi}\wp\left(\frac{1}{t-t_i}\right)\right]. \tag{18}$$

The function $\Phi(t)$ thus depends on the shape of the incident pulse and on the order of the ordinary differential equation, for which $\Phi(t)$ is a solution of the homogeneous one.

On the other hand, the solution of the wave equation in terms of the Green function in going from a point A to a point B is

$$\left(\nabla_{\mathbf{q}}{}^2 - \frac{1}{\beta^2}\frac{\partial^2}{\partial t^2}\right)G(\mathbf{q}_a, \mathbf{q}; 0, t) = \delta(\mathbf{q}_a - \mathbf{q})\delta(t). \tag{19}$$

In the frequency domain the solution of eq. (19) can be written as

$$G(\mathbf{q}_a, \mathbf{q}; k) = H(k)g^{+}(\mathbf{q}_a, \mathbf{q}; k^2) + H(-k)g^{-}(\mathbf{q}_a, \mathbf{q}; k^2), \tag{20}$$

where

$$g^{\pm}(\mathbf{q}_a, \mathbf{q}; k^2) = -\frac{1}{4\pi}\left|\frac{d\Omega_{\mathrm{A}}}{dS_{\mathrm{B}}}\right|^{1/2}\exp(\pm i(\sqrt{k^2 l^2} - \phi)) \tag{21}$$

and $H(k)$ is the Heaviside step function, l is the distance between A and B and ϕ is a phase shift that amounts to π if the path has suffered a reflection or $\frac{\pi}{2}$ if a caustic has been crossed.

In the time domain, eq. (20) reads

$$G(\mathbf{q}_a, \mathbf{q}; 0, t) = -\frac{1}{4\pi}\left|\frac{d\Omega_{\mathrm{A}}}{dS_{\mathrm{B}}}\right|^{1/2}\left[\delta(t_i - t)\cos\phi + \frac{1}{\pi}\wp\left(\frac{1}{t_i - t}\right)\right], \tag{22}$$

where $\wp$ is Cauchy's principal value of the distribution $1/(t_i - t)$ and the term $d\Omega_{\mathrm{A}}/dS_{\mathrm{B}}$ is related to the trajectories that in the solid angle $d\Omega_{\mathrm{A}}$ intercept a surface dS_{B} (Berry & Mount 1972).

It is now straightforward to see the parallelism between eqs (18) and (22), so that we can generalize and interpret $c(t)$ as the Green function of a heterogeneous medium for which ϕ corresponds to the phase change and t_i to the traveltime of the impulse time function.

As a summary, the propagation of seismic pulses in stratified/scattering media can be viewed as a dynamic system in which, the initial pulse corresponds to the steady-state response of the medium, and the effect of the stratification/scatterers-distribution upon a propagating pulse corresponds to the external force that drives the dynamics.

3 THE REPRESENTATION OF A SEISMIC PULSE IN TERMS OF AN OSCILLATOR

Let us see how the proposed new point of view can be used to derive an approximate solution for a propagating seismic pulse. For simplicity, assume the following source time function (Ben-Menahem & Singh 1981)

$$\Phi(t) = \mathrm{e}^{-t/B}\sin\omega t. \tag{23}$$

This pulse is represented in Fig. 2 for the values $B = 1.333$ and $\omega = 0.1053$, and will be used in further numerical simulations. Eq. (23) is a solution of the ordinary differential equation

$$\frac{d^2\Phi}{dt^2} + A\frac{d\Phi}{dt} + E\Phi = 0, \tag{24}$$

where $A = \frac{2}{B}$ and $E = \omega^2 + \frac{1}{B^2}$.

As this pulse has travelled through a stratified medium, the conclusions of Section 2 apply. Hence, the corresponding equation of the oscillator can be written as

$$\frac{d^2\Phi}{dt^2} + A\frac{d\Phi}{dt} + E\Phi = c(t), \tag{25}$$

where

$$c(t) = \sum_{i=0}^{n}\prod_{i}|C_j|\left[\cos\left(\sum_i \Theta_i\right)\delta(t-t_i) + \sin\left(\sum_i \Theta_j\right)\frac{1}{\pi}\wp\left(\frac{1}{t-t_i}\right)\right] \tag{26}$$

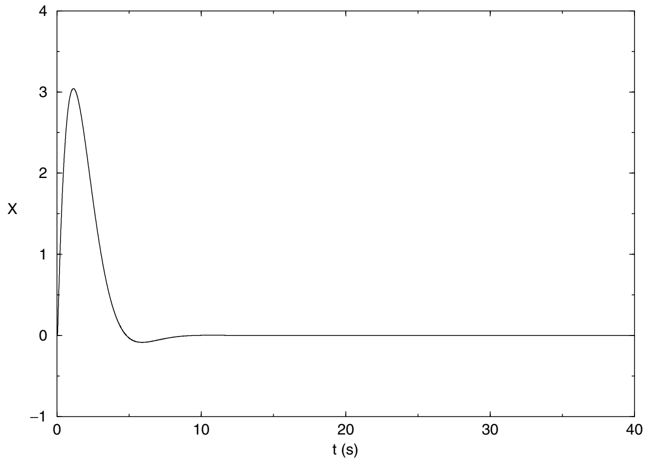

Figure 2. Waveform of the pulse $e^{-t/B}\sin\,\omega\,t$ with $B = 1.330$ and $\omega = 0.1053$.

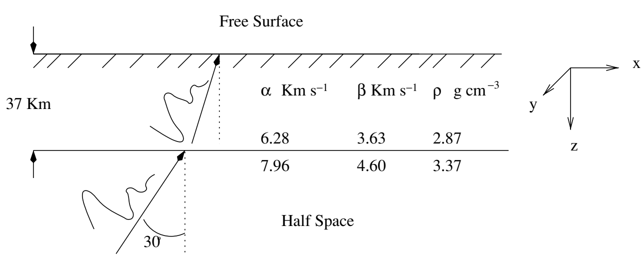

Figure 3. Model of a layer over a half-space as used in numerical simulations.

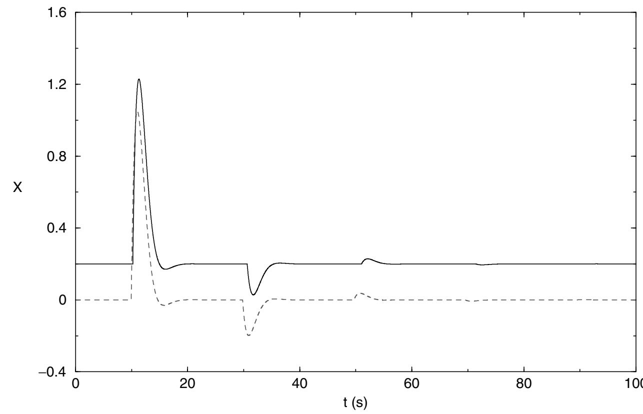

Figure 4. Waveform of the pulse multireflected in the layer shown in Fig. 3. The dashed line denotes the convolution of the pulse shown in Fig. 2 where the Green function corresponds to Fig. 3, and with solid line the solution of the equation $d^2\Phi/dt^2 + 1.5d\Phi/dt + \Phi = c(t)$, where $c(t)$ has been computed from eq. (26).

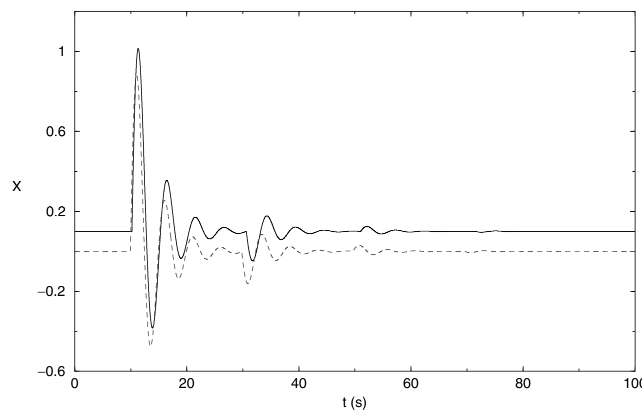

Figure 5. Same as Fig. 4 for another source pulse, defined by $A = 4.0$ and $\omega = 2.0$. The solid line is the solution of the differential equation $d^2\Phi/dt^2 + 0.5\,d\Phi/dt + 1.579\,\Phi = c(t)$, and dashed line corresponds to the convolution.

The above equation is a generalization of (18) for n interactions of the seismic pulse to n discontinuities. An equivalent expression was found by Arons & Yennie (1950).

Hence, we have two ways of characterizing the wavefield as recorded at a fixed seismic station: (1) by convolving the source term eq. (23) with the Green function of a layered medium and (2) by solving eq. (25) and considering (26) as corresponding to the layered medium.

As a first example, consider the propagation of a seismic pulse through the medium shown in Fig. 3. We have generated the Green function for an *SH* seismic pulse in the frequency domain by means of the Haskell method, which has been convolved with the seismic pulse (23) and transformed back to the time domain. The result is presented in Fig. 4 with a dashed line. Then, we have evaluated eq. (26) for the medium presented in Fig. 3, for which the path will consist of the first transmission from the half-space, $C_0 = 1.168$, $\theta_0 = 0$ and $t_0 = 10.193$, and n reflections at the free surface, characterized by $C_n = 0.167$, $\theta_n = \pi$ and $t_n = t_0 + n20.386$, where θ_n is the phase change at the free surface. This solution is presented in Fig. 4 as a solid line. The global effect consists of a phase change due to the reflection at the free surface and the multiplication by the reflection coefficient due to the reflection at the interface (there is no phase change at the half-space boundary) for the layer $n = 20$. Fig. 5 is the same as Fig. 4 except that we have used another pulse shape, characterized by $B = 4.0$ and $\omega = 0.2$.

Figs 6 and 7 display two more examples corresponding to a more complex medium. In the first case, Fig. 6, the medium consists of a stack of 20 layers of variable thicknesses amounting to a total of 70 km, with a shear velocity randomly fluctuating around a mean value of 3.5 km s^{-1}. The second case, Fig. 7, corresponds to a stack of three layers of variable thickness (19.1415, 19.0134 and 18.3739 km) with associated shear velocities of (3.206, 3.276 and 1.864 km s^{-1}) overlying an infinite medium with the same characteristics as Fig. 3. For this example a direct computation of $c(t)$ from eq. (26) is very complicated. Taking into account the equivalence between $c(t)$ and the Green function eq. (22), we compute $C(\omega)$ by the Haskell method, transform back to time domain and solve eq. (25).

The corresponding result is represented with a solid line in Figs 6 and 7, and the convolution of the Green function with the source term with dashed lines. We can clearly observe that both methods are equivalent.

4 DISCUSSION AND CONCLUSION

A new procedure is presented that is able to model records of seismic waves that have propagated through complex media, based on a new interpretation of the representation of scattered waves, actually for the case of wave propagation in a layered medium. The product, in the frequency domain of an incident wave by the coefficient of reflection (or transmission) is equivalent to a particular solution of an inhomogeneous ordinary differential equation, where the source time function is related to an operator that accounts for the influence

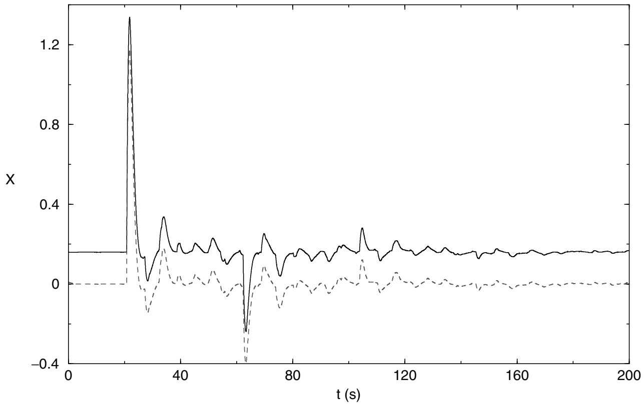

Figure 6. Same as Fig. 4 for a more complex layered medium (see text). The dashed line denotes the results of the convolution, and in solid line those of eq. (25), in which $c(t)$ has been reinterpreted as $G(t)$, the Green function of the layered medium.

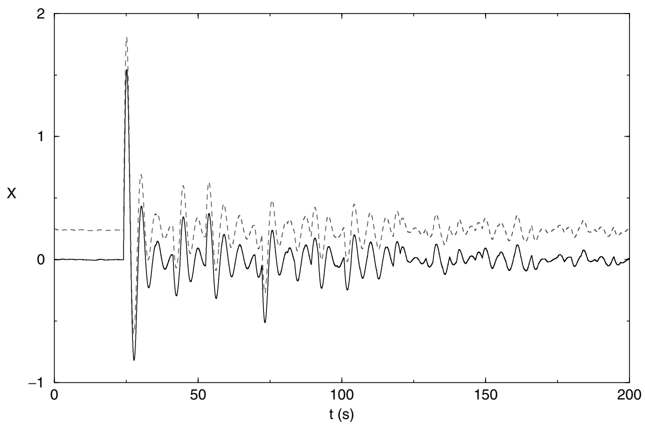

Figure 7. The combination of the source term of Fig. 5 and a more complex layered medium (see text).

of the scatters on the incident wavefield. The resulting time-series is the seismogram that would be recorded at a given seismic station.

Through numerical simulations we have shown that for the source (23), the ground motion at the recording place can be written as

$$\ddot{\Phi}(t) + \alpha\dot{\Phi}(t) + \beta\Phi(t) = G(t). \tag{27}$$

Comparing with eqs (26) and (22) we realize that the external force term $G(t)$ can be assimilated to the Green function (a few factors apart, such as geometrical spreading and attenuation, that could be introduced easily), whereas the solution of the homogeneous equation represents the steady-state response of the medium (the 'initial pulse'). As time goes on, the medium response (the 'initial pulse') will be modified by the external force $G(t)$. This interpretation corresponds to a Lagrangian description of the motion, in which the observer is travelling along with the pulse and 'sees' how the medium interacts with the pulse. This is in contrast to the usual, Eulerian description, in which the observer is located at a fixed point. Of course, both descriptions are equivalent.

From a conceptual point of view, eq. (12) can be viewed as the solution of the wave equation in an inhomogeneous medium, for which Φ_r would correspond to the solution of the perturbed field, computed in terms of the homogeneous equation of the initial field forced by the Green function. This point of view is encountered when solving the motion through a perturbative development of the wave equation, either in the field of quantum mechanics (Morse & Feshback 1953, p. 1069) or in the field of classical mechanics (as used by Aki & Richards 1980, p. 730). Usually, Φ denotes the solution of the wave equation in an unbound medium. In contrast, in this paper Φ corresponds to the solution of to the pulse that propagates through a scattering medium.

Frequently, the propagation of the wavefield is obtained by first computing the medium Green function for each frequency in terms of stationary plane waves and convolving the Green function by the appropriate source time function; under the assumption of plane waves, no complex reflection/transmission coefficients are allowed. Other approximations, such as those proposed by Kriegsmann & Luke (1994) and Freilikher & Tarasov (2001) use numerical simulations to propagate the initial pulse. However, in both cases the mechanism that causes the deformation of the propagating pulse is not apparent.

In this study we propose a procedure that permits us to deal with complex media in a more simple and transparent way, by reinterpreting the usual scenario. The scattered wavefield is characterized by the deformation suffered by the original wave packet. The medium is now considered globally, and what we emphasize is the interaction scatterer-pulse. For example, as done in this paper, for the case of a layered medium the spatial distribution of scatterers will be characterized by t_i, the traveltime, the reflection/transmission coefficients and by phase delays, so that the problem is reduced to the obtention of a set of parameters t_i, Φ_i and C_i, representing the last ones the interaction scatterer-pulse.

As already mentioned, we have applied this new interpretation to a scattering medium consisting of a stack of layers, and have shown through numerical simulations it is equivalent to the conventional method consisting of the convolution of a source time function with a Green function. We have seen that, for this particular case, that of a layered medium, the problem is reduced to solve a second-order linear ordinary differential equation. However, this procedure can be straightforwardly generalized to any kind of operator and to any function $c(t)$, thus opening a wide set of possibilities that will be explored in future studies.

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REFERENCES

Achenbach, J.D., 1973. *Wave Propagation in Elastic Solids*, North-Holland, Amsterdam.

Aki, K. & Chouet, B., 1975. Origin of coda waves: source, attenuation and scattering effects, *J. geophys. Res.,* **80,** 3322–3342.

Aki, K. & Richards, P.G., 1980. *Quantitative Seismology*, Freeman, San Francisco.

Arons, A.B. & Yennie, D.R., 1950. Phase distortion of acoustic pulses obliquely reflected from a medium of higher sound velocity, *J. acoust. Soc. Am.,* **22,** 231–237.

Ben-Menahem, A. & Singh, S.J., 1981. *Seismic Waves and Sources*, Springer-Verlag, Berlin.

Berry, M.V. & Mount, K.E., 1972. Semiclassical approximations in wave mechanics, *Rep. Progr. Phys.,* **35,** 315–378.

Chapman, C.H. & Orcutt, J.A., 1985. The computation of body wave synthetic seismograms in laterally homogeneous media, *Rev. Geophys.,* **23,** 105–163.

Correig, A.M. & Urquizú, M., 1996. Chaotic behavior of coda waves in the eastern Pyrenees, *Geophys. J. Int.,* **126,** 113–122.

Elgotz, L., 1983. *Ecuaciones diferenciales y cálculo variacional*, Mir, Moscow.

Ewing, W.M., Jardetzky, W.S. & Press, F., 1957. *Elastic Waves in Layered Media*, McGraw-Hill, New York.

Freilikher, V.D. & Tarasov, Yu.V., 2001. Propagation of wave packets in randomly stratified media, *Phys. Rev.,*E, **64**, 056620-1.

Kriegsmann, G.A. & Luke, J.H.C., 1994. Rapid pulse responses for scattering problems, *J. Comput. Phys.,* **111,** 390–398.

Morse, P.M. & Feshback, H., 1953. *Methods of Theoretical Physics*, McGraw-Hill, New York.

Snieder, R., Grt, A.S., Douma, H. & Scales, J., 2002. Coda wave interferometry for estimating nonlinear behavior in seismic velocity, *Nature,* **295,** 2253–2255.

APPENDIX

Consider the general case of a seismic pulse incident beyond the critical angle. Following Arons & Yennie (1950) (see also Ewing *et al.* 1957; Ben-Menahem & Singh 1981) the reflection coefficient $C(\omega)$ at an interface for any angle of incidence is given by

$$C(\omega) = -C\mathrm{e}^{-i2\epsilon}. \tag{A1}$$

As we are now dealing with seismic pulses, the whole interval of frequencies $(-\infty, \infty)$ has to be considered. The angular phase shift 2ϵ only depends upon the angle of incidence and the physical constants of the medium, and will vanish for an angle of incidence less than the critical. As a result of this phase shift, a time factor $2\epsilon/|\omega|$ will have to be added to time for any value of ω, and, as a consequence, an incident pulse of arbitrary shape will change its shape upon reflection at any angle of incidence greater than the critical. Eq. (A1) can thus be written in a more general form as

$$C(\omega) = |C| \exp\left(i\frac{\omega\Theta}{|\omega|}\right), \tag{A2}$$

where $\Theta = 2\epsilon$.

Transforming eq. (A2) to the time domain we obtain

$$C(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |C| \exp\left(i\frac{\omega\Theta}{|\omega|}\right) \mathrm{e}^{i\omega s}\, d\omega, \tag{A3}$$

Following Arons & Yennie (1950) the above integral can be written as

$$C(s) = \frac{|C|}{\sqrt{2\pi}} \cos\Theta \int_{-\infty}^{\infty} \mathrm{e}^{-i\omega s}\, d\omega + \frac{i|C|}{\sqrt{2\pi}} \sin\Theta \left(-\int_{-\infty}^{0} \mathrm{e}^{i\omega s}\, d\omega + \int_{0}^{\infty} \mathrm{e}^{i\omega s}\, d\omega\right) \tag{A4}$$

Taking into account the following representation of the Dirac delta

$$\delta(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{i\omega s}\, d\omega \tag{A5}$$

and the definition of the principal value

$$\wp \frac{1}{i\pi} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{i\omega\zeta}}{\zeta} d\zeta = \begin{cases} -1 & \text{if } \omega < 0 \\ 0 & \text{if } \omega = 0, \\ 1 & \text{if } \omega > 0 \end{cases} \tag{A6}$$

where $\wp$ is the principal part of the value of the integral, the coefficient of reflection can be written in the time domain (reduced time $s = t - t_i$) as

$$C(s) = |C| \cos\Theta\delta(s) + |C| \sin\Theta G(s), \tag{A7}$$

where

$$G(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{i\omega s} \wp \int_{-\infty}^{\infty} \frac{\mathrm{e}^{i\omega\zeta}}{\zeta}\, d\zeta\, d\omega. \tag{A8}$$

By reversing the order of integration and taking (A6) into account eq. (25) can be rewritten as

$$G(s) = \frac{1}{\sqrt{2\pi}} \wp \int_{-\infty}^{\infty} \frac{1}{\zeta} \int_{-\infty}^{\infty} \mathrm{e}^{i\omega(\zeta+s)}\, d\zeta\, d\omega \frac{1}{\pi} \wp \times \int_{-\infty}^{\infty} \frac{1}{\zeta} \delta(s+\zeta) d\zeta = \frac{1}{\pi} \wp \left(\frac{1}{s}\right). \tag{A9}$$