The virial-based theory of the earth global dynamics

Ferronsky V.I. (ferron@aqua.laser.ru)

Water Problems Institute of the Russian Academy of Sciences

Since the time of Newton and Clairaut it was adopted that the Earth is an inert, powerless body and the planet's rotation is effected by the inertial forces. The reason of that was as follows. While developing the theory of the Earth's figure and considering the planet as a rotating liquid body being in the solar uniform force field, it was concluded that the resultant of the body gravitational forces is equal to zero [1]. On that basis a speculative idea of hydrostatic equilibrium of the planet's mass and its inertial rotation has been accepted. That idea up to the present remains valid.

During the last four decades, while studying the Earth's gravitational field by means of the geodetic artificial satellites, multiple measurements of zonal and tesseral gravitational moments were carried out. Analysis of the obtained results shows that the Earth does not stay in hydrostatic equilibrium. The Earth figure deviates from the normal ellipsoid of rotation corresponding to hydrostatic equilibrium of the planet by a value equal to about square of its oblateness, i.e. by $\sim(1/300)^2$. The measurements also show that the gravitational field lines decline from the normal throughout the Earth surface and do not create the central force field [2, 3, 4]. Moreover, the potential energy of the inert Earth model appears to be by three order of magnitude less then the kinetic one. This contradicts to the fundamental demand of the virial theorem, where the potential energy must be twice as much of the kinetic energy.

The problems related to the Earth rotation were discussed recently by the NATO Workshop [5]. Two of them, namely, fluctuation in the length of day and the observed Chandler's pole wobble with a component of 14 months period (against the Euler's rigid body model giving 10 months), were found as remaining 'largely unsolved'. These facts and also the registered changes in the gravitational force field, in angular velocity of the Earth rotation, in plate tectonics and in many other dynamical effects evidence that new approaches for the problem solution should be developed.

It is actually nonsense to solve the problem of the Earth dynamics on a basis of the hydrostatics. In fact the Earth is a self-gravitating body. The own force field at the body surface is by three orders of magnitude stronger of the Sun's. And the main point is, that the gravitational as well as inertial forces, by their physical nature, develop volumetric, 4π effects.

The planet's internal forces can not be reduced to a resultant vector by their definition. We prove below, that the forces manifest inside pressure and are reduced to a resultant spatial envelope of pressure with a fixed radius. Hence, the own internal force field, which is induced by gravitational interaction of the Earth masses, should to be the cause of dynamical effects of the same body masses. The above physical ideas underlie in the developed approach to solve the problem of the Earth dynamics in the own force field, which is studied by the method of moments in the framework of classical mechanics of conservative (uniform) and dissipative (nonuniform) continuous media.

2. Reduction of the gravitational and inertial forces to a resultant spheroid (ellipsoid) of the force pressure

Consider the Earth as a self-gravitating sphere with uniform and one-dimensional interacting media. The motion of the Earth proceeds both in its own and the Sun's force field. It's known from theoretical mechanics that any motion of a body can be represented by translation of its center of mass, rotation around the center of mass and changes in the shape and structure of the body mass [6]. In the two-body problem the last two effects are neglected due to their smallness.

In order to study the Earth motion in the own force field the translation (orbital) motion relative to the fixed point (the Sun) should be separated from the two others. After that the rotation around the geometric center of the Earth masses under the action of the own force field and changes in shape and structure (oscillation) can be considered. Such separation is required only for the moment of inertia, which depends on what frame of reference is selected. The force function depends on a distance between the interacted masses and does not depend on selection of a frame of reference [6]. The moment of inertia of the Earth relative to the solar reference frame should be split into two parts. One is the moment of the body mass center relative to the same frame of reference and the moment of inertia of the planet's mass relative to the own mass center.

So, set up the absolute Cartesian coordinates $O_c\xi\eta\zeta$ with the origin in the center of the Sun and transfer it to the system *Oxyz* with the origin in the geometrical center of the Earth's mass (Fig.1). Then, the moment of inertia of the Earth in the solar frame of reference is

$$I_s = \sum m_i \; R_i^2 \,, \tag{1}$$

where m_i is the Earth' piece of mass; R_i is its distance from the origin in the same frame.

The Lagrange's method is applied to separate the moment of inertia (1). The method is based on his algebraic identity

$$\left(\sum_{1\le i\le n} a_i^2\right)\left(\sum_{1\le i\le n} b_i^2\right) = \left(\sum_{1\le i\le n} a_i b_i\right)^2 + \frac{1}{2}\sum_{1\le i\le n}\sum_{1\le j\le n}\left(a_i b_j - b_i a_j\right)^2, \tag{2}$$

where a_i and b_i are whichever values; n is any positive number.

Jacobi was the first who performed the mathematical transformation for separation of the moment of inertia of n interacting mass points into two algebraic sums [6, 7, 8]. He has shown that if we denote (Fig.1)

$$\xi_i = x_i + A; \qquad \eta_i = y_i + B; \qquad \zeta_i = z_i + C;$$

$$\sum m_i = M,\ \sum m_i\xi_i = MA;\ \sum m_i\eta_i = MB;\ \sum m_i\zeta_i = MC, \tag{3}$$

where A, B, C are coordinates of the mass center in the solar frame of reference, then, using identity (2), one has

$$\sum m_i r_i^2 = \sum m_i\xi_i^2 + \sum m_i\eta_i^2 + \sum m_i\zeta_i^2 = \sum m_i x_i^2 + 2A\sum m_i x_i A^2 \sum m_i +$$
$$+\sum m_i y_i^2 + 2B\sum m_i y_i + B^2\sum m_i + \sum m_i z_i^2 + 2C\sum m_i z_i + C^2\sum m_i.$$

Since

$$MA = \sum m_i\xi_i = \sum m_i x_i + \sum m_i A = \sum m_i x_i + MA,$$

then

$\sum m_i x_i = 0$, and also $\sum m_i y_i = 0$, $\sum m_i z_i = 0$.

Now, the moment of inertia (1) acquires the form

$$\sum m_i R_i^2 = M\left(A^2 + B^2 + C^2\right) + \sum m_i\left(x_i^2 + y_i^2 + z_i^2\right), \tag{4}$$

where

$$M(A^2 + B^2 + C^2) = MR^2_m, \tag{5}$$

$$\sum m_i\,(x_i^2 + y_i^2 + z_i^2) = M\,r^2_m \tag{6}$$

M is the Earth's mass; R_m and r_m are radii of inertia of the Earth in the Sun's and the Earth's frame of reference.

Thus, we separated the moment of inertia of the Earth's mass, rotating around the Sun in the inertial frame of reference into two algebraic terms. The first one (5) is the Earth's moment of inertia in the solar reference system $O_c\xi\eta\zeta$. The second term (6) presents the moment of inertia of the Earth in the own frame of reference *Oxyz*. The Earth mass here is distributed over the spherical surface with a reduced radius of inertia r_m. In literature geometrical center of mass *O* in the Earth reference system is erroneously identified with the center of inertia and center of gravity of the planet.

For farther consideration of the problem of the Earth dynamics we accept the polar frame of reference with its origin in center *O*. Then expression (6) for the Earth polar moment of inertia I_p acquires the form

$$I_p = \sum m_i \,(x_i^2 + y_i^2 + z_i^2) = \sum m_i \, r_i^2 = M r_m^2. \qquad (7)$$

Now the reduced radius of inertia r_m, which draws a spherical surface, is

$$r_m^2 = \frac{\sum m_i r_i^2}{M}, \qquad (8)$$

where $\sum m_i = M$ is the Earth's mass relative to the own frame of reference.

Taking into account the spherical symmetry of the uniform and one-dimensional Earth, we consider the sphere as concentric spherical shells with mass $dm(r)=4\pi\, r^2 \rho(r)dr$. Then the expression (8) in the polar reference system can be rewritten in the form

$$r_m^2 = \frac{1}{M}\int_0^R r^2 4\pi r^2 \rho(r)dr = \frac{4\pi R^2}{MR^2}\int_0^R r^4 \rho(r)dr, \qquad (9)$$

or

$$\frac{4\pi r_m^2}{4\pi R^2} = \frac{4\pi \int_0^R r^4 \rho(r)dr}{MR^2} = \frac{\beta^2 MR^2}{MR^2} = \beta^2, \qquad (10)$$

where $r_m^2 = \beta^2 R^2$, $\rho(r)$ is the law of radial density distribution; R is radius of the sphere; β^2 is a dimensionless coefficient of the reduced spheroid (ellipsoid) of inertia $\beta^2 MR^2$.

The value of β^2 depends on the density distribution $\rho(r)$ and earlier [8] it was defined as a structural form-factor of the polar moment of inertia.

Analogously, the reduced radius of gravity r_g, expressed as a ratio of the moment of gravitational forces of the spherical shell with density $\rho(r)$ to the moment of the interacting forces of mass distributed over the shell with radius R can be written as

$$\frac{4\pi}{4\pi}\frac{r_g^2}{R^2} = \frac{\frac{4\pi G}{r}\int_0^R r\rho(r)m(r)dr}{\frac{GM^2}{R^2}} = \frac{\alpha^2 \frac{GM^2}{R^2}}{\frac{GM^2}{R^2}} = \alpha^2. \qquad (11)$$

The gravity radius, expressed through the force function is

$$\frac{4\pi}{4\pi}\frac{r_g^2}{R^2} = \frac{4\pi G\int_0^R r\rho(r)m(r)dr}{\frac{GM^2}{R}} = \frac{\alpha^2 \frac{GM^2}{R}}{\frac{GM^2}{R}} = \alpha^2, \qquad (12)$$

where $m(r)=4\int_0^r r^2 \rho(r)dr$, $r_m^2 = \alpha^2 R^2$, G is the gravitational constant.

The value of α^2 is a dimensionless coefficient of the reduced spheroid (ellipsoid) of gravity $\alpha^2 GM^2/R$. It depends on the density distribution $\rho(r)$ and earlier [8] was defined as a structural form-factor of the force function.

Numerical values of the dimensionless form-factors α^2 and β^2 for a number of density distribution laws $\rho(r)$, obtained by integration of the numerators in Eqs.(10) and (12) for the polar moment of inertia and the force function, are presented in Table 1 [8].

Table I. Numerical values of form-factors α and β^2 for radial distribution of mass density and for polytropic models

Distribution law Index of polytrope	α	$\beta^2_\perp$	β^2
Radial distribution of mass density			
$\rho(r)=\rho_0$	0.6	0.4	0.6
$\rho(r)=\rho_c(1-r/R)$	0.74	0.27	0.4
$\rho(r)=\rho_c(1-r^2/R^2)$	0.71	0.29	0.42
$\rho(r)=\rho_c\exp(1-kr/R)$	0.16k	$8/k^2$	$12/k^2$
$\rho(r)=\rho_c\exp(1-kr^2/R^2)$	$\sqrt{\frac{k}{2\pi}}$	1/k	1.5/k
$\rho(r)=\rho_c\delta(1-r/R)$	0.5	0.67	1.0
Politrope model			
0	0.6	0.4	0.6
1	0.75	0.26	0.38
1.5	0.87	0.20	0.30
2	1.0	0.15	0.23
3	1.5	0.08	0.12
3.5	2.0	0.045	0.07

Note that the values of the polar I_p and axial I_a moments of inertia are related as $I_p=3/2I_a$.. It follows from the Table I that for a uniform sphere with $\rho(r)=const$ its reduced radius of inertia coincides with the radius of gravity. Here both dimensionless structural coefficients α^2 and β^2 are equal to 3/5, and the moments of gravitational and inertial forces are equilibrated.

Thus,

$$\frac{r_m^2}{R^2}=\frac{r_g^2}{R^2}=\frac{3}{5}, \qquad (13)$$

from where

$$r_m = r_g = \sqrt{3/5R^2} = 0{,}7745966R. \qquad (14)$$

For the nonuniform sphere at $\rho(r) \neq const$ from Eqs.(10)-(12) one has

$$0 < \frac{r_m^2}{R^2} < \frac{3}{5} < \frac{r_g^2}{R^2} < 1. \qquad (15)$$

It follows from inequality (15) that in comparison with the uniform sphere, the reduced radius of inertia of the nonuniform body decreases and the reduced gravity radius increases. Because of $r_m \neq r_g$ and $r_m < 0.77R < r_g$, torque appears between the unbalanced gravitational and inertial volumetric forces of the shells. Then from Eq.(15) one has

$$r_g = r_{go} + r_{gt} \quad \text{and} \quad r_m = r_{mo} - r_{mt}\,, \qquad (16)$$

where subscripts *0* and *t* relate to the uniform and nonuniform sphere

In accordance with (15) and (16) rotation of shells of a one-dimensional body should be hinged-like and asynchronous. In the case of mass density increases to the body surface, then the signs in (15) and (16) are reversed. This remark is important because direction of rotation of a self-gravitating body is function of its mass density distribution.

The main conclusion from the above consideration is that the inner force field of a self-gravitating body is reduced to a closed envelope (spheroid or ellipsoid) of gravitational pressure, but not to a resulting force passing through the geometric center of the masses. In the case of a uniform body the envelopes have spherical shape and both gravitational and inertial radii coincide. For a nonuniform body radius of inertia does not coincides with the radius of gravity, the reduced envelope is closed but has non-spherical (ellipsoidal) shape. Analytical solutions done below justify the above.

3. Dynamical equilibrium of motion and equations of rotation and oscillation

In order to derive conditions of dynamical equilibrium and to write analytical equations of motion the structure of the potential and kinetic energy of the body should be expanded. Relationship (16) prompts, that in order to solve dynamical problem of a nonuniform gravitating body, the force function and the moment of inertia should be separated into additive components related to the uniform part of a system and its nonuniformities. Such a separation of the dimensionless form-factors α^2 and β^2 was done by Garcia Lambas et al. [9] with our interpretation [10] by introduction of an auxiliary function

$$\Psi(s) = \int_0^s \frac{(\rho_r - \rho_0)}{\rho_0} x^2 dx\,,$$

where $s = r/R$ is ratio of the running radius to radius of the sphere R; ρ_o the mean density of the sphere of radius r ; ρ_r is the radial density; x is the running coordinate; the value (ρ_r - ρ_o) satisfies $\int_o^R (\rho_r - \rho_o) r^2 dr = 0$ and the function $\Psi(1)=0$.

The function $\Psi(s)$ expresses radial change of mass density of the nonuniform sphere relative to its mean value at the distance r/R. After the variable change by the above function the potential U and kinetic $K=J_p\omega^2$ energies of the nonuniform self-gravitating sphere were expanded into the form[10]

$$U=\alpha \frac{GM^2}{R} = \left[\frac{3}{5} + 3\int_0^1 \psi x dx + \frac{9}{2}\int_0^1 \left(\frac{\psi}{x} \right)^2 dx \right] \frac{GM^2}{R}, \tag{17}$$

$$K=\beta^2 MR^2\omega^2 = \left[\frac{3}{5} - 6\int_0^1 \psi x dx \right] MR^2\omega^2. \tag{18}$$

or after reduction

$$U=(\alpha^2{}_0 + \alpha^2{}_t + \alpha^2{}_\gamma) \frac{GM^2}{R}, \tag{19}$$

$$K=(\beta_0^2 - 2\beta_t^2)MR^2\omega^2, \tag{20}$$

where $\alpha^2{}_0 = \beta_0^2$ and $2\alpha^2{}_t = \beta_t^2$, and subscripts o, t, γ mean the radial, tangential and dissipative components of the considered values.

Because the potential and kinetic energies of a uniform body are equal ($\alpha_0^2=\beta_0^2$=3/5) , then

$$U_0=K_0 , \tag{21}$$

$$E_0=U_0+K_0 = 2U_0 . \tag{22}$$

From the same Eqs.(17)-(18) the dynamical equilibrium for the interaction between uniform part and nonuniformities (tangential component) are written as

$$2U_t=K_t , \tag{23}$$

$$E_t=U_t+K_t=3U_t, \tag{24}$$

where E_0 , E_t ,U_0 , K_0 , U_t , K_t are the corresponding total, potential and kinetic oscillation and rotation energies of the above interaction.

Eqs.(21)-(24) present the averaged virial theorem for a self-gravitating uniform and nonuniform system, which serves as the condition of their dynamical equilibrium [11]. Evidently, the potential energy U_γ in (19) irradiating from the body's outer shell is irretrievably lost and provides mechanism of the body evolution. In accordance with classical mechanics, for the above considered nonuniform gravitating body, being a dissipative system, the torque N is not equal to zero, the angular momentum L of the sphere is not a conservative parameter, and the energy is continuously spent during the motion, i.e.

$$N = \frac{dL}{dt} > 0, \ L \neq const., \ E \neq const. > 0.$$

Physically "a system cannot be conservative if friction or other dissipation forces are present, because $F ds$ due to friction is always positive and integral cannot vanish"[12]:

$$\oint F \cdot ds > 0.$$

After we found that the resultant of the body's gravitational field is not equal to zero and the system's dynamical equilibrium is kept by virial relationship between the potential and kinetic energies, the equations of a self-gravitating body motion can be derived.

Earlier the Jacobi's virial equation was applied to study dynamics of a self-gravitating sphere [8, 10, 13]. Jacobi [7] derived it from the Newton's equations of motion of n mass points and reduced the *n-body* problem to the particular case of one-body task with two independent variables, namely, with the force function and the polar moment of inertia, in the form [6, 8]

$$\ddot{\Phi} = 2E - U \tag{25}$$

where $\Phi = 1/2I$ is the Jacobi function; I is the polar moment of inertia; $E=U+K$ is the total energy; U and K are the potential and kinetic energy.

Eq.(25) represents the energy conservation law and describes, through scalar U and Φ characteristics, the gravitational interaction of n particles ($n \to \infty$) constituting the system. Eq.(25) is also derived from the Euler's equations for continuous medium, and from equations of Hamilton, Einstein, and quantum mechanics [8]. Its time averaging form U=2K at $\ddot{\Phi} = 0$ gives the Clausius' virial theorem. It's known that Clausius deducing the theorem for application to thermodynamics and, in particular, to assessing and designing of the Carnot's machines. As the machines operate in the Earth' outer force field, Clausius introduced the coefficient 1/2 to the term of the kinetic energy, i.e.

$$K = \frac{1}{2} \sum_i m_i v_i^2 .$$

As Jacobi has noted, the meaning of the introduced coefficient was to take into account only the kinetic energy generated by the machine, but not by the Earth gravitational force [7]. That was demonstrated by the work of a steam hammer for driving in piles. The machine just rises up the hammer, but it falls down under the action of the force of gravity. That is why the coefficient 1/2 of the kinetic energy of a uniform self-gravitating body in Eqs. (21)-(22) has disappeared. In the own force field the body moves due to release of the own energy.

Because of two independent variables U and Φ Eq.(25) was used in analytical dynamics mainly for qualitative analysis of stability of gravitating systems [6, 7, 14].

Earlier [8] by means of relation $U\sqrt{\Phi} \approx \text{const}$, an approximate solution of Eq.(25) for a nonuniform body was obtained. Now, after expansion of the force function and polar moment of inertia, Eq.(25) at $U_\gamma=0$ can be written separately for the radial and tangential components. In accordance with (22) and (24) the two equations are as follows

$$\Phi_0 = \frac{1}{2}E_0 - U_0, \qquad (26)$$

$$\ddot{\Phi}_t = \frac{1}{3}E_t - U_t. \qquad (27)$$

Taking into account the relationship (21) and (23) between the potential energy and the polar moment of inertia through the structural coefficients $\alpha_0=\beta_0^2$ and $2\alpha_t=\beta_t^2$, both Eqs. (26), and (27) are reduced to an equation with one variable and have rigorous solution:

$$\ddot{\Phi} = -A + \frac{B}{\sqrt{\Phi}}, \qquad (28)$$

where A and B are constant values.

The general solution of Eq.(28) is [8]

$$\sqrt{\Phi} = \frac{B}{A}\left[1 - \varepsilon\cos(\xi-\varphi)\right], \qquad (29)$$

$$t = \frac{4B}{(2A)^{3/2}}\left[\xi - \varepsilon\sin(\xi-\varphi)\right]. \qquad (30)$$

Here ε and φ are integration constants depending on the initial values of Jacobi function Φ and its first derivative $\dot{\Phi}$ at the time moment t_0; ξ is auxiliary independent variable (here, the time is independent variable); $A=A_0=-1/2E_0>0$; $B=B_0=U_0\sqrt{\Phi_0}$ for radial oscillations; $A=A_t=-1/3E_t>0$; $B=B_t=U_t\sqrt{\Phi_t}$ for rotation of the body.

The expressions for the Jacobi function and its first derivative in the explicit form can be obtained after transforming into the Lagrangia series [8]

$$\Phi = \frac{B^2}{A^2}\left[1 + \frac{3}{2}\varepsilon^2 + \left(-2\varepsilon + \frac{\varepsilon^3}{4}\right)\cos L - \frac{\varepsilon^2}{2}\cos 2L - \frac{\varepsilon^3}{4}\cos 3L + \ldots\right],$$

$$\dot{\Phi} = \sqrt{\frac{2}{A}}\varepsilon B\left[\sin L + \frac{1}{2}\varepsilon\sin 2L + \frac{\varepsilon^2}{2}\sin L\left(2\cos^2 L - \sin^2 L\right) + \ldots\right].$$

Radial frequency of oscillation ω_{or} and angular velocity of rotation ω_{tr} of the shells are [8]

$$\omega_{or} = \sqrt{\frac{U_{or}}{J_{or}}} = \sqrt{\frac{\alpha_{0r}GM_r}{\beta_{or}^2 r^3}} = \sqrt{\frac{4}{3}\pi G\rho_{0r}}, \qquad (31)$$

$$\omega_{tr} = \sqrt{\frac{2U_{tr}}{J_{tr}}} = \sqrt{\frac{2\alpha_{tr} GM_r}{\beta_{tr}^2 r^3} k_{er}} = \sqrt{\frac{4}{3}\pi G\,\rho_{0r} k_{er}}\ , \qquad (32)$$

where U_{0r} and U_{tr} are radial and tangential components of the force function (potential energy); J_{0r} and $J_{tr}=2/3J_{0r}$ are the polar and axial moment of inertia; $\rho_{0r} = \frac{1}{V_r}\int_{V_r}\rho(r)dV_r$; $\rho(r)$ is the law of the radial density distribution; ρ_{0r} is mean density value of the sphere with a radius r; V_r is the sphere volume with radius r; $2\alpha_{tr}=\beta_{tr}^2$; k_{er} is the dimensionless coefficient of the energy dissipation or tidal friction of the shells equal to the shell oblateness.

The relations (29)-(32) express the Kepler's laws of rotation. In the case of uniform mass density distribution the frequency (31) of oscillation of the sphere's shells with radius r is $\omega_{or} = \omega_o = const$. It means that here all the shells are oscillating with the same frequency. Thus, it appears that the only nonuniform systems are rotating.

The oblateness. coefficient k_{er} for the outer shell is determined by Eqs.(31)-(32) and is equal to ratio of the radial oscillation frequency to the angular velocity, i.e.

$$k_e = \frac{\omega_t^2}{\omega_0^2} = \frac{\omega_t^{\ 2}}{\frac{4}{3}\pi G\rho_0}\,.$$

It was found, that in general case of a triaxial (*a, b, c*) ellipsoid with the ellipsoidal law of density distribution, the dimensionless coefficient $k_e \in [0,1]$ is equal to [8]

$$k_r = \frac{F(\varphi, f)}{\sin\varphi} \Big/ \frac{a^2+b^2+c^2}{3a^2},$$

where $\varphi = \arcsin\sqrt{\frac{a^2-c^2}{a^2}}$,$f = \sqrt{\frac{a^2-b^2}{a^2-c^2}}$, and $F(\varphi,f)$ is an incomplete elliptic integral of the first degree in normal Legendre's form.

4. Oscillations and rotation of the Earth

Eq.(31) shows that the body's radial oscillations are not dependant on phase state of the mass and are determined by its mean density. Correctness of Eq.(31) was confirmed by the observation results. By Eq.(31) and by our measurements period of radial oscillation of the Earth's outer shell is 1.4 h [8, 15, 16] and the Sun's one is ~2.8 h [17]. By Eq. (32) period of rotation of the Earth' outer shell is ~ 24 h. Here the coefficient of dynamical oblateness is k_{er} = 1/289.37 which was determined by the satellite measurements [2].

The Earth axis obliquity of the upper shell is determined by ratio of torques (potential energy) of the uniform (equilibrated) and nonuniform (unbalanced) body state

$$Cos\Theta = \frac{\sum N_{og}}{\sum N_g} = \frac{U_o}{U} = \frac{\alpha_o^2}{\alpha^2} = \frac{0.6}{0.66 - \alpha_p} = 0.918, \quad \Theta = 23.5^o, \tag{33}$$

where α_p is the perturbation coefficient of the Sun and Moon which we accept equal to 0.006.

In order to understand physics of the Earth precession and wobbling effects, its mass density differentiation should be clarified.

5. Gravitational mass density differentiation

The Earth has spherical shell structure. In order to understand physics of the gravitational differentiation in terms of the Earth mass density and the nature of the Archimedes and Coriolis forces, let us consider interaction effects between the body envelopes.

It's known from the Newton's theorem of the gravitational interaction between a material point and a spherical layer, that the latter does not affect on the point located inside. To the contrary, the outside located material point is affected by the layer. Roche's tidal dynamics is based on the above theorem. His approach is as follows.

There are two interacting bodies of masses *M* and *m* (Fig.2a). Let $M >> m$ and $R >> r$, where *r* is the radius of the body *m,* and *R* is the distance between the bodies *M* and *m.* Assuming that the mass of the body *M* is uniformly distributed within the sphere of radius *R,* we can write the accelerations of the points *A* and *B* of body *m* as

$$q_A = \frac{GM}{(R-r)^2} - \frac{Gm}{r^2}, \qquad q_B = \frac{GM}{(R+r)^2} + \frac{Gm}{r^2}.$$

The relative tidal acceleration of the points *A* and *B* is

$$q_{AB} = G\left[\frac{M}{(R-r)^2} - \frac{M}{(R+r)^2} - \frac{2m}{r^2}\right] =$$

$$= \frac{4\pi}{3} G\left[\rho_M R^3 \frac{4Rr}{\left(R^2 - r^2\right)^2} - 2\rho_m r\right] \approx \frac{8\pi}{3} Gr(2\rho_M - \rho_m). \tag{34}$$

Here $\rho_M = M / \frac{4}{3}\pi R^3$ and $\rho_m = m / \frac{4}{3}\pi r^3$ are the mean density distribution for the sphere of radius *R* and *r.* Roche's criterion states that the body with a mass *m* is stable against the tidal force disruption of the body *M* if the mean density of the body *m* is at least double that of the body *M* in the sphere of radius *R.* Roche considered the problem of the interaction between two spherical bodies without any interest to their creation history and to haw the forces appeared. We use the Roche's tidal dynamics to assess the stability of a nonuniform spherical envelope.

Apply Roche's dynamics to a spherical layer of radius R and thickness $r = R_B - R_A$ (Fig. 2b). The layer of mass m and mean density $\rho_m = m/4\pi R_A^2 r$ is affected in point A by tidal force of the sphere of radius R_A. The mass of the sphere is M and its mean density $\rho_M = M/\frac{4}{3}\pi R_A^3$. The tidal force in point B is generated by the sphere of radius $R + r$ and mass $M + m$. Then the accelerations of the points A and B are

$$q_A = \frac{GM}{R_A^2} \quad \text{and} \quad q_B = \frac{G(M+m)}{(R_A+r)^2}.$$

The relative tidal acceleration of the points A and B is

$$\begin{aligned} q_{AB} &= GM\left[\frac{1}{R_A^2} - \frac{1}{(R_A+r)^2}\right] - \frac{Gm}{(R_A+r)^2} = \\ &= \left(\frac{8}{3}\pi G\rho_M - 4\pi G\rho_m\right)r = 4\pi G r\left(\frac{2}{3}\rho_M - \rho_m\right), (R >> r.) \end{aligned} \tag{35}$$

Eqs.(34)-(35) give a possibility to understand, in principle, the nature of attraction and repulsion of the body mass and the nature of the Earth geotectonic and earthquake effects.

It follows from the above considered task of tidal acceleration of an outer nonuniform spherical layer, that at $\rho_M \neq \rho_m$ Eq.(35) reveals a mechanism of the gravitational density differentiation of masses. If $\rho_M < \rho_m$, then the shell immerses (is attracted) up to the level of $\rho_M = \rho_m$. At $\rho_M > \rho_m$ the shell floats up to the level of $\rho_M = \rho_m$ and at $\rho_M > 2/3\rho_m$ the shell becomes a self-gravitating one. Thus, in case when the density increases to the sphere's center, which is the Earth's case, then each overlying stratum appears to be in a suspended state due to repulsion by the Archimedes' forces which, in fact, are a radial component of the gravitational interaction forces.

Effect of the gravitational differentiation of mass explain the nature of creation of the Earth's crust and the oceans, geotectonic, orogenic and seismic processes, including the earthquakes. All those phenomena appears to be a consequence of the continuous process of gravitational differentiation of the planet's mass density. We assume that this effect was one of the dominating during creation of the Earth and the Solar system as a whole. For instance, mean value of the Moon density is less then 2/3 of the Earth' one, i.e. $\rho_M < 2/3\rho_m$. If one assume, that this relation was kept during the Moon formation, then, in accordance with Eq. (35), this body has created at earliest stage of the Earth mass differentiation. Creation of the body from the separated shell should be occurred by means of the cyclonic eddy mechanism, which has been proposed in due time by Descartes and which was unjustly rejected. If we take into account existence in the nonuniform mass of the tangential interaction forces, then the above mechanism seems to be realistic [10].

The mean structural form-factor of the axial moment of inertia, determined by the artificial satellite data, is $\beta^2_{\perp}$=0,3315 [4]. Therefore, in accordance with (19)-(20), the polar radius of inertia of the Earth is r_m=3/2$r^{\perp}_m$=$\sqrt{1.5\cdot 0{,}3315R^2}$ =0,70516R=4.493·10^6 m and the radius of gravity is r_g=$\sqrt{\alpha R^2}$ = 0,8164R=5,201·10^{6}m. Thus, Bullen's interpretation of the Earth's density distribution based on seismic data [2] should be reconsidered. Gravitational differentiation of mass density seems to have direct relation to the Earth creation and evolution.

6. Radial density distribution of the Earth

Let us consider the key subject, namely, the radial distribution of the Earth' density, the function of which is the gravitational potential. As it is known, the modern conception on this problem is based on velocity of the longitudinal and transversal seismic waves. Bullen's approach gives the following picture of interpretation of seismic data [2, 4]. The density of the upper crust is 2.7-2.8 g/cm^3 and it increases towards the Earth's center up to ~13 g/cm^3, changing the values by jump along the Moho discontinuity, the upper and lower mantel and the outer and inner core borders. Despite velocity of the longitudinal seismic waves below the upper core border decreases and the transversal waves in the inner core are absent, the density and hydrostatic pressure in the center accepted up to now is maximum. Bullen proposed that interpretation after, as he said, unfortunate approximation of seismic data by a parabolic curve where the density of the core would decrease to the center. This was because Bullen hasn't had an idea about the planet's self-gravitational effects. Now his conception on the density distribution ought to be reconsidered.

We have analyzed the whole range of possible formal density distribution curves calculated by applying parabolic law in the form $\rho_r=\rho_0(ax^2+bx+c)$, (where $x=r/R$; a, b, c are numerical coefficients; ρ_0 is the mean density of the body). The numerical coefficients were taken by applying the following equation of the total body mass

$$M=4\pi\int_0^R r^2\rho(r)dr = 4\pi\int_0^R r^2\rho_0(-a\frac{r^2}{R^2}+b\frac{r}{R}+c)dr = \frac{4}{3}\pi\rho_0R^3(-\frac{3}{5}a+\frac{3}{4}b+c),$$

where the term $-\frac{3}{5}a+\frac{3}{4}b+c=1$ makes it possible to calculate and plot the curves in the dimensionless form. Fig. 3 shows the curves starting from a linear one with its maximum in the body geometrical center (curve 1) up to linear relationship with the maximum on the surface (curve 7). Curves 1-7 cross the mean density line 10 within the shell of the inertia radius $r_m=0.775R$, which coincides here with the radius of gravitation r_g. The spectrum of curves should reproduce a picture of the mass density redistribution connected with its differentiation during the Earth's history. It follows from (35), that during creation of the Earth its outer stratum of mass should have higher density than the mean density of the underlying layers. Otherwise, farther creation of the body would be broken off. Hence, the possible initial density distribution of the created Earth seems to be defined by the curves 7-8 with density value of ~7-8 g/cm^3. The modern evolutionary stage of the mass density redistribution should be characterized by the curves 5-6 with a density ~ 2-3 g/cm^3 (or lower) at the surface, ~1-2 g/cm^3 in the geometric center and 7-8 g/cm^3 close to the mantel and core border. The density curve, which corresponds to the structural form-factors β^2=0.49725, ($\beta^2_\perp$=0.3315) of the polar and axial moments of inertia and α=0.6601 determined by the artificial satellite orbits, is situated within the same ranges.

7. Radial distribution of the force function and the gravity force

The curves of radial distribution of the gravitational potential and the gravitational force (pressure) for a test mass $m=1$ are calculated using the same density distribution equations. Fig. 4a,b shows these curves in the dimensionless form. The calculation was done with the aid of equations known in the gravitational theory [6]

$$U(r)=\frac{4\pi G}{r}\int_0^r r_i^2\rho(r_i)dr_i+4\pi G\int_0^R r_i\rho(r_i)dr_i=\frac{GM}{R}\left[\frac{3}{20}a\frac{r^4}{R^4}-\frac{1}{4}b\frac{r^3}{R^3}-\frac{1}{2}c\frac{r^2}{R^2}-\frac{3}{4}a+b+\frac{3}{2}c\right],$$

$$q(r)=-\frac{4\pi G}{r^2}\int_0^r r_i^2\rho(r_i)dr_i=-\frac{GM}{R^2}\left[-\frac{3}{5}a\frac{r^3}{R^3}+\frac{3}{4}b\frac{r^2}{R^2}+c\frac{r}{R}\right].$$

Fig. 4a and 4b show that the gravitational potential in the body center is of a maximum value, whereas its first derivative (the gravitational force) is equal to zero. It means that the force pressure in a body increases from center to the surface shell.

Four additional curves of the density distribution, which satisfy $\beta^2_\perp$=0.3315, are presented in Fig. 5. The integral values of the motion and their radial, tangential and dissipative components were calculated (Table II) using the equations of the gravitational theory [6].

Taking into account the tidal condition (35) we come to the conclusion that curve 4 in Fig. 5 and curve 6 in Fig. 4 present in first approximation the Earth's radial density distribution, force function and force distribution. In this case the considered distribution of the Earth's density and force field allows us to assume that the planet really has liquid outer core, as it is discussed in literature [2], and even gaseous inner core . But the liquid mass should have density in the center close to 1g/cm^3 and force pressure of about 1 kg/cm^2 (see Fig. 5, line 4).

Table II. Physical and dynamical characteristics of the Earth used for plotting the radial density distribution curves presented in Fig. A5,

Number of curve	1	2	3	4
ρ_s, g/cm^3	2.76	2.08	1.65	1.03224
ρ_c, g/cm^3	13.8	10.455	6.315	1.6284
ρ_{max}, g/cm^3 /km	13.8 / 0	10.455 / 0	8.26 / 2096	8.57 / 3122
$\beta^2_\perp$	0.33(3)	0.3315	0.3315	0.3315238
β^2	0.50	0.49725	0.49725	0.49725858
β^2_t	0.10	0.10275	0.102752	0.10 2714
α	0.6607142	0.6607374	0.6607374	0.660143
α_t	0.05	0.05	0.0513714	0.0513571
α_γ	0.0107142	0.009366	0.009366	0.0087859
r_g , km	5178.6	5178.7	5178.6	5176.4
r_m , km	4504.9	4492.6	4492.6	4492.7

Specification: ρ_s , ρ_c , ρ_{max} are the densities at the surface, center and maximal value; $\beta^2_\perp$, β^2, β^2_t are the form-factors of the axial, polar and tangential parts of the kinetic energy; α, α_t , α_γ are the form-factors of the polar, tangential and dissipative parts of the potential energy; r_g , r_m are the reduced gravitational and inertial radii.

8. Precession and wobbling of the axis

It is observed by the seismic data that there are jump changes in density at the borders of the lithosphere and upper mantel (~350-400 кm), the upper and lower mantel (~ 1000 km), the lower mantel and outer core (2700-2900 кm) and between the outer and inner core (5400-6370 кm). In accordance with Eq. (32) the above borders can be considered as surfaces of the shells angular velocity and the oblateness change. It is obvious, that by Eq. (33) integral effect of the shells rotation appears in precession of the planet's axis, which also reflects rotation of the ellipsoids of inertia and gravitation. The upper and lower mantel having majority mass of the planet should make the main contribution to that effect. The inner core having zero velocity of the transverse seismic waves represents a liquid or gaseous creature with small density and law pressure (~1.5 bar). The observed daily rotation of the Earth relates to an upper shell, which seems to extend up to the Moho discontinuity (350 km).

The Earth crust and oceans are "floating" in suspended state on the underlying upper shell. It follows from Eqs. (29)-(32), that the shell is rotating according to the Kepler's lows. It means that during half turn of each turnover the angular velocity of the shell accelerates and the second half turn it is slowing down. At the same time the crust and the oceans, being in suspended state and because of inertial effect of their mass, are damping at the shell acceleration and, vice versa, they accelerates during the shell is slowing down. These effects of inertial wobbling of the Earth's crust and oceans are observed as daily and half daily nutation. Analogously the Earth's outer force field accelerates and slows down the Moon's orbital motion and initiates the biweekly and monthly nutation of the Earth's axis. There is the same effect in the Earth orbital motion about the Sun. It gives annual and semiannual and the Chandler 420 day wobble. The last effect is proportional to 27 days period of the Moon and the Sun rotation and is equal to $365/27 \cong 14$ month.

The inertial wobble of the Earth's crust and oceans with their reverse acceleration and damping is known in geophysics as the Coriolis' effect. In addition to the wobbling it also explains the plate tectonics, ocean's currants, irregularity in the Earth rotation, short periodicity in the wether and climate change and synchronism in wobbling and tidal effect.

The Sun and the Moon have definitely the same nonuniform (shell) structure. By the same reason their shells axes precess accordingly. If so, the Moon's precession effect gives the major Earth's axis nutation with period of 18.6 yr. The precession of the Sun's upper shell gives effect of the planet's orbit rotation.

That is the nature of the Earth's axis precession and wobbling (nutation) following from dynamics of the of self-gravitating planet.

9. Conclusion

It follows from the presented solution that Mac Millan's two component model [18] for explanation of the nature of interaction potential energy and Coimmi's approach [19-21] in deriving Hubble's flow from interaction of the homogeneous substratum and angular momentum from interaction of the heterogeneities with homogeneous substratum are found successful application in celestial body dynamics and geophysics in the framework of classical mechanics methods. Solution of the problem of dynamics of a self-gravitating body has great number of new applications in the Earth and environmental sciences.

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Captions

Fig. 1. Separation of the Earth moment of inertia from its relative motion in the solar force field.

Fig. 2. Roche's tidal forces for two bodies (a) and for nonuniform spherical layer (b).

Fig. 3. Formal parabolic curves of the Earth radial density distribution.

Fig. 4. Formal curves of the Earth radial distribution of force function (a) and gravity forces (b) for the test mass m=1 at parabolic law of mass density distribution.

Fig. 5. Possible curves of the Earth radial density distribution at the value of form- factor $\beta_{\perp}^2 = 0.3315$ found by analysis of the artificial satellite orbits.

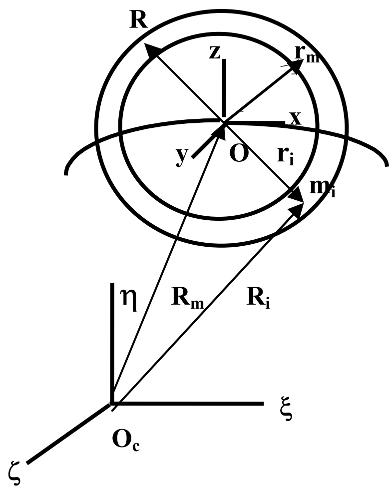

Fig. 1

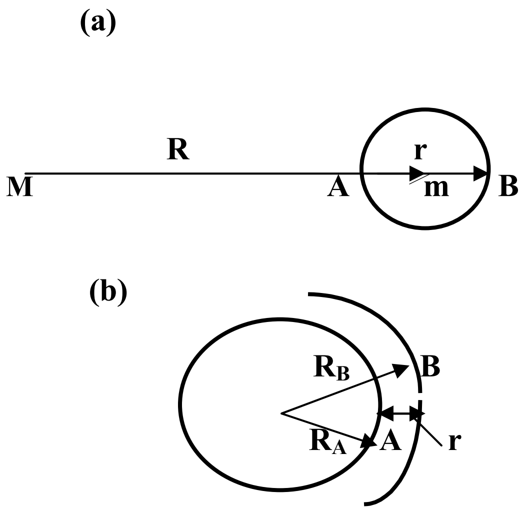

Fig. 2

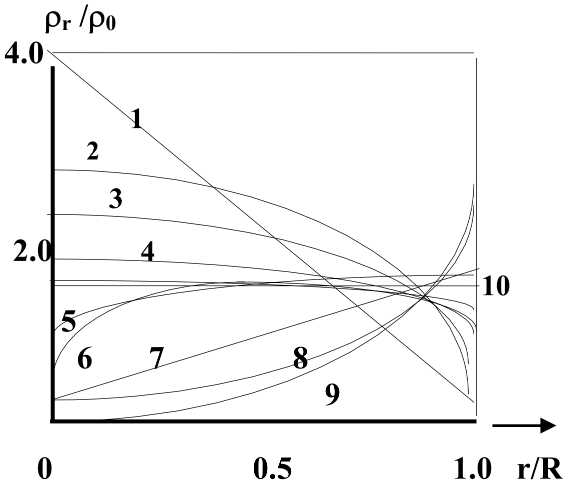

Fig. 3

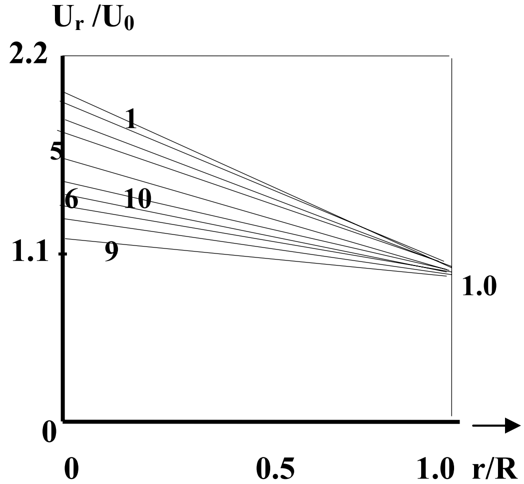

Fig. 4 a

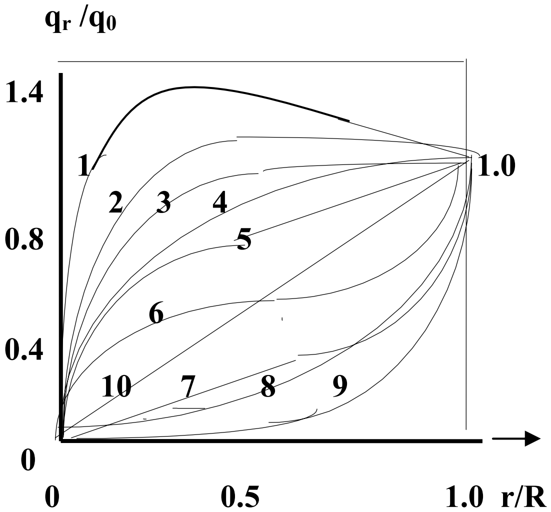

Fig. 4b

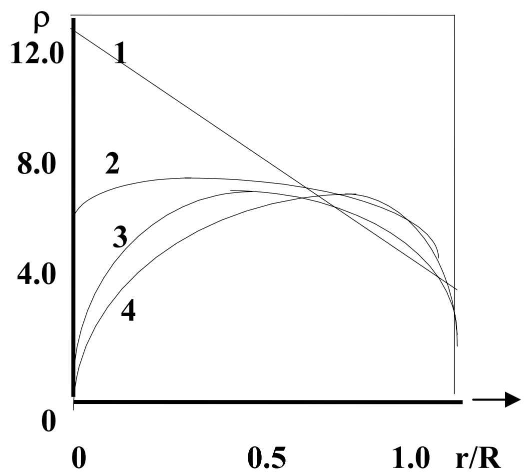

Fig. 5