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Multidimensional Self-Affine Distribution With Application in Geochemistry[1]

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In this paper, we present the conception of the multidimensional self-affine distribution and show that the multidimensional self-affine distribution possesses the fractal property of scale-invariance under truncation, which means that theoretical study of fractals has expanded from univariate cases to multivariate cases. Application of the multidimensional self-affine distribution is illustrated by means of geochemical Au and Ag elements data sets. The fractal dimension is a parameter which can quantitatively explain the variation of geochemical elements data on some orientation. This method is applied to Au data and Ag data, but also suited for other geochemical elements data or geological data. Theory of multivariate fractal can be applied for the study of change courses of fractal system, that is, fractal dynamics.

KEY WORDS: fractal, fractal dimension, scale-invariance, power law, variogram.

INTRODUCTION

The term fractal was founded by the Mathematician Mandelbrot (1983). A fractal is an object made of parts similar to the whole in some way, either exactly the same except for scale or statistically the same. The character of self-affine fractal is anisotropic with varying fractal body, that is, different orientations have different scale factors, while self-similar fractal is a special case of self-affine fractal, that is, different orientations have the same scale factor.

The chaos dynamic mechanism of enrichment or migration of geochemical elements in the crust and the interaction of nonlinear process in geological environment may be an essential cause of uneven distributions of geochemical elements, which results in fractal structure of geochemical elements, ore reserves, and their spatial distributions. Many geological phenomena possess scale similarity

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(scale-invariance) properties. They include rock fragments, faults, earthquakes, volcanic eruptions, mineral resources, and oil pools.

The concept of fractals has been developed and applied for the study of the spatial distribution of physical and chemical quantities and entities with properties of self-similarity or statistical self-similarity (Agterberg, 1994; Cheng, 1997a,b; Cheng and Agterberg, 1996a,b; Feder, 1988; Mandelbrot and Evertsz, 1991; Schertzer and Lovejoy, 1991; Stanley and Meakin, 1988). Multifractal models also have been used in geoscience to characterize various geological and geophysical quantities such as gold concentration values in rocks, and density of mineral deposits in a mineral district (Agterberg, Cheng, and Wright, 1993; Cheng and Agterberg, 1996a,b; Cheng, Agterberg, and Ballantyne, 1994). Meng and Zhao (1991) considered that geochemical data (for example Au data or Ag data) have fractal structure. Cheng, Agterberg, and Ballantyne (1994) studied separation of geochemical anomalies from background by fractal methods.

The current paper consists of four sections: (1) review of theory of multifractals; (2) definition of a self-affine fractal; (3) multidimensional self-affine distribution; and (4) this method and example of geochemical Au and Ag elements date sets in Shangdong province, People's Republic of China.

REVIEW OF THEORY OF MULTIFRACTALS

The concept of multifractals has been developed and applied recently in physics and chemistry where this approach was shown to be useful for the study of the spatial distribution of physical and chemical quantities with geometrical support. For a continuous random variable this support may consist of cells created by a partitioning of k-dimensional space R^k ($k = 1, 2,$ or 3). Suppose that each cell is regarded as a set. Then the measurements are assumed to be the intersections of the sets for the cells with a set of spatial objects; for example, points, line segments, or area objects (polygons in R^2).

Multifractals are intertwined (spatially) fractals with a continuous spectrum of fractal dimensions (Agterberg and others, 1996; Evertsz and Mandelbrot, 1992; Feder, 1988; Hentschel and Procaccial, 1983; Stanley and Meakin, 1988). From the multifractal model, fractal models can be derived which may have different fractal dimensions including the box-counting, information, and correlation or cluster dimensions. In general, different fractal sets, defined on the same complex geometrical pattern with the property of self-similarity, can be interrelated by multifractal theory.

Suppose $\mu(\mathrm{S})$ represents the measure of a set S in R^k, $k = 1, 2,$ or 3. The space R^k can be subdivided into cells of the same linear size ε (equal intervals in R^1, squares of side ε in R^2). The measure $\mu(\mathrm{S})$ can be regarded as the total mass or the combined length of all line segments considered. The study region may be subdivided into square cells with length of side ε; local fracture intensity can be

represented by the measure $\mu_i(\varepsilon)$ on the intersection of S with the ith cell of size ε. The partition function $\chi_q(\varepsilon)$ of order q for cell size ε (Evertsz and Mandelbrot, 1992) is defined as

$$\chi_q(\varepsilon) = \sum_{i=1}^{N(\varepsilon)} \mu_i^q(\varepsilon), \tag{1}$$

where $N(\varepsilon)$ is the total number of cells of size ε. If the measure $\mu_i(\varepsilon)$ satisfies the multifractal model, the partition function of Eq. (1) has a simple power-law relation with cell size ε for any $\chi_q(\varepsilon)$ with $-\infty \leq q \leq \infty$, or

$$\chi_q(\varepsilon) \propto \varepsilon^{\tau(q)}, \tag{2}$$

where $\propto$ represents proportionality, and $\tau(q)$ is the mass exponent of order q. The value of q is not an integer number necessarily. Equation (2) implies approximate self-similarity (scale-independence) for $\mu_i(\varepsilon)$ and its spatial variability.

The box-counting dimension can be derived from $\chi_0(\varepsilon)$, which represents the total number of cells with size ε, and is equal to $-\tau(0)$. Because the total mass in the study region should be independent of ε, or $\mu(\mathrm{S}) = \chi_1(\varepsilon)$, it follows that $\tau(0) = 0$ (principle of conservation of total mass). If the assumption of multifractality is satisfied, the log–log plot $\chi_q(\varepsilon)$ versus ε consists of a number of different straight lines, one for every value of q used. The slopes of these lines would be obtained also for a simple fractal model, but then all values of $\tau(q) = -\tau(-q + 2)$ for any value of q.

If ε is small, the relation $\mu(\varepsilon) = \varepsilon^{\alpha}$ representing self-similarity of the measure itself can used to define the singularity exponent α in the immediate vicinity of any point in the study area. All small areas with the same value of α form a fractal set with dimension $f(\alpha)$. Evertsz and Mandelbort (1992) discuss that, for multifractals with self-similar patterns of change, α is a function of q, and be obtained from

$$\alpha(q) = \frac{d\tau(q)}{dq}. \tag{3}$$

For a simple fractal, $\alpha(q)$ would be the same for all values of q. The multifractal spectrum $f(\alpha) = f\{\alpha(q)\}$ follows from

$$f(\alpha) = q\alpha(q) - \tau(q). \tag{4}$$

The function $f(\alpha)$ can be interpreted as the negative of the Legendre transform of $\tau(q)$ (Evertsz and Mandelbrot, 1992). As mentioned already, each point along the curve for the spectrum $f(\alpha)$ represents the fractal dimension of a subset of S with approximately the same singularity exponent α. The maximum value $f_{\max}\{\alpha(0)\} = -\tau(0)$, which is reached for $q = 0$, corresponds to the box-counting fractal dimension. For a two-dimension set, $f_{\max}\{\alpha(0)\} \leq 2$. The value $f\{\alpha(1)\}$ for $q = 1$ is the entropy dimension. The clustering of separate correlation of

continuous random variables for cells are determined primarily by the second-order mass exponent $\tau(2)$. The correlation dimension $f\{\alpha(2)\}$ follows from $\alpha(2)$ and $\tau(2)$.

DEFINITION OF A SELF-AFFINE FRACTAL

A statistically self-similar fractal is by definition isotropic (Turcotte, 1997). In two dimensions defined by x and y coordinates the results do not depend on the geometrical orientation of the x- and y-axes. A formal definition of a self-similar fractal in a two-dimensional xy-space is that $f(rx, ry)$ is statistically similar to $f(x, y)$ where r is a scaling factor. This is quantified by applications of the fractal relation

$$N = Cr^{-D}, \tag{5}$$

where r is a characteristic linear dimension, $D =$ fractal dimension (real number > 0), $C =$ constant of proportionality (prefactor parameter), $N = N(\geq r)$ (number of objects with characteristic linear dimension $\geq r$). The number of boxes with dimensions x_1, y_1 required to cover a rocky coastline is N_1; the number of boxes with dimensions $x_2 = rx_1, y_2 = ry_1$ required to cover a rocky coastline is N_2. If the rocky coastline is a self-similar fractal, we have $N_2/N_1 = r^{-D}$.

A formal definition of a self-similar fractal in a n-dimensional $x_1 \cdots x_n$-space is that $f(rx_1, \ldots, rx_n)$ is statistically similar to $f(x_1, \ldots, x_n)$ where r is a scaling factor.

A statistically self-affine fractal is not isotropic. A formal definition of a self-affine fractal in a two-dimensional xy-space is that $f(rx, r^H y)$ is statistically similar to $f(x, y)$ where H is known as the Hausdorff measure; In the box-counting method square boxes become more and more rectangular as their size is increased.

A formal definition of a self-affine fractal in a n-dimensional $x_1 \cdots x_n$-space is that $f(r^{d_1}x_1, \ldots, r^{d_n}x_n)$ is statistically similar to $f(x_1, \ldots, x_n)$ where $d_i (i = 1, \ldots, n)$ are constants.

MULTIDIMENSIONAL SELF-AFFINE DISTRIBUTION

N-Dimensional Self-Affine (I-Type) Distribution

The random variables $X_1, X_2, \ldots, X_n$ have the n-dimensional self-affine (I-type) distribution if their probability density function is of form

$$f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = \prod_{j=1}^{n} a_j k_j^{a_j} x_j^{-(a_j+1)}$$

$$(a_j > 0;\ x_j > k_j > 0;\ j = 1, 2, \ldots, n), \tag{6}$$

where $a_j (j = 1, 2, \ldots, n)$ are shape parameters and $k_j (j = 1, 2, \ldots, n)$ are scale parameters.

The n-dimensional self-affine (I-type) cumulative distribution function of random variables $X_1, X_2, \ldots, X_n$ is

$$\begin{aligned} F_{X_1,\ldots,X_n}(x_1, \ldots, x_n) &= P(X_1 \le x_1, \ldots, X_n \le x_n) \\ &= \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} f_{X_1,\ldots,X_n}(x_1, \ldots, x_n)\, dx_1 \cdots dx_n \\ &= \prod_{j=1}^{n} \left[1 - \left(\frac{x_j}{k_j} \right)^{-a_j} \right] \\ &\qquad (a_j > 0;\ x_j > k_j > 0;\ j = 1, 2, \ldots, n), \quad (7) \end{aligned}$$

As a consequence of (7), the complementary cumulative distribution function of random variables $X_1, X_2, \ldots, X_n$ is

$$\begin{aligned} P(X_1 \ge x_1, \ldots, X_n \ge x_n) &= \int_{x_1}^{+\infty} \cdots \int_{x_n}^{+\infty} f_{X_1,\ldots,X_n}(x_1, \ldots, x_n)\, dx_1 \cdots dx_n \\ &= \prod_{j=1}^{n} \left(\frac{x_j}{k_j} \right)^{-a_j} \\ &\qquad (a_j > 0;\ x_j > k_j > 0;\ j = 1, 2, \ldots, n). \quad (8) \end{aligned}$$

Under lower truncation, that is, $(a_j > 0;\ x_j > k'_j > k_j > 0;\ j = 1, 2, \ldots, n)$, we obtain

$$\begin{aligned} &P(X_1 \ge x_1, \ldots, X_n \ge x_n \mid X_1 \ge k'_1, \ldots, X_n \ge k'_n) \\ &\quad = \frac{\prod_{j=1}^{n} (x_j/k_j)^{-a_j}}{\prod_{j=1}^{n} (k'_j/k_j)^{-a_j}} = \prod_{j=1}^{n} \left(\frac{x_j}{k'_j} \right)^{-a_j} \\ &\qquad (a_j > 0;\ x_j > k'_j > k_j > 0;\ j = 1, 2, \ldots, n), \quad (9) \end{aligned}$$

which is itself a n-dimensional self-affine (I-type) distribution with scale parameter $k'_j (j = 1, 2, \ldots, n)$. Thus, the original n-dimensional self-affine (I-type) distribution is self-similar under lower truncation.

Another important additional property of the conditioned or truncated variable is that it is scale invariant in the sense that its distribution does not depend upon the original scale parameter $k_j (j = 1, 2, \ldots, n)$. The n-dimensional self-affine (I-type) distribution characterizes this property. The proof of this can be based

on the necessary condition of scale-invariance under lower truncation ($a_j > 0$; $x_j > k'_j > k_j > 0$; $j = 1, 2, \ldots, n$) which implies the n-dimensional self-affine (I-type) distribution

$$\begin{aligned} &P(X_1 \geq x_1, \ldots, X_n \geq x_n \mid X_1 \geq k'_1, \ldots, X_n \geq k'_n) \\ &\quad = P(X_1 \geq cx_1, \ldots, X_n \geq cx_n \mid X_1 \geq ck'_1, \ldots, X_n \geq ck'_n) \\ &\qquad\qquad (a_j > 0;\ x_j > k'_j > k_j > 0;\ j = 1, 2, \ldots, n), \end{aligned} \tag{10}$$

where c can be any positive number. The n-dimensional self-affine (I-type) distribution of the rescaled truncation variable is independent of c.

From Eq. (6) we have the marginal density function

$$f_{X_j}(x_j) = a_j k_j^{a_j} x_j^{-(a_j+1)} \quad (j = 1, 2, \ldots, n) \tag{11}$$

and the marginal cumulative distribution function

$$F_{X_j}(x_j) = 1 - \left(\frac{x_j}{k_j}\right)^{-a_j} \quad (a_j > 0;\ x_j > k_j > 0;\ j = 1, 2, \ldots, n). \tag{12}$$

In fact, Eqs. (11) and (12) are the Pareto probability density function and the Pareto distribution function, respectively (Johnson and Kotz, 1970, p. 234 or see Appendix A).

The expectation or mean of the marginal distribution function (12) are

$$E(X_j) = a_j k_j (a_j - 1)^{-1} \quad (a_j > 0;\ x_j > k_j > 0;\ j = 1, 2, \ldots, n), \tag{13}$$

The variance of the marginal distribution function (12) are

$$D(X_j) = a_j k_j^2 (a_j - 2)^{-1} (a_j - 1)^{-2} \quad (a_j > 0;\ x_j > k_j > 0;\ j = 1, 2, \ldots, n). \tag{14}$$

Since $f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = f_{X_1}(x_1) \cdots f_{X_n}(x_n)$ the random variables X_1, $X_2, \ldots, X_n$ are independent variables.

N-Dimensional Self-Affine (II-Type) Distribution

The random variables $X_1, X_2, \ldots, X_n$ have the n-dimensional self-affine (II-type) distribution if their probability density function is of form

$$f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = \prod_{j=1}^{n} a_j k_j^{-a_j} x_j^{a_j - 1} \quad (a_j > 0;\ 0 < x_j < k_j;\ j = 1, \ldots, n), \tag{15}$$

where $a_j (j = 1, 2, \ldots, n)$ are shape parameters and $k_j (j = 1, 2, \ldots, n)$ are scale parameters.

The n-dimensional self-affine (II-type) cumulative distribution function of random variables $X_1, X_2, \ldots, X_n$ is

$$F_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = P(X_1 \le x_1, \ldots, X_n \le x_n)$$
$$= \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} f_{X_1,\ldots,X_n}(x_1, \ldots, x_n)\, dx_1 \cdots dx_n = \prod_{j=1}^{n} \left(\frac{x_j}{k_j}\right)^{a_j}$$
$$(a_j > 0;\ 0 < x_j < k_j;\ j = 1, \ldots, n). \quad (16)$$

The n-dimensional self-affine (II-type) complementary cumulative distribution function of random variable $(X_1, X_2, \ldots, X_n)$ is

$$P(X_1 \ge x_1, \ldots, X_n \ge x_n) = \int_{x_1}^{+\infty} \cdots \int_{x_n}^{+\infty} f_{X_1,\ldots,X_n}(x_1, \ldots, x_n)\, dx_1 \cdots dx_n$$
$$= \prod_{j=1}^{n} \left[1 - \left(\frac{x_j}{k_j}\right)^{a_j}\right]$$
$$(a_j > 0;\ 0 < x_j < k_j;\ j = 1, \ldots, n). \quad (17)$$

Under upper truncation, that is, $(a_j > 0;\ 0 < x_j < k'_j < k_j; j = 1, 2, \ldots, n)$, we obtain

$$P(X_1 \le x_1, \ldots, X_n \le x_n \mid X_1 \le k'_1, \ldots, X_n \le k'_n)$$
$$= \frac{\prod_{j=1}^{n} (x_j/k_j)^{a_j}}{\prod_{j=1}^{n} (k'_j/k_j)^{a_j}} = \prod_{j=1}^{n} \left(\frac{x_j}{k'_j}\right)^{a_j}$$
$$(a_j > 0;\ 0 < x_j < k'_j < k_j;\ j = 1, 2, \ldots, n), \quad (18)$$

which is itself a n-dimensional self-affine (I-type) distribution with scale parameter $k'_j (j = 1, 2, \ldots, n)$. Thus, the original n-dimensional self-affine (II-type) distribution is self-similar under upper truncation.

$$P(X_1 \le x_1, \ldots, X_n \le x_n \mid X_1 \le k'_1, \ldots, X_n \le k'_n)$$
$$= P(X_1 \le cx_1, \ldots, X_n \le cx_n \mid X_1 \le ck'_1, \ldots, X_n \le ck'_n)$$
$$(a_j > 0;\ 0 < x_j < k'_j < k_j;\ j = 1, 2, \ldots, n), \quad (19)$$

where c can be any positive number.

From the above equation we can see that the n-dimensional self-affine (II-type) distribution of the rescaled truncation variable is independent of c.

From Eq. (15) we have the marginal density function

$$f_{X_j}(x_j) = a_j k_j^{-a_j} x_j^{a_j - 1} \quad (j = 1, 2, \ldots, n) \tag{20}$$

and the marginal cumulative distribution function

$$F_{X_j}(x_j) = \left(\frac{x_j}{k_j}\right)^{a_j} \quad (a_j > 0;\ 0 < x_j < k_j;\ j = 1, 2, \ldots, n). \tag{21}$$

In fact above Eqs. (20) and (21) are the power-function probability density function and the power-function distribution function, respectively (Johnson and Kotz, 1970, p. 247 or see Appendix B).

The expectation or mean of the marginal distribution function (21) are

$$E(X_j) = a_j k_j (a_j + 1)^{-1} \quad (a_j > 0;\ 0 < x_j < k_j;\ j = 1, 2, \ldots, n). \tag{22}$$

The variance of the marginal distribution function (21) are

$$D(X_j) = a_j k_j^2 (a_j + 2)^{-1} (a_j + 1)^{-2} \quad (a_j > 0;\ 0 < x_j < k_j;\ j = 1, 2, \ldots, n). \tag{23}$$

Since $f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = f_{X_1}(x_n) \cdots f_{X_n}(x_n)$ the random variables $X_1, X_2, \ldots, X_n$ are independent variables.

The multidimensional self-affine distribution can be applied to the case of anisotropic fractal bodies, that is, different orientations have different scale factors. The vertical coordinate is statistically related to the horizontal coordinate but systematically has a small magnitude. Vertical cross sections of this type are often examples of self-affine fractals.

APPLICATION: GEOCHEMICAL Au AND Ag ELEMENTS DATA

We now apply the variogram method and the multidimensional self-affine (II-type) distribution to geochemical Au and Ag elements data sets in Shangdong province, People's Republic of China. There are 21334 samples of Au and 21334 samples of Ag.

The variogram method is widely used in the determination of the fractal dimension of surfaces (Burrough, 1981) and appears to have properties—in particular its ease of use—which make it a preferred method over spectral analysis (Carr and Benzer, 1991; Klinkenberg and Goodchild, 1992). Although it has been much less commonly applied to strictly linear phenomena, the method is very easy to implement when analyzing self-affine profiles. By sampling a large number of

pairs of points (of differing spacings) along the profile and computing the differences in their vertical values (e.g., z values) the fractal dimension is derived easily from the log–log plot of expected differences in z squared versus distance between the point pairs.

A semivariogram quantifies the spatial correlation of a set of measurements by examining the variability of pairs of measurements in terms of the separation distance. The semivariogram is defined as

$$\gamma(h) = \frac{[E(Z(x) - Z(x+h))^2]}{2}, \tag{24}$$

Where $\gamma(h)$ is the semivariogram at separation or lag distances h, $Z(x)$ (a second order stationary random function) is the value at a point x in the region, $Z(x+h)$ is the value at a distance h from x, and E is the expected value of the quantity in brackets.

A small value of semivariogram indicates that pairs of measurements for a particular separation distance are similar, or have low variability. High values of semivariogram indicate that the values of measurements in the lag pairs are, on average, very dissimilar.

Let $Z(x)$ be a second order stationary random function (for example, $Z(x)$ is a sample or measurement of geochemical elements in a two-dimensional space). We have the following equation

$$\gamma(h) = \frac{[E(Z(x) - Z(x+h))^2]}{2} = Ch^D \quad (h > 0; 0 < D < 2), \tag{25}$$

where C and D are unknown parameters. The function $\gamma(h)$ is also called the fractal variance function, that is, the power semivariogram. In fact, Eq. (25) is a nonlinear regression model with unknown parameters C and D. By means of nonlinear least-square regression analysis (Shen, 1997), we can evaluate the estimates of the parameters C and D from the data sets $(h_1, h_2, \ldots, h_n)$ and $(\hat{\gamma}(h_1), \hat{\gamma}(h_2), \cdots, \hat{\gamma}(h_n))$ which are obtained by experimental semivariograms

$$\hat{\gamma}(h_j) = \frac{1}{2n(h_j)} \sum_{i=1}^{n(h_j)} [Z(x_i + h_j) - Z(x_i)]^2, \quad (j = 1, 2, \cdots, n), \tag{26}$$

where $Z(x_i)$, for $i = 1, 2, \ldots, N$, is a sampling of size N and $n(h_j)$ is the number of pairs of variables at distance $h_j (j = 1, 2, \ldots, n)$ apart.

Estimate $\hat{D}$ is called the generalized fractal dimension (Shen and Zhao, 1998). In fact, the parameter D is a shape parameter $a_j - 1$ of the n-dimensional self-affine (II-type) density function. Fractal dimension D is an index which controls the variation or trend of variable $Z(x)$ on certain orientation. The greater the fractal dimension D, the greater the variation of variable $Z(x)$ for $h > 1$. For example, the variation of thickness of seam is lower than that of the grade of metallic ore deposits.

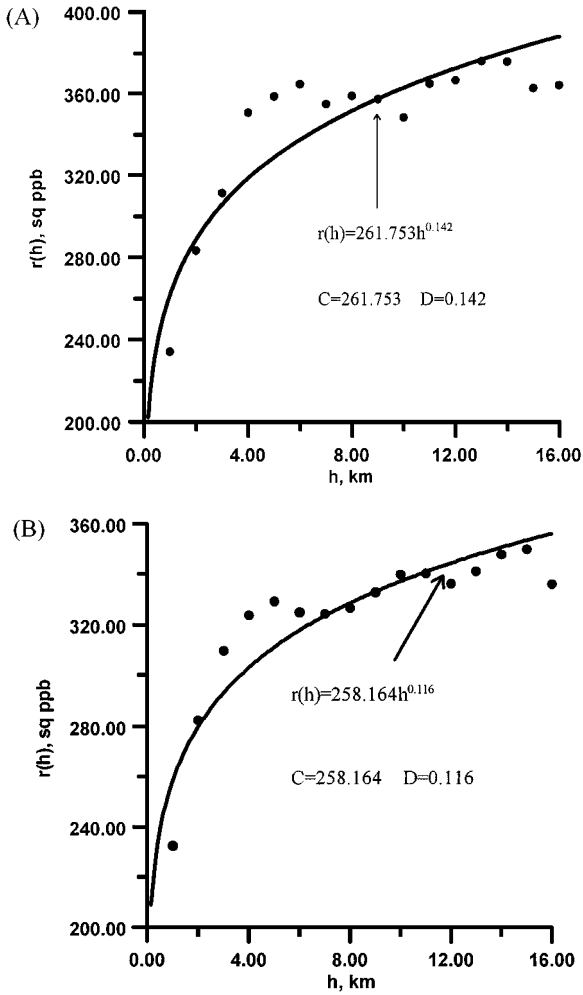

Figure 1. Empirical data is shown as the solid points and a fractal variance function (the power semivariogram) fitting the data shown by the solid line. (A) Plot of fitting a fractal variance function to Au date eastward, (B) Plot of fitting a fractal variance function to Au date northward, (C) Plot of fitting a fractal variance function to Ag date eastward, and (D) Plot of fitting a fractal variance function to Ag date northward.

Figure 1(A)–(D) are plot of fitting a fractal variance function to geochemical Au elements data eastward, northward, geochemical Ag elements date eastward, and northward, respectively, in Shangdong province, People's Republic of China.

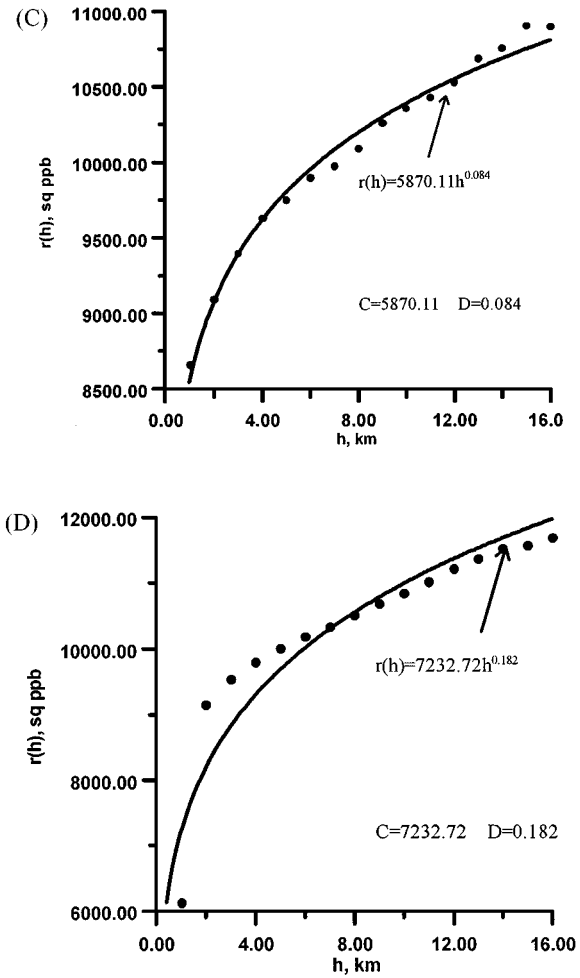

Figure 1. (*Continued*)

Note that orientations above don't require orthogonality (eastward, northward) and may be arbitrary orientations.

The Figure 1(A)–(D) suggests that the fractal dimension ($D = 0.142$) for Au elements data eastward is more than that ($D = 0.116$) of northward, that is, the variation of Au elements data eastward is higher than that of northward, and that the fractal dimension ($D = 0.182$) for Ag elements data northward is more than that ($D = 0.084$) of eastward, that is, the variation of Ag elements data northward is higher than that of eastward.

From Eq. (20) (in two-dimension case) we have

(a) The marginal density function of the 2-dimensional self-affine (II-type) distribution of Au elements data eastward

$$f_{\mathrm{WE}}(x_1) = 0.0473x_1^{0.142} \quad (0 < x_1 < 16); \tag{27}$$

(b) The marginal density function of the two-dimensional self-affine (II-type) distribution of Au elements data northward

$$f_{\mathrm{NS}}(x_2) = 0.0498x_2^{0.116} \quad (0 < x_2 < 16); \tag{28}$$

(c) The marginal density function of the two-dimensional self-affine (II-type) distribution of Ag elements data eastward

$$f_{\mathrm{WE}}(x_1) = 0.0531x_1^{0.084} \quad (0 < x_1 < 16); \tag{29}$$

(d) The marginal density function of the two-dimensional self-affine (II-type) distribution of Ag elements data northward

$$f_{\mathrm{NS}}(x_2) = 0.0437x_2^{0.182} \quad (0 < x_2 < 16). \tag{30}$$

Supposing the variables X_1 (eastward) and X_2 (northward) are independent, from Eqs. (27) and (28) we can obtain the two-dimensional self-affine (II-type) distribution function of Au elements data X_1 (eastward) and X_2 (northward)

$$f_{\mathrm{WE,NS}}(x_1, x_2) = 0.00236x_1^{0.142}x_2^{0.116} \quad (0 < x_1 < 16;\ 0 < x_2 < 16) \tag{31}$$

and from Eqs. (29) and (30) the two-dimensional self-affine (II-type) distribution function of Ag elements data X_1 (eastward) and X_2 (northward)

$$f_{\mathrm{WE,NS}}(x_1, x_2) = 0.00232x_1^{0.084}x_2^{0.182} \quad (0 < x_1 < 16;\ 0 < x_2 < 16). \tag{32}$$

From equations above we can analyze the distribution of geochemical elements data and the variation (or trend) of geochemical elements data in a two-dimensional x_1x_2-space.

CONCLUSIONS

The conception of the multidimensional self-affine distribution (I-type and II-type) is presented in this paper. We prove that the original n-dimensional self-affine (I-type) distribution is self-similar under lower truncation, the

original n-dimensional self-affine (II-type) distribution is self-similar under upper truncation and the n-dimensional self-affine (I-type or II-type) distribution is the distribution which the rescaled truncation variable has a distribution independent of any positive number c. The variogram method and the multidimensional self-affine (II-type) distribution are applied for geochemical Au and Ag elements data sets in Shangdong province, People's Republic of China. The fractal dimension D is an index which can quantitatively explain the variation or trend of variable $Z(x)$ on certain orientation. The method above is applied to Au data and Ag data but also suited for other geochemical elements data or geological data.

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APPENDIX A: THE PARETO DISTRIBUTION AND ITS MOMENTS

The Pareto probability density function of X is

$$f(x) = ak^a x^{-a-1} \quad (a > 0;\ x \geq k > 0), \tag{A1}$$

where a is a shape parameter and k is a scale parameter.

The Pareto distribution function of X is

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx = 1 - \left(\frac{x}{k}\right)^{-a} \quad (a > 0;\ x \geq k > 0). \tag{A2}$$

Provided r is less than a, the rth moment about zero is

$$\mu_r' = \frac{ak^r}{(a-r)} \tag{A3}$$

The expected value or mean of X is

$$E(X) = ak(a-1)^{-1} \quad (a > 1). \tag{A4}$$

The variance of X is

$$D(X) = ak^2(a-2)^{-1}(a-1)^{-2} \quad (a > 2). \tag{A5}$$

APPENDIX B: THE POWER-FUNCTION DISTRIBUTION AND ITS MOMENTS

If X has the probability density function (the Pareto density function)

$$f(x) = ak^a x^{-a-1} \quad (a > 0;\ x \geq k > 0), \tag{B1}$$

then $Y = X^{-1}$ has the density function

$$g(y) = ak^a y^{a-1} \quad (a > 0;\ 0 < y \leq k^{-1}). \tag{B2}$$

The power-function distribution function of Y is

$$G(y) = P(Y \leq y) = \int_{-\infty}^{y} g(y)\,dy = (ky)^a \quad (a > 0;\ 0 < y \leq k^{-1}). \tag{B3}$$

This distribution, which is a special Pearson Type I distribution, is called the power-function distribution. The rth moment about zero is, of course, simply the negative moment of the corresponding Pareto distribution, so that

$$\mu_r' = \frac{ak^{-r}}{(a+r)}. \tag{B4}$$

The expected value is $ak^{-1}(a+1)^{-1}$ $(a > 0)$ and the variance is $ak^{-2}(a+2)^{-1}$ $(a+1)^{-2}$ $(a > 0)$.