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GEOPHYSICS

A New Type of Double-Diffusive Instability

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In paper [1], the title of which differs from that of the present paper by a question mark at the end, Welander suggested the possibility of existence of a new (unstudied) type of convective instability in a two-component medium, in particular, in seawater stratified both in temperature and admixture (salt) concentration. According to this hypothesis, a medium stably stratified by density can, nevertheless, lose its stability not due to the difference in the transport coefficients of heat and admixture (a known mechanism) but owing to the difference in boundary conditions at the horizontal boundary for two substances. The possibility of such an instability would be of significant interest owing to the following reason: unlike the known mechanism (double or differential diffusion), it could also be realized by turbulent exchange, when effective transport coefficients for heat and salt are practically equal. This process corresponds to a greater degree to the real conditions in the upper oceanic layer. However, to our knowledge, the hypothesis mentioned above was neither proved theoretically nor confirmed experimentally. The author of [1] considered a simplified theoretical scheme (inviscid fluid, strongly idealized and strictly fixed boundary conditions, and so on). In the present paper, we perform a sufficiently strict analysis of linear stability with respect to monotonous perturbations in a semibounded problem. We demonstrate the real possibility of appearance of instability, although significantly different from that suggested by Welander [1]. In particular, we found monotonous (rather than the previously supposed oscillatory) instability, which appears during heating from above rather than from below.

We consider semibounded medium layer $z \leq 0$ (the z axis is directed vertically upward) stratified by temperature and admixture concentration (for definiteness, we shall speak about seawater) so that the hydrostatic balance is stable (however, temperature and salinity stratifications can be unstable separately, but the general density stratification is stable).

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The physical idea is as follows. We assume, for example, that at stable temperature stratification and unstable salinity stratification, a volume of medium near the surface would slightly displace upward. Since the density stratification is stable, this volume should apparently obtain negative buoyancy (because it is cooler than the environment) and be subject to the influence of a returning force. However, its buoyancy also depends on the exchange with the environment. If the temperature of the horizontal surface of the medium at $z = 0$ is fixed more strictly than salinity (boundary conditions for the two substances differ), the temperature deviation in the displaced volume considered here would relax faster than salinity perturbation, other conditions being the same. Since the latter perturbation in this case makes a positive contribution to the buoyancy of the volume considered here and is better conserved than the negative temperature perturbation, a principal possibility of the positive feedback is seen. In some respects, such mechanism is similar to the instability caused by double diffusion [2–4], but the effect is related to the difference between the boundary conditions rather than between the exchange coefficients.

According to the usually applied approximation, we suppose that the density of the medium depends linearly on the perturbations of temperature T and concentration of admixture (salinity) s:

$$\rho = \rho_0(1 - \alpha T + \beta s),$$

where ρ_0 is the mean (reference) density of the medium, α is the thermal coefficient of expansion, and β is the coefficient of haline contraction s. The linearized system of dynamics, as well as transport of heat and admixture, in the Boussinesq approximation is written as [2, 3]:

$$\frac{\partial \mathbf{v}}{dt} = -\frac{1}{\rho_0}\nabla p + \nu\nabla^2\mathbf{v} + g(\alpha T - \beta s)\mathbf{e}_z, \quad \nabla\mathbf{v} = 0,$$
$$\frac{\partial T}{dt} + \gamma_T\mathbf{v}\cdot\mathbf{e}_z = \lambda\nabla^2 T, \quad \frac{\partial s}{\partial t} + \gamma_s\mathbf{v}\cdot\mathbf{e}_z = \chi\nabla^2 s. \tag{1}$$

Here, $\mathbf{v}$ is the vector of velocity field perturbation, t is time, p is the pressure perturbation, $\mathbf{e}_z$ is a unit vector in the direction of the z axis, ν is the kinematic coefficient of viscosity, λ is the coefficient of temperature conductivity, χ is the coefficient of admixture diffusion, and g is acceleration due to gravity. As mentioned above, constant values of background vertical gradients for each of the substances γ_T and γ_s are assumed to maintain the hydrostatic stability of the background state [2–4].

In the analysis of the possibility of instability appearance related to surface effects, we shall consider the perturbations attenuating at a distance from the surface at $z \to -\infty$. We neglect the deformations of the fluid surface so that the vertical component of velocity $z = 0$ at surface w turns to zero. We also assume that the following conditions are satisfied at the surface:

$$\frac{\partial u}{\partial z} = 0, \quad \frac{\partial T}{\partial z} = -\frac{T}{h_T}, \quad \frac{\partial s}{\partial z} = -\frac{s}{h_s} \quad \text{at } z = 0. \quad (2)$$

Here, u is the component of velocity perturbation in the direction of horizontal coordinate x (for simplicity, we limit ourselves with a 2D problem, which contains all of the main new results); h_T and h_s are given scales of length.

The formulated stability problem was studied with respect to the monotonous perturbations using the standard method of normal modes. We seek a solution with the following form:

$$w(x, z, t) = W(z)\cos(kx)\exp(\omega t) \quad (3)$$

(similarly for other unknown variables). Excluding all unknown variables from the initial system, except for w, at $\omega = 0$ (keeping in mind the calculation of neutral curves), we obtain the following equation:

$$\left(\frac{d^2}{dz^2} - k^2\right)^3 W = k^6 S W. \quad (4)$$

Here,

$$S = \frac{1}{\nu k^4}\left(\frac{N_T^2}{\lambda} + \frac{N_s^2}{\chi}\right), \quad (5)$$

where $N_T = (\alpha g \gamma_T)^{1/2}$ and $N_s = (-\beta g \gamma_s)^{1/2}$ are thermal and haline buoyancy frequencies (Brunt–Väisälä frequencies). Dimensionless parameter S is an analog and generalization of the Rayleigh number [2, 3] for the case of a two-component medium. However, the parameter includes a horizontal scale of perturbation k^{-1} instead of the fluid layer thickness (which is infinite in the problem considered here). As mentioned above, we consider the situations when the system is stable without account for the surface effects. For example, unstable salinity stratification is excessively compensated by the stable temperature stratification: $\gamma_T > 0$, $\gamma_s > 0$, $N_T^2 > 0$, $N_s^2 < 0$, $N_s^2 + N_T^2 > 0$, and $\frac{N_T^2}{\lambda} + \frac{N_s^2}{\chi} > 0$. The latter inequality is one of the conditions related to the known effects of double (differential) diffusion, which can destabilize the medium even under stable density stratification [2–4]. According to the latter condition, we consider only positive values of parameter S.

We seek the solution of Eq. (4) as a sum of exponents. Taking attenuation into account, the solution for the vertical velocity at $z \to -\infty$ is the sum of three exponents. In the general case, the expressions for the roots of the characteristic equation and the corresponding analysis of stability are somewhat cumbersome. However, in order to demonstrate the main new physical result (possibility of appearance of instability even under an arbitrarily strong hydrostatic stability of the medium), it is sufficient to consider asymptotic $S \gg 1$. In this case, the roots of the characteristic equation are approximately equal to

$$q_1 \approx S^{1/6}, \quad q_{2,3} \approx S^{1/6}\exp\left(\pm\frac{1}{3}\pi i\right),$$

and the solution for vertical velocity can be presented as

$$w \approx C_1 e^{Kz}(e^{Kz} - \cos\sqrt{3}Kz - \sqrt{3}\sin\sqrt{3}Kz)\cos kx.$$

Here, $K = \frac{kS^{1/6}}{2}$, and C_1 is one of the integration constants (two other constants are written using the boundary conditions for w and u). Depth dependence of the solution for the velocity components (as well as for the perturbations of pressure and density of the medium) is a linear combination of three functions: $\exp(2Kz)$, $\exp(Kz)\cos\sqrt{3}\,Kz$, and $\exp(Kz)\sin\sqrt{3}\,Kz$.

The exponents in these functions attenuate with depth at scales H of the order of $K^{-1} \sim (kS^{1/6})^{-1}$. In the considered approximation $S \gg 1$, this is significantly smaller than in the characteristic horizontal scale of perturbations $L \equiv \frac{2\pi}{k}$. The wavelength of sinusoids in the two latter functions mentioned above is of the same order H. It is easy to understand that in the solutions for temperature and salinity perturbations, the three named functions are supplemented with exponent $\exp(kz)$, which decreases with depth significantly more slowly (at the scales of the order of L):

$$T = \frac{\gamma_T}{\lambda k^2 S^{1/3}}[C_2 e^{kz} + C_1 e^{Kz}(e^{Kz} + 2\cos\sqrt{3}Kz)]\cos kx,$$

$$s = \frac{\gamma_s}{\chi k^2 S^{1/3}}\left[\frac{\chi\alpha\gamma_T}{\lambda\beta\gamma_s}C_2 e^{kz} + C_1 e^{Kz}(e^{Kz} + 2\cos\sqrt{3}Kz)\right]\cos kx, \quad (6)$$

where C_2 is one more integration constant. Using boundary conditions (2), we obtain a system of two linear equations for C_1 and C_2. The situation when its determinant turns to zero corresponds to the threshold of monotonous instability. Let us formulate the result: the instability region corresponds to inequality

$$\frac{\alpha\gamma_T\lambda^{-1}}{\beta\gamma_s\chi^{-1}} < \frac{(1+kh_T)(\varepsilon+kh_s)}{(1+kh_s)(\varepsilon+kh_T)} = \frac{1+kh_T}{\varepsilon+kh_T}\frac{\varepsilon+kh_s}{1+kh_s}, \quad (7)$$

where $\varepsilon = \frac{3}{2S^{1/6}}$ is a dimensionless parameter (small in the asymptotic considered here). We note that dimensionless parameters kh_T and kh_s are inverse to the corresponding analogs of the Bio number [5]. In the expression for temperature perturbation (6), amplitudes C_1 and C_2 of the additives, which decrease with depth rapidly and slowly, respectively, are interrelated as

$$C_2 = -2S^{1/6}\frac{\varepsilon+kh_T}{1+kh_T}C_1 = -\frac{3}{\varepsilon}\frac{\varepsilon+kh_T}{1+kh_T}C_1. \quad (8)$$

In particular, at $h_T = 0$ (temperature perturbations turn to zero at the boundary), $C_2 = -3C_1$. This expression depends strongly on the boundary conditions: the fraction in the right part of (7) increases significantly with increasing h_T when parameter kh_T exceeds a small value ε.

At equal boundary conditions for two substances ($h_T = h_s$), the right part of (7) is equal to unity. This corresponds to the known criterion of instability caused by double diffusion in a binary mixture [2–4]. However, if the boundary conditions for heat and admixture differ, condition (7) can be much milder: the instability region can be expanded significantly so that the appearance of instability becomes possible at arbitrary stable density stratification (at $S \to \infty$) even at equal values of the exchange coefficients for heat and salt.

Let us consider the dependence of the obtained criterion on the length scales h_T and h_s. The first fraction in the right part of (7) depends only on h_T and is maximal (equal to ε^{-1}) at $h_T = 0$. The second fraction depends only on h_s and increases monotonously together with this parameter (tends to unity). Thus, the right part of (7) is maximal (tends to a large value ε^{-1}) at $h_T = 0$, $h_s \to \infty$. In other words, the development of instability is most favored by the limiting case, in which one substance is subject to perturbations at the boundary conditions of the first kind (the temperature at the surface $z = 0$ is strongly fixed), while the other substance is subject to perturbations at boundary conditions of the second type. A similar situation is discussed in [1], in which a more general case of the boundary conditions of the third type (2) is not considered. As we shall see (Fig. 1), the solution in the mentioned limiting case is very sensitive to small variations in the boundary conditions $\left(\text{variations of parameter } \delta = \frac{h_T}{h_s}\right)$.

The most "dangerous" mode corresponds to value $k = k_* = \left(\frac{\varepsilon}{h_T h_s}\right)^{1/2}$, at which the right part of (7) reaches the maximal value $\left[\frac{1+\sqrt{\varepsilon\delta}}{\sqrt{\varepsilon}+\sqrt{\delta}}\right]^2$. The horizontal scale of this mode slightly exceeds the geometrical mean of scales h_T and h_s:

$$k_*^{-1} = S^{1/12}\sqrt{\frac{2}{3}h_T h_s}.$$

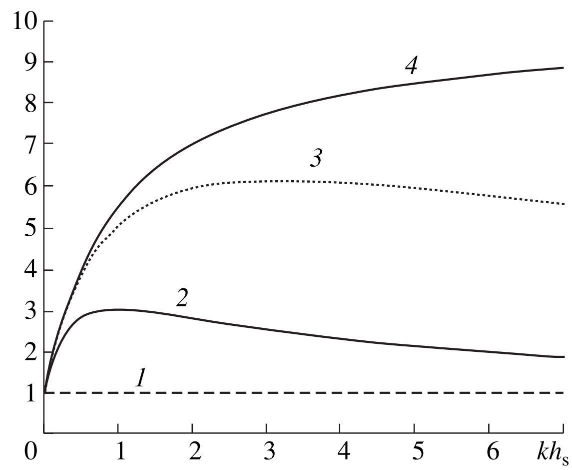

Fig. 1. Right part of inequality (7) as a function of dimensionless parameter kh_s at ε = 0.1 and different values of ratio $\delta \equiv \frac{h_T}{h_s}$: 0.1 (*2*), 0.01 (*3*), 0 (*4*). Dashed line *1* corresponds to equal boundary conditions for two substances (δ = 1). Regions of instability are below the corresponding curves.

Figure 1 shows expansion of the instability region when differences between boundary conditions for the two substances increase (at decreasing dimensionless ratio $\delta \equiv \frac{h_T}{h_s}$ from unity to zero). The abscissa axis corresponds to dimensionless value kh_s. The region below straight line *1* corresponds to the known instability mechanism caused by double diffusion. At $kh_s \to \infty$, curve *4* asymptotically approaches the ε^{-1} value mentioned above.

Figure 2 shows an example of vertical profiles of neutral perturbations for the case of very stable background density stratification (an analog of the Rayleigh number $S = 3 \cdot 10^7$). We consider the case only when salinity makes a small destabilizing contribution to the background stable density stratification ($\varepsilon \approx 0.1$) and the exchange coefficients for the two substances are equal (e.g., effective coefficients of turbulent exchange). Such situation corresponds, for example, to the following values of the parameters: $\alpha = 2 \cdot 10^{-4}$ K^{-1}, $\beta = 0.76 \cdot 10^{-3}$ ‰$^{-1}$, $\gamma_T = 1.5$ K/m, $\gamma_s = 0.04$ ‰ m^{-1}, $\nu = \lambda = \chi = 10^{-3}$ m^2/s, and $k = 0.1$ m^{-1}. In this case, the known mechanisms of instability have no effects. The profiles of temperature and salinity perturbations in Fig. 2 are normalized so that they are related to the buoyancy per-

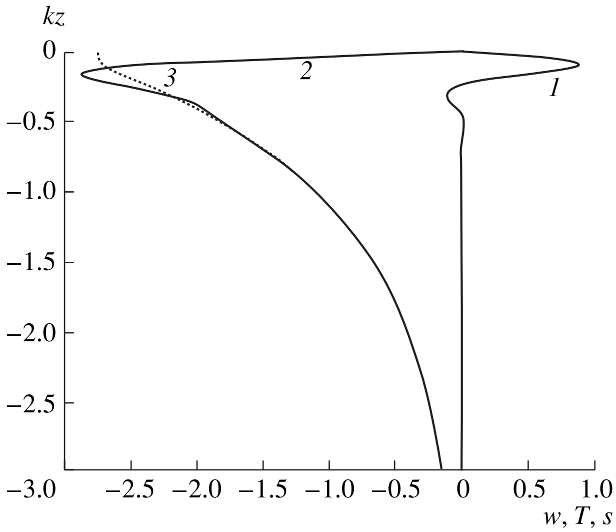

Fig. 2. Example of dimensionless vertical profiles of neutral perturbations at $S = 3 \cdot 10^7$, $kh_T \ll \varepsilon$, $kh_s \gg 1$, $x = 0$. Vertical velocity (curve *1*) is normalized by C_1; perturbations of temperature and salinity (*2, 3*) are normalized by $\dfrac{\gamma_T C_1}{\lambda k^2 S^{1/3}}$ and $\dfrac{\alpha \gamma_T C_1}{\beta \lambda k^2 S^{1/3}}$, respectively.

turbations proportionally to the contributions of these substances. As seen from the figure, owing to the difference between the boundary conditions for perturbations of the two substances ($h_T \ll \varepsilon k^{-1}$, $h_s \gg k^{-1}$), the deviation of buoyancy at the surface of the medium appears positive according to the physical mechanism described above. Thus, upwelling motions should appear at the surface (curve *1*). Below the layer with a thickness of the order of $H \sim K^{-1} \sim (kS^{1/6})^{-1} \ll L$, the signs of the perturbations of buoyancy and velocity change. Neutral perturbations in the velocity field are a vertical series of circulation cells, which attenuate rapidly with depth. The perturbations of buoyancy and pressure attenuate with the same rate (at the scales of the order of H). However, the perturbations of temperature and salinity considered separately attenuate with depth significantly more slowly (at a scale of the order of the horizontal wavelength L). Thus, although the instability found here is related to the boundary effects, the perturbations can generally penetrate sufficiently deep into the medium.

In addition to monotonous instability, the earlier-known mechanisms of double diffusion can lead to the appearance of oscillatory instability (that depends first of all on the relation between the background stratification of the two substances). It is not excluded that an analog of such region of oscillatory instability exists for the mechanism considered above. Welander's hypothesis [1] is to a greater degree related to such version, which strictly has not been studied so far.

Thus, the analysis described above points to the existence of a new type of convective instability of hydrostatically stable binary mixture. Previously, instability of such media was considered possible only when the transport coefficients differ strongly ($\chi \ll \lambda$). However, instead of the known condition $\dfrac{\chi}{\lambda}\dfrac{\alpha\gamma_T}{\beta\gamma_s} < 1$, we obtained in this paper condition (7), which, generally speaking, can be significantly milder. This means that if the transport coefficients are equal, even a weakly unstable stratification of one of the substances ($|N_s| \ll |N_T|$) can, in principle, destabilize a medium stably stratified by density. We also note a principal possibility for manifestation of instability of the type considered here not only in saline water but also in humid air, where the traditionally considered effects of double diffusion are excluded due to virtual equality of the exchange coefficients of heat and moisture.

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