Mathematical Geology, Vol. 38, No. 8, May 2006 (© 2006)
DOI: 10.1007/s11004-005-9020-x

Theoretical Estimation of the Critical Sampling Size for Homogeneous Ore Bodies with Small Nugget Effect[1]

K. Modis[2] and K. Papaodysseus[3]

The aim of this work is to investigate whether it is possible to determine a critical sampling grid density for a given ore body, above which further improvement in the accuracy of the estimated ore reserves would be small or negligible. The methodology employed is based on the theory of information. First, it is proven that the range of influence, when appears in the variogram function, is a measure of the maximum variability frequency observed in the ore body. Then, a simple application of the well-known sampling theorem shows that, under certain assumptions, it is possible to define a critical sampling density as mentioned before. An approximate rule of thumb can then be stated: that critical sampling grid size is half the range of influence observed in the variogram.

KEY WORDS: Geostatistics; sampling density; accuracy of estimation; sampling theorem.

INTRODUCTION

Geostatistics and the probabilistic approach in general, is particularly suited to the study of natural phenomena (Journel and Huijbregts, 1978, p. 2). In the mining field, geostatistics provides a coherent set of probabilistic techniques, which are available to each person involved in the mining project. The acceptance of geostatistics rests on the coherence and effectiveness of the solutions it provides to various problems encountered in practice. This is achieved through an effective utilization of "hard" sample data or, even "soft" probabilistic data in the case of modern spatiotemporal geostatistics (Christakos, 2000, p. 83). In the first case, the minimization of estimation error is assured by the projection of the unknown variable on the space of surrounding data (Journel and Huijbregts, 1978, p. 557), while in the second case one tries to achieve maximization of the expected information.

---

[1]Received 30 July 2003; accepted 1 August 2005; Published online: 31 August 2006.
[2]School of Mining and Metallurgical Engineering, National Technical University of Athens, Zografou Campus, 15 780 Athens, Greece; e-mail: kmodis@mail.ntua.gr.
[3]School of Electrical and Computer Engineering, National Technical University of Athens, Greece.

While highly sophisticated in cases, theoretical estimators are affected by the quality of the sampling campaign. An important question can be asked here on how representative of the mineralization the samples can be. Apart from core recovery and accuracy of chemical analyses that can be improved using appropriate technology, the main parameter affecting the accuracy of estimation is the size of the sampling grid. Up to date, the usual practice for the determination of an optimum grid size is to employ the estimation variance as a criterion of efficiency (Brooker, 1975; 1977; David, 1976; Dowd and Milton, 1987; Dunlop, 1979). One experiments with various grid setups and plots the average estimation variance as a function of sampling density. The optimal grid size is the threshold for which further decrease offers no estimation variance improvement.

The contribution of this work to the above subject is that it calculates the theoretical optimum sampling grid size, sufficient for the accurate representation of the mineralization by its samples. The adoption of this grid size assures the minimum error in the mean square sense. So this work is complementary to the methodology followed up to date. The mathematical approach is based on the theory of information.

RELATED NOTIONS AND THEOREMS FROM INFORMATION THEORY

Originally developed for deterministic electrical signals (Whittaker, 1915; Shannon, 1949), information theory states that a random waveform can be reconstructed by its samples if the sampling rate is greater or equal to a critical value depending on the signal characteristics (Lloyd, 1959). The specification of this rate, called the "Nyquist rate" is done in the frequency domain. The transformation from space to frequency domain is achieved by taking the Fourier transform of the correlation function, which equals to the power spectrum of the random field model of the mineralization.

A necessary and sufficient condition for the existence of a finite Nyquist rate is that the random function representing the mineralization is band limited, meaning that the Fourier transform of its correlation function takes zero values outside a band defined by a minimum and a maximum frequencies. The theorem directly extends to two or more dimensions (Peterson and Middleton, 1962; Jain, 1989, p. 84), usually on a rectangular sampling grid. In this paper, the common class of stationary random field models with isotropic variogram functions with finite or asymptotic range of influence is considered in order to

(a) examine whether the correlation functions are band limited,
(b) check whether the Nyquist rate corresponds to a reasonable sampling lag size.

Note that, even if sometimes the calculated optimal grid size results small, still the obtained value offers a very good estimation of the inaccuracy, should a greater grid size be chosen.

The restriction to stationary models is set, without essential loss of generality, since usual estimation models filter out trend by the application of authorized linear combinations (Matheron, 1970, p. 141).

Also, in the following, centered random field models will be considered for ease of calculations:

$$\mathbf{X}_{\text{centered}}(s) = \mathbf{X}(s) - \overline{\mathbf{X}(s)} \tag{1}$$

where $\overline{\mathbf{X}(s)}$ is the constant mean value of the random field.

LINEAR VARIOGRAM MODEL WITH SILL

The simplest stationary random field model with a range of influence is a homogeneous model represented by a linear variogram model with a sill (Fig. 1a). Since C(h) = C(0) − γ(h) and R(h) = C(h) because of (1), the corresponding correlation function (Fig. 1b) is given by

$$R(h) = \begin{cases} 1 - \frac{|h|}{a} & |h| \le a \\ 0 & |h| > a \end{cases} \tag{2}$$

Taking the Fourier transform of the above function, the power spectrum of the underlying random function model is given by Equation (3)

$$S(\omega) = \frac{4\sin^2\left(\frac{\omega a}{2}\right)}{a\omega^2} \tag{3}$$

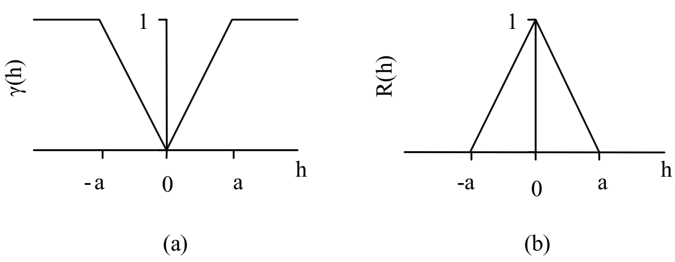

Figure 1. Linear variogram and correlation models with sill.

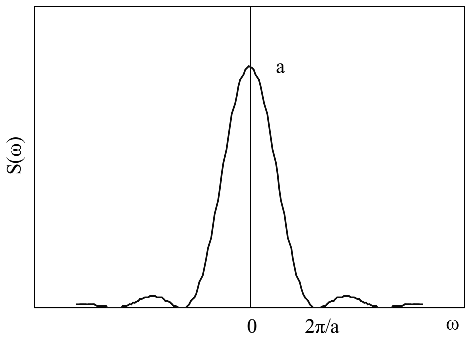

Figure 2. Fourier transform of the linear correlation model.

As seen from the graph in Figure 2, even though the correlation function is not strictly band limited, the value $\omega = 2\pi/a$ could be considered as an approximate vanishing point. But, according to the sampling theorem, the lag size T is

$$T \leq \frac{\pi}{\omega}, \quad \text{or} \quad T \leq \frac{a}{2} \tag{4}$$

By Equation (4), it is apparent that if the mineralization is sampled by a square grid with lag half the size of range of influence, then approximately no information loss accounts and the reconstructed random field is optimal in the mean square sense.

An important thing seen from Figures 1 and 2 is that the value of a priori variance does not affect the value of the vanishing point of the power spectrum, and the size of the optimum sampling grid depends only on the range of influence a.

MODEL WITH ASYMPTOTIC SILL

While the previous model effectively reaches its sill for a finite distance $h = a =$ range of influence, there are variogram models reaching their sill only asymptotically. A good example is the exponential model with corresponding correlation function (Fig. 3) as in (5):

$$R(h) = \exp\left(-\frac{|h|}{a}\right) \tag{5}$$

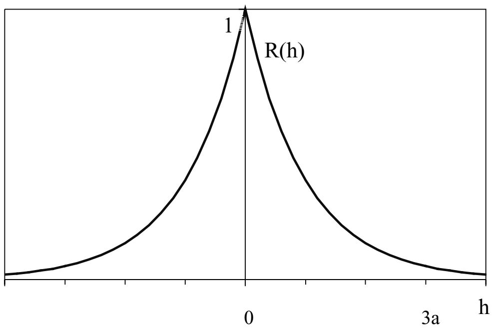

Figure 3. Exponential correlation model with asymptotic sill.

Taking the Fourier transform of (5), the power spectrum of the underlying random function model is given by Equation (6):

$$S(\omega) = \frac{\frac{2}{a}}{\left(\frac{1}{a}\right)^2 + \omega^2} \tag{6}$$

The correlation model in (5) reaches its sill asymptotically, but a good approximation would be (Journel and Huijbregts, 1978, p. 164) to take a range of influence $a_{\mathrm{ap}} = 3a$. For $h = a_{\mathrm{ap}}$, $R(h) = e^{-3} = 0.05 \approx 0$.

As seen again in Figure 4, the correlation function is not strictly bandlimited but it vanishes asymptotically. In analogy to the previous case, taking the value:

$$\omega = 2\pi/a_{\mathrm{ap}} = \frac{2\pi}{3a} \approx 2/a,$$

formula (6) gives

$$S\left(\frac{2}{a}\right) = \frac{\frac{2}{a}}{\left(\frac{1}{a}\right)^2 + \frac{4}{a^2}} = \frac{1}{5}2a$$

But from Figure 4, $2a$ is the maximum value of the spectrum and the point $\omega = \frac{2\pi}{3a}$ could be approximately considered as its vanishing point.

By the sampling theorem, the lag size T is

$$T \leq \frac{\pi}{\omega}, \quad \text{or} \quad T \leq \frac{\pi}{\frac{2\pi}{3a}} = \frac{3a}{2} = \frac{a_{\mathrm{ap}}}{2} \tag{7}$$

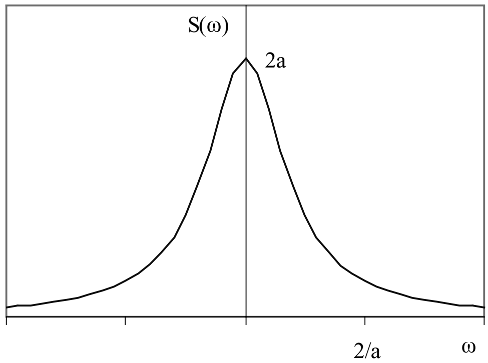

Figure 4. Fourier transform of the exponential correlation model.

By (7), it is seen that, as previously, in the case of an exponential variogram model, the approximative value of half the size of range of influence, still holds for the optimum sampling grid size.

THE CASE OF THE SPHERICAL VARIOGRAM MODEL

The spherical scheme is the most commonly used covariance (variogram) model in mining practice. Its formula is based on the geometric covariogram (Matheron, 1970, p. 86) of a sphere of radius a. The corresponding correlation function (Fig. 5) is given in Equation (8), after being normed to 1, i.e., corresponding to random function with unit a priori variance.

$$R(h) = \begin{cases} 1 - \frac{3|h|}{2a} + \frac{|h|^3}{2a^3} & |h| \leq a \\ 0 & |h| > a \end{cases} \tag{8}$$

Taking the Fourier transform of (8), the power spectrum of the underlying random function model is given by Equation (9):

$$S(\omega) = \frac{3}{a\omega^2} + \frac{12}{a^3\omega^4}\sin^2\left(\frac{\omega a}{2}\right) - \frac{6}{a^2\omega^3}\sin\omega a \tag{9}$$

As seen in Figure 6, the spherical correlation function is not strictly bandlimited but again it vanishes asymptotically. In this figure we can test the “rule of thumb” developed in the previous cases: the value $\omega = 2\pi/a \approx 6$ could be approximately considered as its vanishing point for the spectrum. This “rule of thumb” is

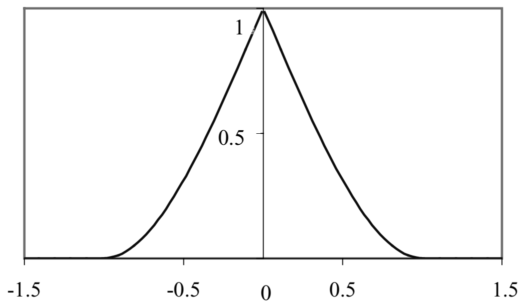

Figure 5. Spherical correlation model normed to 1, with range of influence equal to1.

true for every value of the range of influence "a." Figure 7 gives the power spectrum of the underlying RF as a function of "a." Also, according to the sampling theorem again, the sampling lag size T, as in (4), must be less or equal to half the size of the range of influence.

APPLICABILITY TO REAL WORLD PROBLEMS

The symmetry imposed by the isotropy of the underlying variogram function, set as a prerequisite in the second section of the present work, ensures direct applicability of all theoretical results, regardless of the number of dimensions entered each time. Problems involving nonisotropic variogram models should be

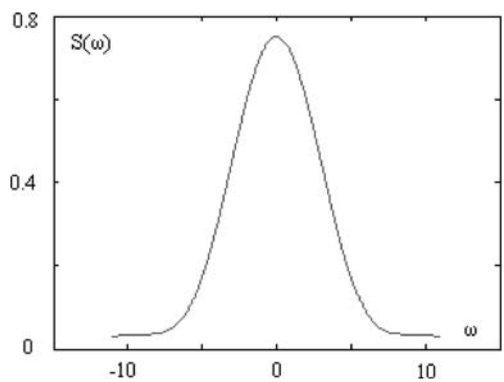

Figure 6. Fourier transform of the spherical correlation model ($a = 1$).

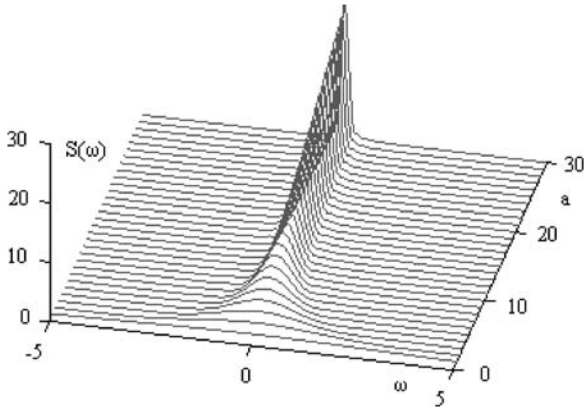

Figure 7. Power spectrum as a function of range of influence "*a*.".

studied as separate problems, while not all the results of the paper may be directly applicable depending on the case.

Local variability is also a critical parameter in practical design of a drilling pattern. The existence of a nugget effect in geostatistics is treated, whether or not, as a noise effect that increases the estimation variance. Every possible microstructure appearing in the form of a nugget effect acts as an error factor that will be finally transferred to the estimation variance. The error is proportional to the ratio between nugget effect and a priori variance. The effect of this noise to the proposed methodology is analogous: If the contribution of local variability to the a priori variance is relatively small (e.g., less than 10%) it can be neglected without much to lose at this early stage of ore estimation process. Besides, the scope of this stage is to calculate the size of the exploration grid that will not leave out any of the structures observed in the variogram. If, on the other hand, the contribution of local variability to the a priori variance is high (e.g., more than 60%) then, by neglecting again the nugget effect, the proposed methodology will still provide the appropriate grid size for the structures reflected on the variogram, but the miner will be aware that the final results of ore reserves estimation using this grid will be less accurate.

APPLICATION IN A COPPER ORE BODY IN CYPRUS

The above theoretical results are validated in the "Phoenix" copper ore body located in Cyprus Island and exploited since 1996 by the Hellenic Copper Mines Ltd (HCM). The HCM hydrometallurgical plant, first of its kind in Europe, has

Figure 8. View of the “Phoenix” mine.

great impact in the development of mining industry in Cyprus. The ore body and the hydrometallurgical plant are located in Skouriotissa area, 50 km west of Nicosia, the capital of Cyprus, near the roots of Troodos Mountain. According to the exploitation program of the mine, a total of 2–3 million tons of ore and 2 million tons of burdens are to be extracted each year. The ore is extracted with the application of open pit mining method in a closed excavation (Fig. 8). The access to the ore is effected through a network of earth haul roads joining the stopes to the heap areas and the mineral processing plant. The ore is selectively extracted with the application of controlled blasting. The purpose of blasting is the relaxation of the rock mass, which is subsequently extracted and transported by mine trucks. After extraction, the ore is fed to the nearby hydrometallurgical plant for further processing. The final product is 99.999% copper.

From the geological point of view, the mineralization belongs to the well-known metalliferous ophiolithic complex of Troodos Mountain. According to the currently accepted theory, the formation of all sulfuric ore bodies in Cyprus is closely related to tectonic grabens. The geological ore reserves are estimated to 20,000,000 tons with 0.4% average copper content (Adamides, 2001; Xydas and others, 2000).

A total of 325 vertical drill holes containing 24528 samples of 1.5 m average length were employed for the estimation of ore reserves. The drill holes are located in a dense, approximately square grid, of average size 15×15 m. Five experimental variograms were calculated in the following directions: North–South, East–West, Northeast–Southwest, Northwest–Southeast and down the hole (DTH). A close examination of these variograms reveals a nonrandom spatial distribution of copper content, since nugget effect is practically zero. Isotropy is also evident in all directions. Finally, a spherical scheme with a sill Co $=$ 0.15 Cu $(\%)^2$ and a range of influence $\alpha = 50$ m was fitted to the experimental data

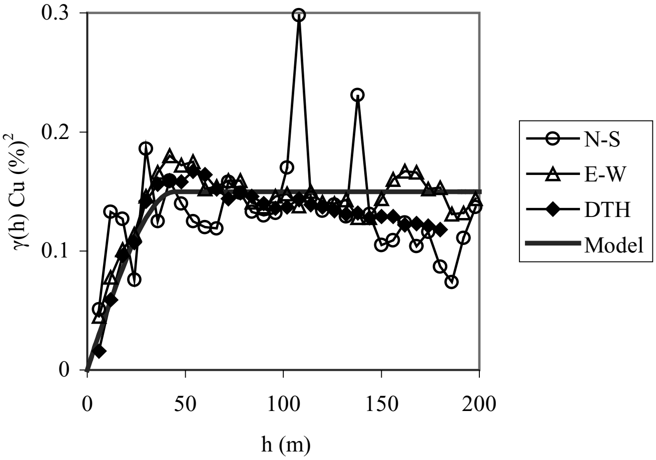

Figure 9. Experimental variograms and fitted model.

(Fig. 9). According to the results of this study, the ore body is over sampled and a reduction of sample density would have a small effect in the results of estimation (Fig. 10).

Initially, ore reserves were estimated using all available samples. The overall block model is a $900 \times 600 \times 350$ m horizontal parallelepiped with center coordinates at -39350E, 10200N and 175 m altimeter. The unit block size is $10 \times 10 \times 3$ m. In order to validate the previous theoretical results, two subsequent estimations were carried out by gradually increasing the average grid size. First, the average grid size was increased to 25 m, which in fact is the critical sampling grid size, by removing 80 properly placed drill holes (245 drill holes were left). The greatest difference observed between this model and the original was negligible (less than 4%). On the contrary, when the grid size was further increased towards 50 m by removing 73 more drill holes (172 drill holes left), the new estimated ore body model differs considerably from the original (up to 27%). The comparison between the results of the three scenarios carried out by plotting the average Cu grade calculated in 27 benches, 3 m high each, is shown in Figure 11. It is apparent that no significant differences occur between estimations in the first two cases (grids 15 and 25 m), where the average sampling grid size is less or equal to half the size of range of influence. The third estimation, though, is clearly of lower quality since it fails to follow local fluctuations of the model. The reason for this discrepancy is that the 50 m grid size exceeds the Nyquist limit. Therefore, the aforementioned experimental results are in agreement with the previous theoretical analysis.

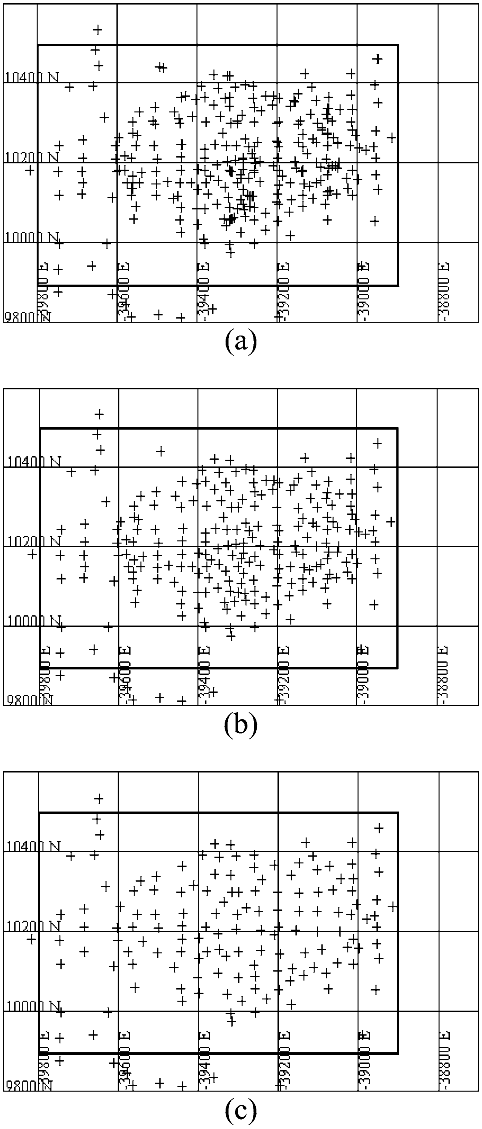

Figure 10. Estimation of ore reserves using different sampling grid sizes: (a) 15×15 m, (b) 25×25 m, (c) 50×50 m.

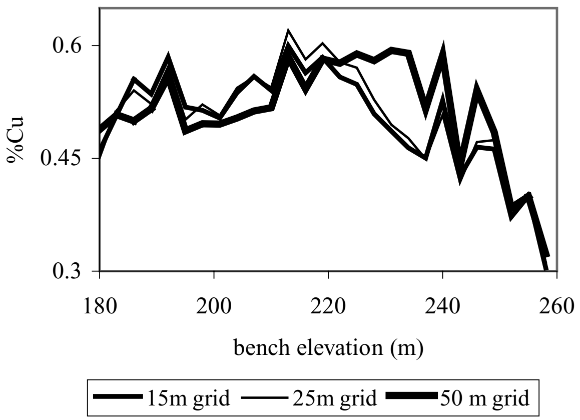

Figure 11. Comparison of results between the three estimation scenarios.

CONCLUSIONS

Calculation of the power spectrum of various representative models of stationary isotropic correlation functions with finite a priori variance shows that the underlying random function models are approximately band limited. Considering the approximate spectrum vanishing frequency ω each time, one can develop a "rule of thumb" for the sampling lag size T, that must be less or equal to half the size of the range of influence a. This approximation rule is developed after the application of the sampling theorem for random functions and assures that the mineralization can be reconstructed by its samples if the sampling rate is greater or equal to the above-specified value.

Therefore, when the prerequisites of Geostatistics are satisfied, then, using the results of the present paper one can apply a practical algorithm in connection with the appropriate sampling grid size:

1. Start with a reasonable sampling density.
2. Compute the corresponding covariance model.
3. Determine the practical upper limit of the spectrum of the model in hand.
4. Using (iii), determine the "optimal" sampling density.
5. If this density is reasonable, repeat the drilling process, if necessary.
6. If the "optimal" density is financially prohibited, clearly perform drilling up to the available funds.

REFERENCES

Adamides, N. G., 2001, The geology of Phoenix with an estimate of available resource: Internal Report, Hellenic Copper Mines Ltd, Nicosia, 26 p.

Brooker, P. E., 1975, Avoiding unnecessary drilling: Proceedings, Australasian Institute of Mining and Metallurgy, No. 253, p. 21–23.

Brooker, P. E., 1977, Block estimation at various stages of deposit development: Proceedings, 14th APCOM Symposium, American Institute of Mining, Metallurgical and Petroleum Engineers, p. 995–1003.

Christakos, G., 2000, Modern spatiotemporal geostatistics: Oxford University Press, Oxford, 288 p.

David, M., 1976, What Happens If?—Some Remarks on Useful Geostatistical Concepts in the Design of Sampling Patterns: Proceedings, Australasian Institute of Mining and Metallurgy, Symposium on Sampling Practices in the Minerals Industry, p. 1–15.

Dowd, P. A., and Milton, D. W., 1987, Geostatistical estimation of a section of the perseverance nickel deposit, *in* Matheron, G. and Armstrong, M., eds., Geostatistical case studies, Reidel, Dordrecht, p. 39–67.

Dunlop, J. S. F., 1979, Geostatistical modeling of an Australian Iron Ore body: Proceedings, 16th APCOM Symposium, American Institute of Mining, Metallurgical and Petroleum Engineers, p. 226–239.

Jain, A. K., 1989, Fundamentals of digital image processing: Prentice Hall, Englewood Cliff, NJ, 569 p.

Journel, A. G., and Huijbregts, Ch. J., 1978, Mining geostatistics: Academic Press, London, 600 p.

Lloyd, S. P., 1959, A sampling theorem for stationary (wide sense) stochastic processes: Trans. Am. Math. Soc., v. 92, p. 1–12.

Matheron, G., 1970, La Theorie des Variables Regionalisees et ses Application: Les Cahiers du Centre de Morphologie Mathematique, Fasc. 5, CGMM Fontainebleau, 212 p.

Peterson, D. P., and Middleton, D., 1962, Sampling and reconstruction of wave number limited functions in N-dimensional Euclidean spaces: Inform. Contr., v. 5, p. 279–323.

Shannon, C. E., 1949, Communications in the presence of noise: Proc. IRE v. 37, p. 10–21.

Whittaker, E. T., 1915, On the functions, which are represented by the expansions of the interpolation theory: Proc. Roy. Soc., Edinburgh, Section A 35, p. 181–194.

Xydas, K., Vattis, D., Georgaki-Ilia, K., Lamprou, V., and Triandafyllou, M., 2000, The use of information technology for the design of an intensive excavation program at the "Phoenix" copper deposit in Cyprus: Proceedings of the 3d Hellenic Conference for the mineral Wealth-Part B, Athens, p. 55–62, (in Greek).