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Response of a sloping aquifer to constant replenishment and to stream varying water level

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Abstract

The problem of seepage from a stream into an adjacent unconfined aquifer of semi-infinite extent, underlain by an impermeable sloping bed was considered in this study as a problem of one-dimensional unsteady-state groundwater flow. It was assumed that the water level in the stream gradually rises to a certain height, according to a known exponential function of time, while the aquifer was assumed to be replenished at a constant rate from ground surface. Applying the Laplace transformation method derived an analytical solution to an extended and linearized form of the nonhomogeneous Boussinesq equation used to describe the phreatic surface in sloping aquifers. The comparison of the analytical solution with a numerical solution obtained by applying the finite difference Mac Cormack explicit computational scheme to the nonlinear Boussinesq equation illustrates the validity of the new analytical solution and the effectiveness of the linearization. Some nondimensional diagrams are also presented to show the variation of the water table height and the seepage rate as well as their sensitivity to various sets of parameter values.

Keywords: Seepage; Sloping aquifer; Semi-infinite aquifer; Analytical solution; Finite differences; Explicit scheme

1. Introduction

Groundwater flow in an unconfined aquifer owing to seepage from an unlined canal or from a natural stream has attracted the attention of many investigators. This is because the volume of the seepage loss from an unlined canal may be significant and this waste of water may cause the water table to rise within the rootzone in the adjacent agricultural land, and as a result reduce the yield. Additionally, in the case of a natural stream, the recharge of the aquifer takes place through the streambed. Any attempt to promote this recharge must be based on the prediction of the seepage rate towards the aquifer and possibly of the rise of the water table, which, as in the previous case, may affect the yield. Under certain hydrogeological situations and boundary conditions, which make the well-known Dupuit–Forchheimer (D–F) assumptions for horizontal flow valid, the unsteady state unconfined groundwater flow can be considered approximately one-dimensional, as described by the nonlinear Boussinesq equation.

Analytical solutions of the Boussinesq equation, as well as numerical computational schemes, which can be applied to agricultural land drainage problems and to modeling the seepage towards unconfined horizontal aquifers, have been presented in the past. Analytical solutions for the seepage problem have been derived by many researchers using various approaches to overcome the nonlinearity of the Boussinesq equation (Polubarinova-Kochina, 1962;

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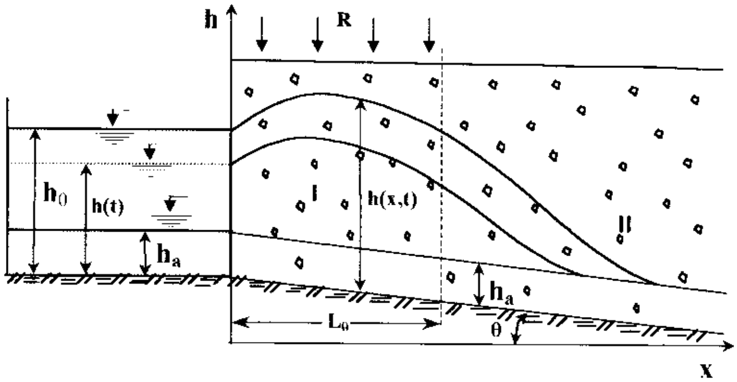


Fig. 1. Seepage towards a sloping unconfined aquifer.

Marino, 1973; Rai and Singh, 1979), though some analytical solutions of the nonlinear Boussinesq equation exist (Lockington, 1997). Upadhyaya and Chauhan (1998) obtained a numerical solution of the nonlinear Boussinesq equation to compare it with some analytical solutions. They found that for practical purposes, the analytical solutions, even those of linearized forms of the Boussinesq equation, may be adopted for predicting water table heights, provided the simplifying assumptions for the linearization are satisfied.

For flow in sloping unconfined aquifers two main approximations have been used in the past. In the first approximation the streamlines are assumed to be horizontal and the hydraulic gradient equal to the absolute slope of the water table. This means that the D–F assumptions are applied without any modification (Polubarinova-Kochina, 1962), leading to a form of the nonlinear Boussinesq equation, also referred to as Boussinesq's second approximation (Chapman, 1980). It is expressed in terms of horizontal and vertical axes with a term, which includes the slope of the impermeable base. By considering the horizontal flow, Polubarinova-Kochina (1962) presented an analytical solution of the linearized homogeneous Boussinesq equation for the seepage towards a sloping unconfined aquifer of semi-infinite extent. For the same problem, Yussuff et al. (1994) obtained a numerical finite difference solution to the nonlinear Boussinesq equation using the unconditionally stable Du Fort-Frankel explicit computational scheme. For the horizontal flow towards drains over a sloping impermeable bed, Shukla et al. (1990) presented a numerical solution of the nonlinear Boussinesq equation, which was found to be reasonably valid for up to 30% slope. Analytical solutions of the linearized homogeneous and nonhomogeneous Boussinesq equation have been also reported by Chauhan et al. (1968), Sewa Ram and Chauhan (1987) and Singh et al. (1991).

According to the second approximation, it is assumed that the streamlines are nearly parallel to the sloping impermeable layer. This assumption was adopted by Childs (1971), who also considered the hydraulic gradient as a function of the slope of the groundwater free surface. Experimental measurements by Jaiswal and Chauhan (1975) verified the validity of Childs' (1971) approximation. Towner (1975) showed that Childs' approximation is in much better agreement with the experimental results than the earlier analysis based on horizontal flow. Chapman (1980) suggested a simplified modification to Childs' (1971) analysis to obtain an equation in terms of horizontal and vertical axes, which we call as an extended form of the nonlinear Boussinesq equation.

It should be emphasized that in all previous work the seepage was assumed to start after an abrupt rise or decline of the water level in the stream. Therefore in this paper to advance this work the influence of a gradually varying flow depth in a stream on the seepage towards an unconfined sloping aquifer of semi-infinite extent is studied by adopting the modification suggested by Chapman (1980). The influence of a constant replenishment over a limited distance from the stream, which is caused by the vertical downward recharge from irrigation or rainfall, is also studied. Applying the Laplace transformation method an analytical solution to a quasi-linear form of the extended Boussinesq equation was derived. A numerical solution of the nonlinear Boussinesq equation was also obtained by applying the explicit and conditionally stable Mac Cormack finite difference computational scheme. The new analytical solution was found to be in good agreement with the numerical solution provided the assumption for the linearization is satisfied. The results we present show the effect of the rate of the water level rising in the stream, the localized recharge rate and the slope of the aquifer on the water table height, and the seepage rate.

2. Mathematical formulation of the seepage problem

The seepage or recharge problem over a sloping impermeable layer is depicted in Fig. 1. The unconfined sloping aquifer is assumed to be homogeneous, isotropic and semi-infinite. The stream penetrates the aquifer depth fully and the stream banks are nearly vertical, having the same hydraulic properties as the aquifer, which are constant. The assumption that the streamlines are nearly parallel to the sloping bed, implying an extended D–F assumption, has been the basis for an extended form of the Boussinesq equation (Chapman 1980), which is a nonlinear, nonhomogeneous partial differential equation, used to describe the one-dimensional unsteady-state groundwater flow. It is written as

$$\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right) - \tan\vartheta\frac{\partial h}{\partial x} = \frac{S}{K\cos^2\vartheta}\frac{\partial h}{\partial t} - \frac{R}{K\cos^2\vartheta} \tag{1}$$

where $h(x, t)$ is the water table height at a distance x from the stream and time t, R the rate of recharge, ϑ the angle of the sloping bed with horizontal, K hydraulic conductivity and S is the specific yield of the aquifer, which are considered to be constant.

Eq. (1) is linearized in terms of h^2 by introducing a weighted mean of the water table height, $\bar{h}$ (Marino, 1973, 1974; Rai and Singh 1979).

Then Eq. (1) is written as

$$\frac{\partial^2 h^2}{\partial x^2} - \frac{\tan\vartheta}{\bar{h}}\frac{\partial h^2}{\partial x} = \frac{S}{K\bar{h}\cos^2\vartheta}\frac{\partial h^2}{\partial t} - \frac{2R}{K\cos^2\vartheta} \tag{2}$$

In Eq. (2) the only simplifying assumption is that $\bar{h}$ is considered constant. This assumption is valid when the change in the water table height, h is small compared with the average height. However, Eq. (2) gives more accurate results when $\bar{h}$ is obtained by successive approximations (Marino, 1973, 1974). In that case $\bar{h}$ is obtained by $\bar{h} = (h_a + h_t)/2$, where h_a is the initial water table height and h_t is the height at time t at the end of which h is to be approximated. It is also considered that the water level in the stream gradually rises to a certain height, according to a known exponential function of time. Moreover, it is considered that the aquifer receives a constant vertical downward recharge from an overlying strip area along the stream having a finite width, L_o. Therefore the two distinguished regions of the aquifer, shown in Fig. 1, are the region of recharge represented by zone I, and the rest of the aquifer represented by zone II. The rate of recharge is assumed to be small compared with the aquifer hydraulic conductivity and therefore the vertically added water is almost refracted in the direction of the slope of the water table.

In the two-zone flow system shown in Fig. 1, we consider Eq.(2) to govern the groundwater flow in the zone of recharge where the water table height is represented by $h_1(x, t)$, while the homogeneous form of Eq. (2) governs the flow in the rest of the aquifer where the water table height is represented by $h_2(x, t)$.

$$\frac{\partial^2 h_1^2}{\partial x^2} - \frac{\tan\vartheta}{\bar{h}}\frac{\partial h_1^2}{\partial x} = \frac{S}{K\bar{h}\cos^2\vartheta}\frac{\partial h_1^2}{\partial t} - \frac{2R}{K\cos^2\vartheta}, \quad 0 < x < L_o \tag{3}$$

$$\frac{\partial^2 h_2^2}{\partial x^2} - \frac{\tan\vartheta}{\bar{h}}\frac{\partial h_2^2}{\partial x} = \frac{S}{K\bar{h}\cos^2\vartheta}\frac{\partial h_2^2}{\partial t}, \quad L_o < x < \infty \tag{4}$$

The corresponding initial and boundary conditions are:

$$h_1^2(x,0) = h_a^2 \tag{5a}$$

$$h_2^2(x,0) = h_a^2 \tag{5b}$$

$$h_1^2(0,t) = [h_o - (h_o - h_a)\,e^{-\lambda_o t}]^2 \tag{5c}$$

$$h_2^2(L_o,t) = h_1^2(L_o,t) \tag{5d}$$

$$h_2^2(\infty,t) = h_a^2 \tag{5e}$$

where h_a is the initial height of the water level in the stream and, according to the assumption that flow is parallel to the sloping bed, is also the initial height of the phreatic surface that is parallel to the sloping bed, h_o is the maximum water level height in the stream, and λ_o is a positive constant that expresses the rate of the water level rising in the stream.

Using the nondimensional variables

$$H_1 = \frac{h_1^2 - h_a^2}{h_o^2 - h_a^2} \tag{6a}$$

$$H_2 = \frac{h_2^2 - h_a^2}{h_o^2 - h_a^2} \tag{6b}$$

$$\xi = \frac{x}{h_o} \tag{6c}$$

$$\tau = \frac{Kt}{Sh_o} \tag{6d}$$

Eqs. (3) and (4) can be written in the following non-dimensional forms

$$\frac{\partial^2 H_1}{\partial \xi^2} - 2\alpha \frac{\partial H_1}{\partial \xi} = \delta \frac{\partial H_1}{\partial \tau} - q \tag{7}$$

$$\frac{\partial^2 H_2}{\partial \xi^2} - 2\alpha \frac{\partial H_2}{\partial \xi} = \delta \frac{\partial H_2}{\partial \tau} \tag{8}$$

where

$$\alpha = \frac{h_o \tan \vartheta}{2\bar{h}}, \tag{9a}$$

$$\delta = \frac{h_o}{\bar{h} \cos^2 \vartheta} \tag{9b}$$

and

$$q = \frac{2Rh_o^2}{K(h_o^2 - h_a^2)\cos^2 \vartheta} \tag{9c}$$

The auxiliary conditions are written as follows:

$$H_1(\xi, 0) = 0, \qquad 0 < \xi < L, \quad \tau = 0 \tag{10a}$$

$$H_2(\xi, 0) = 0, \qquad L < \xi < \infty, \quad \tau = 0 \tag{10b}$$

$$H_1(0, \tau) = 1 - \rho_1 \, e^{-\lambda^2 \tau} + \rho_2 \, e^{-2\lambda^2 \tau}, \quad \xi = 0, \quad \tau > 0 \tag{10c}$$

$$H_2(L, \tau) = H_1(L, \tau), \qquad \xi = L, \quad \tau > 0 \tag{10d}$$

$$\lim_{\xi \to \infty} H_2(\xi, \tau) = 0, \qquad \tau > 0 \tag{10e}$$

where

$$L = \frac{L_o}{h_o} \tag{11a}$$

$$\rho_1 = \frac{2h_o}{h_o + h_a} \qquad \rho_2 = \rho_1 - 1 \tag{11b}$$

and

$$\lambda^2 = \frac{\lambda_o h_o S}{K} \tag{11c}$$

3. Analytical solution

Applying the Laplace transformation method to Eqs. (7), (8) and to the initial and boundary conditions (10a)–(10e), we obtain the solution of the problem as

$$\begin{aligned}
H(\xi, \tau) = {} & \frac{1}{2}\left[\operatorname{erfc}\left(\frac{\xi - 2\frac{\alpha}{\delta}\tau}{2\sqrt{\tau/\delta}} \right) + e^{2\alpha\xi} \operatorname{erfc}\left(\frac{\xi + 2\frac{\alpha}{\delta}\tau}{2\sqrt{\tau/\delta}} \right) \right] \\
& - \frac{\rho_1 \, e^{(\alpha - \sqrt{\alpha^2 - \delta\lambda^2})\xi - \lambda^2 \tau}}{2} \left[\operatorname{erfc}\left(\frac{\xi - \frac{2}{\delta}\sqrt{\alpha^2 - \delta\lambda^2}\,\tau}{2\sqrt{\tau/\delta}} \right) + e^{2\sqrt{\alpha^2 - \delta\lambda^2}\,\xi} \operatorname{erfc}\left(\frac{\xi + \frac{2}{\delta}\sqrt{\alpha^2 - \delta\lambda^2}\,\tau}{2\sqrt{\tau/\delta}} \right) \right] \\
& + \frac{\rho_2 \, e^{(\alpha - \sqrt{\alpha^2 - 2\delta\lambda^2})\xi - 2\lambda^2 \tau}}{2} \left[\operatorname{erfc}\left(\frac{\xi - \frac{2}{\delta}\sqrt{\alpha^2 - 2\delta\lambda^2}\,\tau}{2\sqrt{\tau/\delta}} \right) + e^{2\sqrt{\alpha^2 - 2\delta\lambda^2}\,\xi} \operatorname{erfc}\left(\frac{\xi + \frac{2}{\delta}\sqrt{\alpha^2 - 2\delta\lambda^2}\,\tau}{2\sqrt{\tau/\delta}} \right) \right] \\
& - \frac{q}{2\delta}\left[\left(\tau - \frac{\xi}{2\alpha/\delta} \right) \operatorname{erfc}\left(\frac{\xi - 2\frac{\alpha}{\delta}\tau}{2\sqrt{\tau/\delta}} \right) + \left(\tau + \frac{\xi}{2\alpha/\delta} \right) e^{2\alpha\xi} \operatorname{erfc}\left(\frac{\xi + 2\frac{\alpha}{\delta}\tau}{2\sqrt{\tau/\delta}} \right) \right] \\
& + \frac{q \, e^{2\alpha\xi}}{2\delta}\left(\tau + \frac{L + \xi}{2\alpha/\delta} \right) \operatorname{erfc}\left(\frac{L + \xi + 2\frac{\alpha}{\delta}\tau}{2\sqrt{\tau/\delta}} \right) \\
& - \frac{q}{2\delta}\left(\tau + \frac{L - \xi}{2\alpha/\delta} \right) \left[\operatorname{erfc}(\Omega) - \frac{2\alpha\sqrt{\tau/\delta}}{\Omega} - U(\xi - L)\left(2 - \frac{2\alpha\sqrt{\tau/\delta}}{\Omega} \right) \right]
\end{aligned} \tag{12}$$

where

$$\Omega = \frac{L - \xi + 2\dfrac{\alpha}{\delta}\tau}{2\sqrt{\tau/\delta}} \tag{13a}$$

and

$$U(\xi - L) = \begin{Bmatrix} 0, & \xi < L \\ 1, & \xi > L \end{Bmatrix} \tag{13b}$$

is the unit step function of Heaviside.

The second and third term of the right-hand side of Eq. (12) include the effect of the gradually rising water level in the stream, which is expressed by the parameter λ. These two terms are eliminated for $\lambda \to \infty$, which indicates an abrupt rise in the water level on the stream. The three last terms in Eq. (12) include the effect of the recharge rate, expressed by the parameter R/K, and they are eliminated in the case where the aquifer is not recharged ($R/K = 0$ or $L = 0$).

Based on the D–F assumptions and the assumption that the streamlines are nearly parallel to the sloping bed, the flow rate per unit width, q_o, was obtained by Chapman (1980) as

$$q_o = -Kh_1\left(\frac{\partial h_1}{\partial x} - \tan\vartheta\right)\cos^2\vartheta \tag{14}$$

The nondimensional seepage or recharge rate, $Q(0,\tau)$, at the stream boundary is obtained from Eqs. (12)–(13b) as

$$Q(0,\tau) = \frac{2q_0}{Kh_0}$$

$$= 2\cos^2\vartheta\tan\vartheta\sqrt{1-\left(1-\frac{h_a^2}{h_0^2}\right)(\rho_1\,\mathrm{e}^{-\lambda^2\tau} - \rho_2\,\mathrm{e}^{-2\lambda^2\tau})}$$

$$-\alpha\left(1-\frac{h_a^2}{h_0^2}\right)\cos^2\vartheta\,\mathrm{erfc}(\alpha\sqrt{\tau/\delta})$$

$$+\rho_1\left(1-\frac{h_a^2}{h_0^2}\right)\cos^2\vartheta\,\mathrm{e}^{-\lambda^2\tau}$$

$$\times[\alpha - \sqrt{\alpha^2-\delta\lambda^2}\,\mathrm{erf}(\sqrt{\alpha^2-\delta\lambda^2}\sqrt{\tau/\delta})]$$

$$-\rho_2\left(1-\frac{h_a^2}{h_0^2}\right)\cos^2\vartheta\,\mathrm{e}^{-2\lambda^2\tau}$$

$$\times[\alpha - \sqrt{\alpha^2-2\delta\lambda^2}\,\mathrm{erf}(\sqrt{\alpha^2-2\delta\lambda^2}\sqrt{\tau/\delta})]$$

$$+\frac{2R\tau}{\delta K}\left[\alpha\,\mathrm{erfc}(\alpha\sqrt{\tau/\delta}) - \frac{\mathrm{e}^{-(\alpha^2/\delta)\tau}}{\sqrt{\pi\tau/\delta}}\right.$$

$$\left.-\left(\alpha + \frac{L}{2\tau/\delta}\right)\mathrm{erfc}(Z)\right]$$

$$-\frac{R}{K\alpha}[\mathrm{erfc}(Z) + 1 - \mathrm{erfc}(\alpha\sqrt{\tau/\delta})] \tag{15}$$

where

$$Z = \frac{L + 2\dfrac{\alpha}{\delta}\tau}{2\sqrt{\tau/\delta}} \tag{16}$$

4. Numerical solution

Due to the absence of experimental data a numerical solution was implemented to compare with the analytical solution. The Mac Cormack finite difference computational scheme was used for the numerical solution of Eq. (1) (Mac Cormack, 1969). It is a two-step (predictor–corrector) scheme, explicit, of second-order accuracy, conditionally stable and convergent. It can be used for linear as well as for nonlinear equations. While the analytical and numerical solutions of the linearized equation (2) could be used to validate each other, the numerical solution of the nonlinear equation (1) can be used to show the effectiveness of the applied linearization.

In deriving the finite difference scheme for the nonlinear problem, Eq. (1) is rewritten in the form

$$\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right) - A\frac{\partial h}{\partial x} = D\frac{\partial h}{\partial t} - Q \tag{17}$$

where

$$A = \tan\vartheta, \tag{18a}$$

$$D = \frac{S}{K\cos^2\vartheta}, \tag{18b}$$

$$Q = \frac{R}{K\cos^2\vartheta} \tag{18c}$$

and $h = h_1$ or $h = h_2$ and $Q = 0$.

Forward difference approximations are used to replace the partial derivatives of Eq. (17) in the first step, where predicted values of the unknown function, h, are to be obtained. At this step the approximate form of Eq. (17) may be written as follows

$$h^*_{k,n+1} = h_{k,n} + \frac{Q\Delta t}{D} - A\frac{\Delta t}{D\Delta x}(h_{k+1,n} - h_{k,n}) + \frac{\Delta t}{D(\Delta x)^2} \times [h_{k+1,n}(h_{k+1,n} - h_{k,n}) - h_{k,n}(h_{k,n} - h_{k-1,n})] \tag{19}$$

For the second step (corrector) the partial derivatives with respect to x are replaced by backward difference approximations, while the partial derivative with respect to t is replaced by a forward difference approximation.

Then Eq. (17) may be written as

$$h^{**}_{k,n+1} = h_{k,n} + \frac{Q\Delta t}{D} - A\frac{\Delta t}{D\Delta x}(h^*_{k,n+1} - h^*_{k-1,n+1}) + \frac{\Delta t}{D(\Delta x)^2}[h^*_{k,n+1}(h^*_{k+1,n+1} - h^*_{k,n+1}) - h^*_{k-1,n+1}(h^*_{k,n+1} - h^*_{k-1,n+1})] \tag{20}$$

Finally the corrected value of $h_{k,n+1}$ is obtained as the arithmetic mean of $h^*_{k,n+1}$ and $h^{**}_{k,n+1}$ which is

$$h_{k,n+1} = \frac{1}{2}\left[h_{k,n} + h^*_{k,n+1} + \frac{Q\Delta t}{D} - A\frac{\Delta t}{D\Delta x}(h^*_{k,n+1} - h^*_{k-1,n+1}) + \frac{\Delta t}{D(\Delta x)^2}[h^*_{k,n+1}(h^*_{k+1,n+1} - h^*_{k,n+1}) - h^*_{k-1,n+1}(h^*_{k,n+1} - h^*_{k-1,n+1})]\right] \tag{21}$$

Eqs. (19) and (21) constitute the Mac Cormack computational scheme for the nonlinear Eq. (1), which by mathematical experimentation was found to be stable if

$$\frac{A}{D}\frac{\Delta t}{\Delta x} \le 0.9, \tag{22a}$$

$$\frac{\bar{h}_n \Delta t}{D(\Delta x)^2} \le 0.5 \tag{22b}$$

and

$$\frac{A\Delta x}{\bar{h}_n} \le 1.8 \tag{22c}$$

where $\bar{h}_n$ is an average of h along x at time step n.

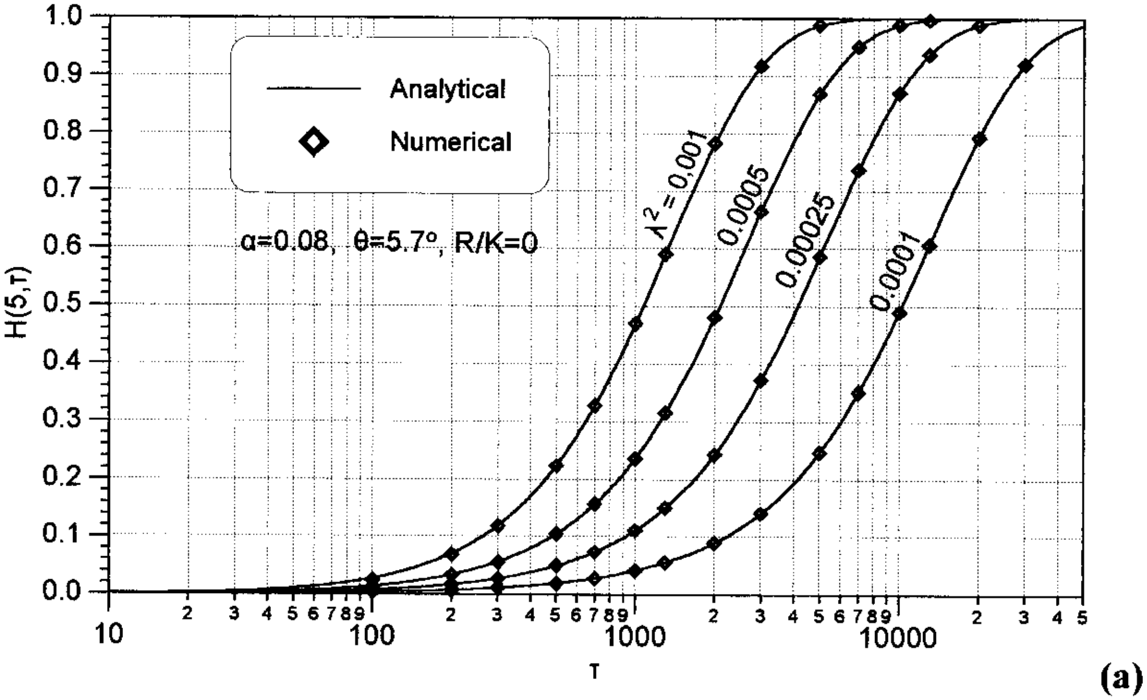
5. Results

In the absence of experimental data, the analytical solutions for the water table height given by Eq. (12), as well as the seepage rate given by Eq. (15), were compared with the numerical solutions of the nonlinear extended Boussinesq equation with the corresponding initial and boundary conditions.

To show the behavior of the analytical solution in response to the gradually rising water level in the stream and to a constant replenishment from above, we consider numerical applications for the various values of replenishment R and the parameter λ that express the rate of the water level rising in the stream. It should be noted that the larger the values of λ, the more abrupt is the rise of the water level at the stream.

Fig. 2a and b shows the results for the dimensionless water table height at $\xi = 5$ calculated by the analytical solution for $R/K = 0$, $\bar{h} = (h_o + h_a)/2$ and for slopes 10 ($\tan\vartheta = 0.1$) and 30% ($\tan\vartheta = 0.3$), that correspond to angles of the sloping bed with horizontal, 5.7 and 16.7°, respectively. The results of the numerical solutions for the same sets of parameters are also presented. It is obvious that the results from the analytical and the numerical solutions are almost identical. It is also shown that, despite the different rates of the water level rising in the stream, the water table height gradually converges to 1. This means that the water table near the stream tends to be parallel to the sloping bed, at a height of $H_1 = 1$ or $h_1 = h_o$ as it is expected since our solution is based on the assumption that flow is parallel to the sloping impermeable layer. As H_1 converges to 1 near the stream the front of the rising water table is moving downslope.

Fig. 3a and b illustrates the variation of the dimensionless seepage rate from the stream vs. time τ, calculated by the analytical solution for $R/K = 0$, $\bar{h} = (h_o + h_a)/2$ and various values of the parameter λ. The results of the numerical solution of the nonlinear equation (1) are also presented. The agreement



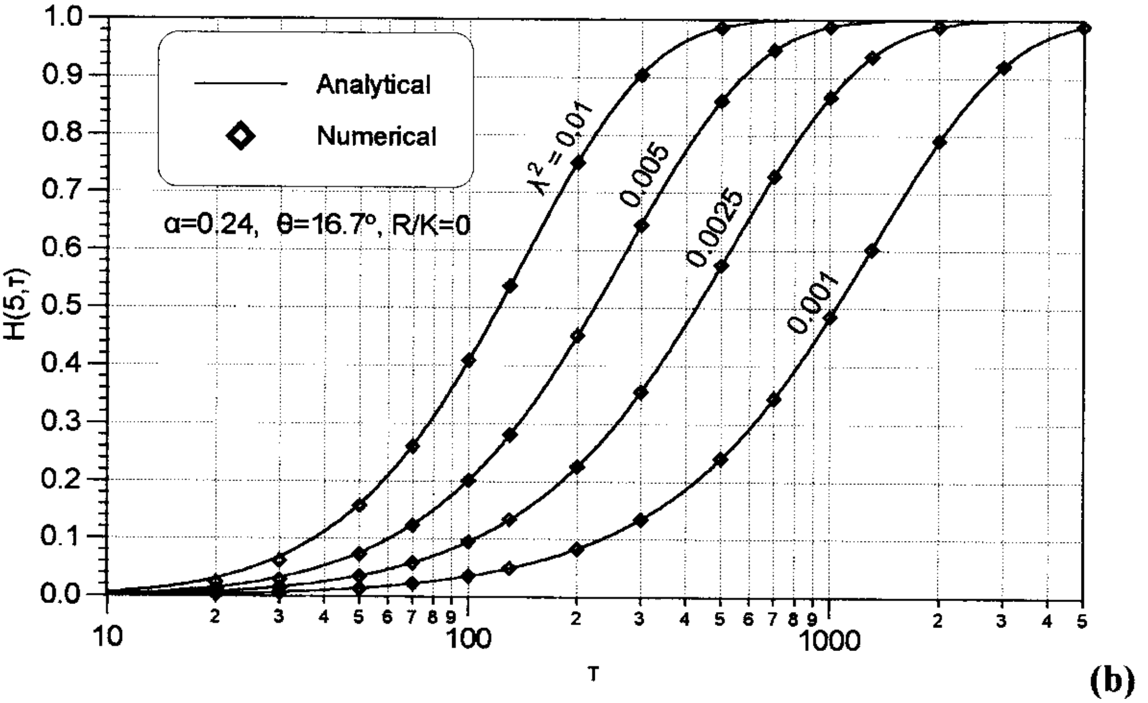


Fig. 2. Analytical and numerical solutions for the water table height at $\xi = 5$ for: (a) $\vartheta = 5.7°$ and (b) $\vartheta = 16.7°$.

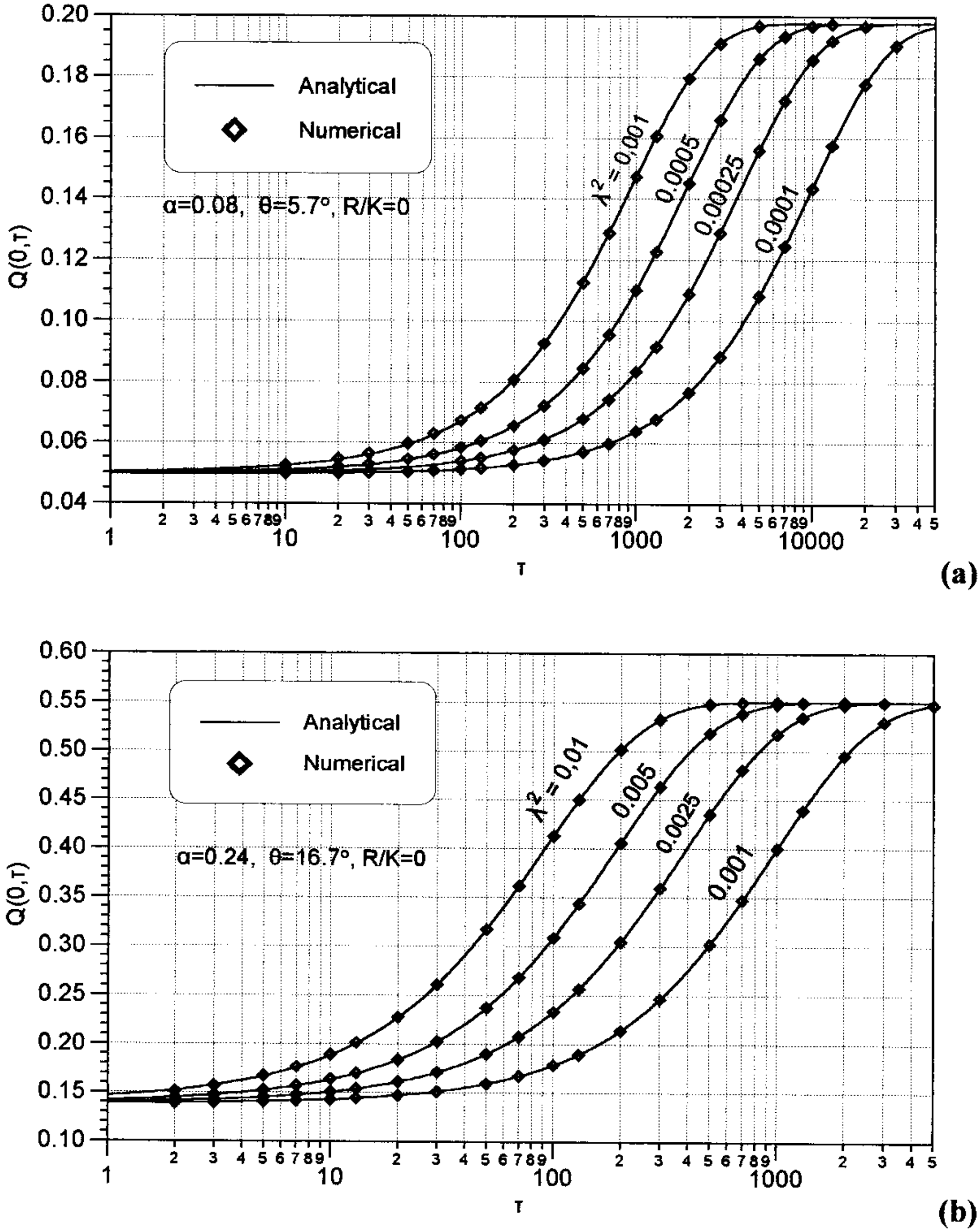
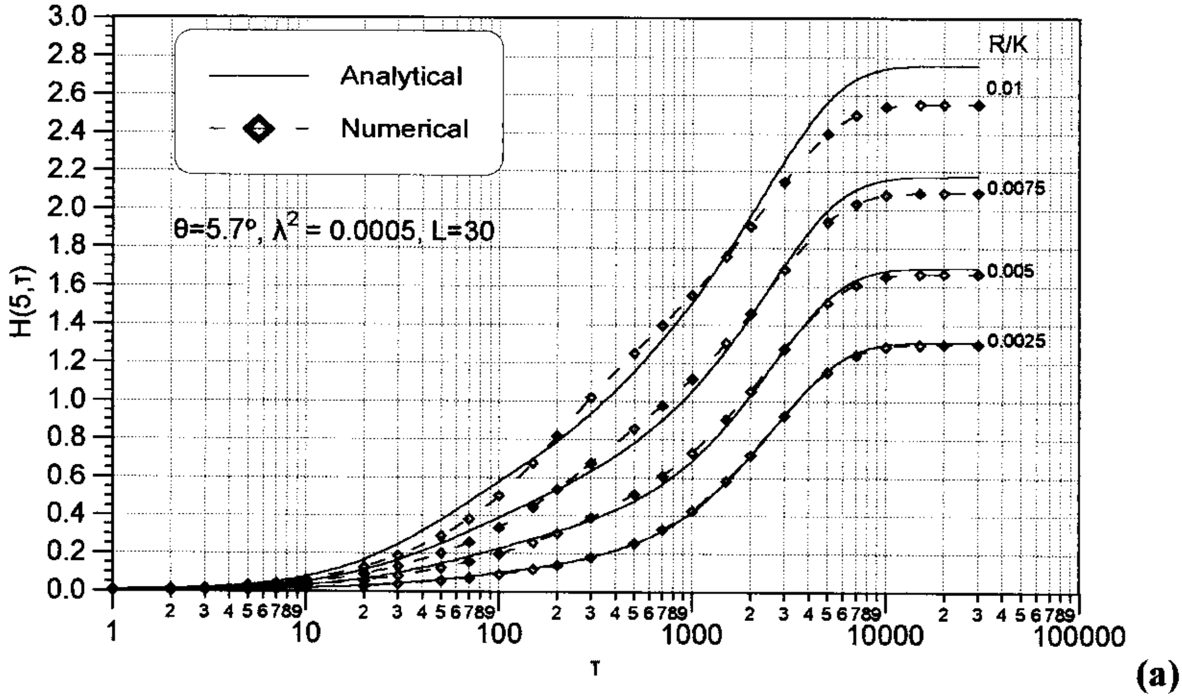


Fig. 3. Analytical and numerical solutions for the seepage rate at $\xi = 0$ for: (a) $\vartheta = 5.7°$ and (b) $\vartheta = 16.7°$.



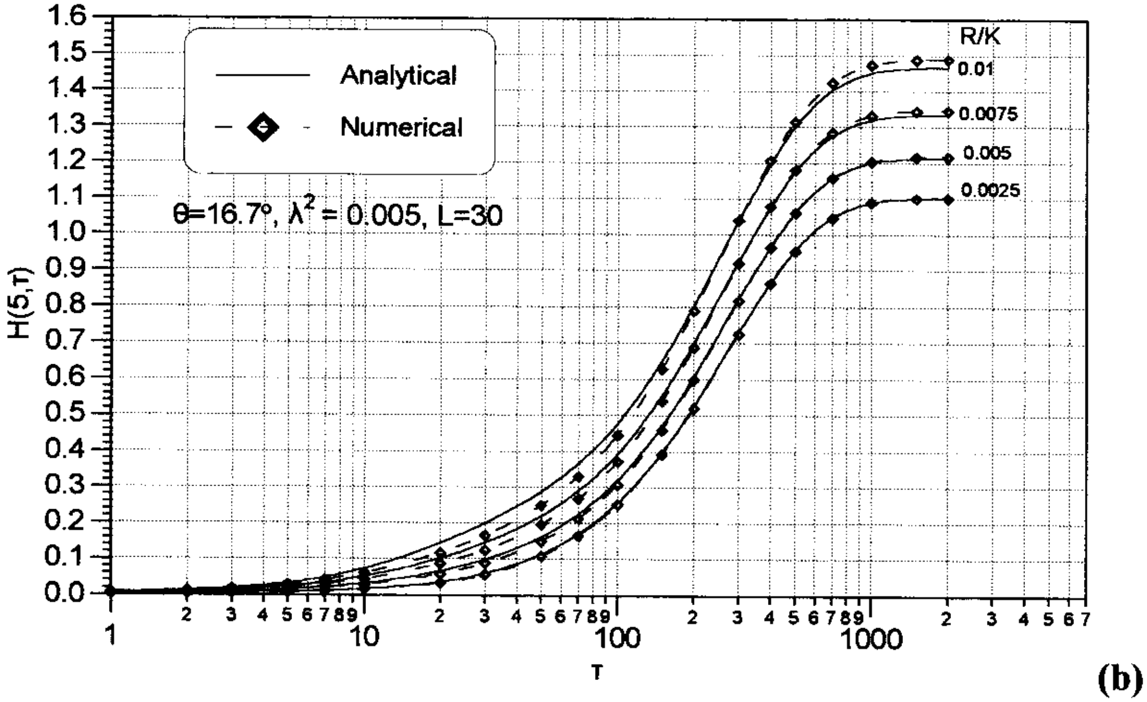
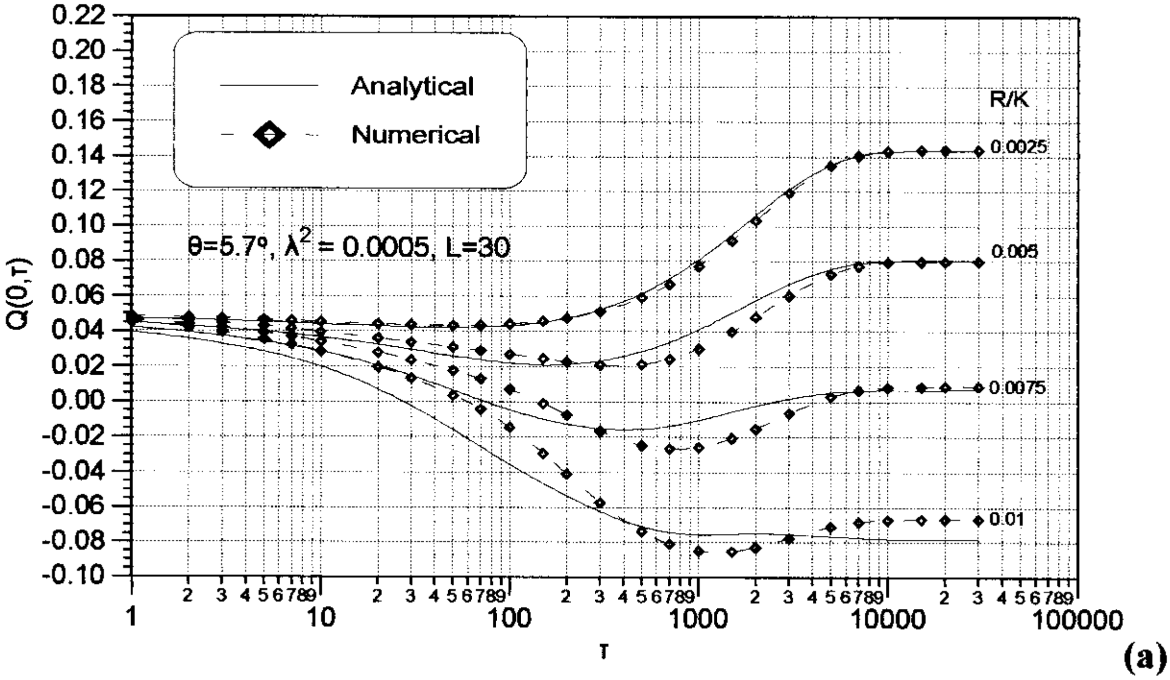


Fig. 4. Analytical and numerical solutions for the water table height at $\xi = 5$ in response to various values of the replenishment rate for: (a) $\vartheta = 5.7°$ and (b) $\vartheta = 16.7°$.



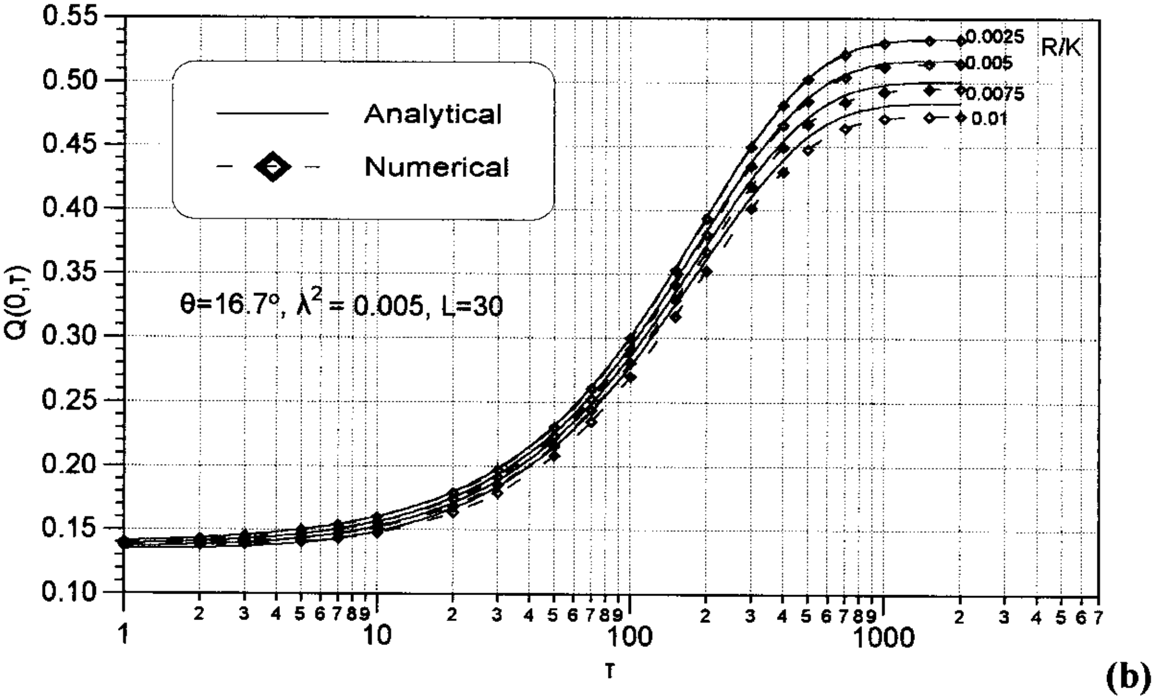


Fig. 5. Analytical and numerical solutions for the seepage rate at $\xi = 0$ in response to various values of the replenishment rate for: (a) $\vartheta = 5.7°$ and (b) $\vartheta = 16.7°$.

between the analytical and numerical solutions is excellent for both slopes of the impermeable bed. The seepage rate has an initial value that corresponds to the initial condition of the problem and gradually increases with time, converging to the values of 0.197657 (Fig. 3a) and 0.550481 (Fig. 3b), which correspond to hydraulic gradients that are equal to the two slopes of the impermeable layer, respectively.

Fig. 4a and b shows the variation of the dimensionless water table height vs. time τ at $\xi = 5$ for various values of the parameter R/K. The replenishment is assumed to occur from a strip along the stream having the dimensionless width $L = L_0/h_o = 30$. In this case, where $R/K > 0$, a constant value of $\bar{h} = (h_o + h_a)/2$ is not sufficient for an accurate calculation of the water table height by the analytical solution. Instead, $\bar{h}$ is obtained by successive approximations in an iterative procedure as $\bar{h} = (h_a + h_t)/2$, where h_t is the height at time t at the end of which h is to be approximated. It is observed that the agreement between the analytical and numerical solution is better for smaller values of the ratio R/K. However, for larger values of the ratio R/K the water table height increases with time more rapidly, converging to higher values and the assumption for the linearization is not satisfied. It is also observed that the effect of the ratio R/K on the water table height is more significant in the case of small slope ($\vartheta = 5.7°$) than in the case of the larger slope ($\vartheta = 16.7°$) of the impermeable bed. In any case the water table height increases with time and finally converges to higher values as the ratio R/K becomes larger. This is because the phreatic surface near the stream reaches an equilibrium flow condition as the water table at a distance larger than L tends to be parallel to the sloping bed and the water table rising is moving downslope. This equilibrium flow condition is reached more rapidly in the case of the larger slope.

Fig. 5a and b shows the variation of the dimensionless seepage rate vs. time τ for various values of the replenishment rate. In the case of the small slope ($\vartheta = 5.7°$) shown in Fig. 5a the seepage rate starts to decrease with time from its initial value. This decrease is more significant for larger values of the ratio R/K. For the larger values of R/K it is observed that an inverse flow takes place from the aquifer towards the stream. This is because the water table rising within the recharging zone, owing to the recharge rate R, is more rapid than the rising of the water level in the stream and reaches to higher values. Finally the seepage rate converges to lower values as the ratio R/K becomes larger. In Fig. 5b we can see that, similarly to the case of the water table height (Fig. 4a and b), the effect of the ratio R/K in the variation of the seepage rate is more significant for the small slope ($\vartheta = 5.7°$) than for the larger slope ($\vartheta = 16.7°$).

Nondimensional diagrams such as those presented in Figs. 2–5, and especially those for the seepage rate, can also be used for easy calculations in solving practical problems. However, it is realized that practically such diagrams can only be made for limited sets of parameter values. Alternatively, the complicated mathematical functions included in the analytical solutions can easily be computed using commercially available mathematical software, which allows automation of the calculations for any set of parameter values.

6. Conclusions

In this study a new analytical solution is presented for the problem of seepage from a stream towards an adjacent sloping unconfined aquifer of semi-infinite extent, due to an exponentially rising with time water level at the stream and to a constant downward replenishment from a finite strip along the stream. For $R/K = 0$ an excellent agreement was found between the analytical solution of an extended and linearized form of the nonhomogeneous Boussinesq equation and the numerical solution of the nonlinear form of the extended Boussinesq equation for both the water table height near the stream and the seepage rate. For $R/K > 0$ the agreement between the analytical and the numerical solution was found to be still satisfactory when the water table rising was small compared with the average height and therefore the assumption for the linearization was satisfied. The variation of the water table height and of the seepage rate vs. time is illustrated in nondimensional diagrams. From these diagrams the effect of the parameter λ and the ratio R/K on the water table height and on the seepage rate can be observed. Nondimensional diagrams, or even better, mathematical software can be used to solve practical problems such as the calculation of the recharge of the aquifer or the seepage loss from an unlined canal.

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