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Review

A review of bivariate gamma distributions for hydrological application

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Abstract

A univariate gamma distribution is one of the most commonly adopted statistical distributions in hydrological frequency analysis. A bivariate gamma distribution constructed from specified gamma marginals may be useful for representing joint probabilistic properties of multivariate hydrological events such as floods and storms. This article presents a review of various bivariate gamma distribution models that are constructed from gamma marginals. Advantages and limitations of each of these models are pointed out. Applicability of a few bigamma distributions whose gamma marginal distributions have different scale and shape parameters is investigated. The dependence of these models is directly or indirectly measured via the Pearson's product-moment correlation coefficient. The scale and shape parameters of the models are estimated from their marginal distributions by the method of moments. Results indicate that these bigamma distribution models will be useful for describing the joint probability distribution of two correlated random variables with gamma marginals. © 2001 Elsevier Science Ltd All rights reserved.

Keywords: Gamma distribution; Bivariate gamma distribution; Joint distribution; Marginal distribution; Correlation; Multivariate frequency analysis

1. Introduction

It has gradually been recognized that complex hydrological events such as floods and storms always appear to be multivariate events that are characterized by a few correlated random variables. Single-variable hydrological frequency analysis can only provide limited assessment of these events and it is not sufficient to represent multiple episodic hydrological phenomena. A thorough understanding of multivariate hydrological events requires the study of the joint probabilistic behavior of two or more correlated random variables that characterize the events. Some meaningful attempts have been made to address this topic. Examples include the work of Ashkar (1980), Ashkar et al. (1998), Hashino (1985), Correia (1987), Sackl and Bergmann (1987), Krstanovic and Singh (1987), Loganathan et al. (1987), Choulakian et al. (1990), Singh and Singh (1991), Bacchi et al. (1994), Lall and Bosworth (1994), Kelly and Krzysztofowicz (1997), Kurothe et al. (1997), Durrans (1998), Goel et al. (1998, 2000), Wilks (1998), Adamson et al. (1999), Yue et al. (1999) and Yue (1999, 2000a).

In practice, many hydrological events may follow

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the gamma distribution (Bobée and Ashkar, 1991; Stedinger et al., 1993). The study of the bivariate gamma distribution constructed from specified gamma marginals may be of usefulness to hydrological engineers in evaluating multivariate hydrological events. In the past, some researchers have investigated a few bivariate gamma distributions with special gamma marginals for hydrological frequency analysis (see for example Blokhinov and Sarmanov, 1968; Crovelli, 1973; Clarke, 1980). These models lack flexibility and are difficult to be implemented for the solution of practical problems. In the literature, several theoretical bivariate gamma distributions of two correlated variables with gamma marginals have been proposed by McKay (1934), Kibble (1941), Cherian (1941), Izawa (1953), Moran (1969), Nagao and Kadoya (1970), Sarmanov (1971), Crovelli (1973), and Smith et al. (1982). However, these models have mainly remained in the form of theoretical developments and seldom succeeded in gaining popularity among practitioners in the field of hydrological frequency analysis. This could be explained by many reasons: (i) the mathematical expressions of some of these models are complex and have computational limitations; (ii) a methodology for investigating the applicability of these models is not straightforward; (iii) hydrological frequency analysis itself has still been mainly centered around single-variable analysis; and (iv) some of these models have only been published in Japanese.

In this paper, the bivariate gamma distribution (BGD) models constructed from specified gamma marginals are summarized. The applicability of a few bigamma models constructed from two different gamma marginals is investigated by both generated data and actual flood data.

2. Bivariate gamma distribution (BGD) models

This section describes a few different BGDs constructed from two different univariate gamma distributions. Other special BGDs are presented in Appendix A.

2.1. Probability density function (PDF) of a two-parameter gamma distribution

For the sake of consistency, the common form of the PDF of a univariate gamma distribution with two parameters is presented by

$$f_Z(z;\alpha_z,\lambda_z) = \frac{1}{\Gamma(\lambda_z)}\alpha_z^{\lambda_z} z^{\lambda_z - 1} e^{-\alpha_z z} \tag{1}$$

where α_z and λ_z are the scale and shape parameters of the gamma distribution, respectively. Then the PDFs of gamma distributed random variables X and Y can be obtained by replacing z with x and y in Eq. (1), respectively. The corresponding cumulative distribution functions (CDFs) of X and Y can be obtained by numerically integrating Eq. (1) as follows:

$$F_Z(z;\alpha_z,\lambda_z) = \int_0^z f_Z(t;\alpha_z,\lambda_z)\,\mathrm{d}t \qquad (z = x, y) \tag{2}$$

2.2. Izawa bigamma model

Izawa (1953) proposed a bivariate gamma model that is constructed from gamma marginals and allows for different scale and shape parameters. Its marginal PDFs are $f_X(x;\alpha_x,\lambda_x)$ and $f_Y(y;\alpha_y,\lambda_y)$, respectively. As this model was published in Japanese, it has not yet attracted an extensive amount of attention in the statistical literature. The joint PDF is (Izawa, 1953; Nagao, 1975):

$$f(x,y) = \frac{(xy)^{(n-1)/2} x^m \exp\left(-\dfrac{\alpha_x x + \alpha_y y}{1-\eta}\right)}{\Gamma(n)\Gamma(m)(\alpha_x\alpha_y)^{-(n+1)/2}\alpha_x^{-m}(1-\eta)\eta^{(n-1)/2}} \cdot \int_0^1 (1-t)^{(n-1)/2} t^{m-1} \times \exp[\alpha_x \eta x t/(1-\eta)] I_{n-1}\left(\frac{2\sqrt{\alpha_x\alpha_y\eta x y(1-t)}}{1-\eta}\right)\mathrm{d}t \tag{3a}$$

$$I_s(h) = \sum_{k=0}^{\infty} \frac{(h/2)^{s+2k}}{k!\,\Gamma(s+k+1)} \tag{3b}$$

$$\eta = \rho\sqrt{\lambda_x/\lambda_y} \qquad (0 \le \rho < 1,\ 0 \le \eta < 1) \tag{3c}$$

$(n = \lambda_y,\ m = \lambda_x - \lambda_y = \lambda_x - n,\ n \ge 0,\ m \ge 0)$

where $I_s(\cdot)$ is the modified Bessel function of the first kind; η is the association parameter between X and Y; ρ is the Pearson's product-moment correlation coefficient, which hereinafter represents the correlation coefficient estimated from original sample data. It is given by

$$\rho = \frac{E[(X-\mu_x)(Y-\mu_y)]}{\sigma_x\sigma_y} \tag{4}$$

in which (μ_x, σ_x) and (μ_y, σ_x) are the population means and standard deviations of X and Y respectively, which are often replaced by the sample mean (M) and sample standard deviation (S). The dependence of the model is measured by the association parameter via the correlation coefficient ρ. The corresponding joint CDF can be obtained by numerically integrating the joint PDF.

2.3. Moran model

Moran (1969) derived another bivariate gamma model from the bivariate normal distribution. Let W and G be two random variables with a bivariate normal distribution whose joint PDF is

$$f(w,g) = \frac{1}{2\pi\sqrt{(1-\rho_N^2)}}\exp\left(-\frac{w^2-2\rho_N wg+g^2}{2(1-\rho_N^2)}\right) \tag{5a}$$

By defining random variables U and V using the following equations:

$$u = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{w} e^{-\frac{1}{2}t^2}\,\mathrm{d}t = \Phi(w) \tag{5b}$$

$$v = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{g} e^{-\frac{1}{2}t^2}\,\mathrm{d}t = \Phi(g) \tag{5c}$$

the variables U and V are jointly distributed with marginal probability distributions whose densities are equal to unit in the interval (0, 1). We can then define gamma distributed random variables X and Y by the following equations:

$$u = \int_0^x f_X(t;\alpha_x,\lambda_x)\,\mathrm{d}t = F_X(x;\alpha_x,\lambda_x) \tag{5d}$$

$$v = \int_0^y f_Y(t;\alpha_y,\lambda_y)\,\mathrm{d}t = F(y;\alpha_y,\lambda_y) \tag{5e}$$

The joint PDF of the bigamma distribution of X and Y with the gamma marginals can be represented by

$$f(x,y) = \frac{1}{\sqrt{(1-\rho_N^2)}}f_X(x;\alpha_x,\lambda_x)f_Y(y;\alpha_y,\lambda_y) \times\exp\left[-\frac{(\rho_N x')^2-2\rho_N x'y'+(\rho_N y')^2}{2(1-\rho_N^2)}\right]$$

$$(|\rho_N|<1) \tag{5f}$$

$$x' = \Phi^{-1}(w) = \Phi^{-1}[F_X(x;\alpha_x,\lambda_x)] \tag{5h}$$

$$y' = \Phi^{-1}(g) = \Phi^{-1}[F_Y(y;\alpha_y,\lambda_y)] \tag{5g}$$

where ρ_{N} represents the correlation coefficient between the normalized variates W and G, or X' and Y', as given by Eq. (4); $\Phi^{-1}(w)$ and $\Phi^{-1}(g)$ are the inverse of the standard normal distributions. The normalized variates X' and Y' are related to the original variates X and Y via the normal quantile transform (NQT), as shown by Eqs. (5h) and (5g) (Kelly and Krzysztofowicz, 1997).

The dependence of the Moran bigamma model is fully measured by the correlation coefficient ρ_N, and it represents a full range of the association between two random variates. In fact, this model is a special case of the bivariate meta-Gaussian model developed by Kelly and Krzysztofowicz (1997). The joint CDF of the Moran model can be computed either by numerically integrating the joint PDF Eq. (5f) or by numerically integrating the binormal the joint PDF of Eq. (5a) in which W and G are replaced by X' and Y', respectively, as pointed out by Kelly and Krzysztofowicz (1997). This study computes the joint CDF by the latter.

2.4. Smith–Adelfang–Tubbs (SAT) model

Smith et al. (1982) developed another bivariate gamma model in which the marginals are $f_X(x;\alpha_x,\lambda_x)$ and $f_Y(y;\alpha_y,\lambda_y)$, respectively. The joint PDF and joint CDF of correlated two random variables X and Y with gamma marginal distributions are

$$f(x,y)\begin{cases}\dfrac{K_1}{K_2}\displaystyle\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}c_{jk}(\alpha_x x)^j(\eta\alpha_y y)^{j+k} & (0<\eta<1)\\ f_X(x;\alpha_x,\lambda_x)f_Y(y;\alpha_y,\lambda_y) & (\eta=0)\end{cases}$$

$$(\lambda_y \geq \lambda_x) \tag{6a}$$

$$F(x,y) = P_r[X \le x,\ Y \le y]$$

$$= \begin{cases} J\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} d_{jk} H[x/(1-\eta), \lambda_x + j] \\ \cdot H[y/(1-\eta), \lambda_y + j + k] & (0 < \eta < 1) \\ F_X(x; \alpha_x, \lambda_x) F_Y(y; \alpha_y, \lambda_y) & (\eta = 0) \end{cases} \quad (6b)$$

where

$$K_1 = (\alpha_x x)^{\lambda_x - 1}(\alpha_y y)^{\lambda_y - 1}\exp\left(-\frac{\alpha_x x + \alpha_y y}{1-\eta}\right) \quad (6c)$$

$$K_2 = (1-\eta)^{\lambda_x}\Gamma(\lambda_x)\Gamma(\lambda_y - \lambda_x) \quad (6d)$$

$$c_{jk} = \frac{\eta^{j+k}\Gamma(\lambda_y - \lambda_x + k)}{(1-\eta)^{2j+k}\Gamma(\lambda_y + j + k)j!k!} \quad (6e)$$

$$\eta = \rho\sqrt{\lambda_y/\lambda_x} \quad (6f)$$

$$J = \frac{(1-\eta)^{\lambda_y}}{\Gamma(\lambda_x)\Gamma(\lambda_y - \lambda_x)} \quad (6g)$$

$$d_{jk} = \frac{\eta^{j+k}\Gamma(\lambda_y - \lambda_x + k)}{\Gamma(\lambda_y + j + k)j!k!} \quad (6h)$$

$$H(z,a) = \int_0^z t^{a-1}e^{-t}\mathrm{d}t \quad (6i)$$

and where $H(\cdot)$ is the incomplete gamma function, η is the association parameter between X and Y, and ρ is the correlation coefficient of X and Y, as given by Eq. (4). The dependence of the model is measured by the association parameter via the correlation coefficient ρ, as shown by Eq. (6f). Among all the bigamma models, only this model provides the explicit formulae for both the joint PDF and joint CDF, to the authors' knowledge. By comparing with the Izawa model, it can be seen that the association parameter of the SAT model is the same as that of the Izawa model in which X is replaced by Y and Y by X.

2.5. Farlie–Gumbel–Morgenstern (FGM) model

In the literature, there exist a family of bivariate distributions that allow for different marginal distribution types, termed as the Farlie–Gumbel–Morgenstern (FGM) model due to Morgenstern (1956), Gumbel (1958), and Farlie (1960). Its joint PDF and joint CDF of two random variables X and Y are:

$$f(x,y) = f_X(x)f_Y(y)\{1 + \eta[2F_X(x) - 1][2F_Y(y) - 1]\} \quad (7a)$$

$$F(x,y) = F_X(x)F_Y(y)\{1 + \eta[1 - F_X(x)][1 - F_Y(y)]\} \quad (7b)$$

$$\eta = 3\rho\ (|\eta| \le 1) \quad (7c)$$

where $(f_X(x), f_Y(y))$ and $(F_X(x), F_Y(y))$ are the marginal PDFs and CDFs of X and Y, respectively. In this study, they are represented by Eqs. (1) and (2), respectively. η is the association parameter of X and Y. ρ is the correlation coefficient between X and Y, which is computed using Eq. (4). Although the FGM model is an interesting family constructed from specified marginals, it is limited to describe weak dependence, i.e. $|\rho| \le 1/3$, as documented by Schucany et al. (1978), Long and Krzysztofowicz (1992) and others.

3. Investigation of applicability of the Izawa, Moran, SAT, and FGM models for representing two correlated gamma distributed random variables

3.1. Parameters estimation

For a bivariate distribution constructed from marginals, the marginal distributions contain all the information about the parameters of the marginal distributions, as pointed out by Moran (1969). Theoretically, it is difficult, or even impossible, to understand how much extra information about the parameters of the marginal distribution of X can be given by the other variable Y. Therefore, the simplest way for deriving the parameters of the bivariate distribution constructed from marginals might be to estimate the scale and shape parameters separately using the marginal distributions, and to estimate the association parameter via the product-moment correlation coefficient, as employed in the work of Yue et al. (1999). As the correlation coefficient is computed by the method of moments (MM) [see Eq. (4)], for the consistency of the methodology, the other parameters

Table 1
Statistics and derived parameters ($\rho = 0.35$)

	Statistics		Parameters of gamma distribution	
	M	S	λ	α
X (Y for SAT)	108.34	39.33	7.59	0.070
Y (X for SAT)	9.68	6.12	2.50	0.258

should also be estimated by the MM. The scale and shape parameters of a gamma distribution are given by (Bobée and Ashkar, 1991; Stedinger et al., 1993)

$$\alpha = \frac{M}{S^2} \tag{8a}$$

$$\lambda = \frac{M^2}{S^2} \tag{8b}$$

Actually, all the parameters of a bigamma distribution may be estimated via the maximum likelihood (ML) method, as proposed by Moran (1969). However, the parameter estimation via the ML becomes much more complicated. The present study does not consider this approach as an alternative.

Basically, if a selected theoretical model is the true one from which sample data are drawn, then theoretical probabilities computed from the selected theoretical model should provides a good fit to empirical probabilities estimated from sample data. On the basis of this principle, the above reviewed four models are examined. Simulation can provide various combination scenarios of two variates, which may be difficult to find in the practical world. Generated data are employed to check the suitability of these models for representing two correlated gamma distributed variates. A flood data set from an actual river basin is also used to investigate their suitability, which includes different two-way combinations: joint distribution of flood peak and volume and joint distribution of flood volume and duration.

Table 2
Statistics and derived parameters ($\rho = 0.80$)

	Statistics		Parameters of gamma distribution	
	M	S	λ	α
X (Y for SAT)	9.76	4.24	5.30	0.543
Y (X for SAT)	15.74	7.56	4.33	0.275

Table 3
Statistics and derived parameters ($\rho = -0.37$)

	Statistics		Parameters of gamma distribution	
	M	S	λ	α
X (Y for SAT)	34.37	18.24	3.55	0.103
Y (X for SAT)	9.58	5.76	2.77	0.289

3.2. Generation of gamma distributed random variables

All the reviewed bigamma models are constructed from gamma marginal distributions. To mimic this situation, a number of univariate gamma distributed random variables with sample size of 75 were generated using the Johnk's gamma generator that is written in MATLAB code. The detailed description of the method was given by Devroya (1986). The product-moment correlation coefficient between two sample data of these generated samples were estimated using Eq. (4). Then three pairs (X, Y) with correlation coefficients of 0.35, 0.80 and −0.37 were selected. The sample means (M) and standard deviations (S) of the random variables X and Y are presented in Tables 1–3, respectively. The scale and shape parameters for the gamma distribution models were estimated using Eqs. (8a) and (8b). They are also listed in Tables 1–3. The association parameters for the Izawa, SAT, and FGM models were computed using Eqs. (3c), (6f), and (7c), respectively, which are presented in Table4. For the FGM model, its theoretical limitation is $|\eta| \le 1$. When $|\eta| > 1$, its joint pdf becomes negative in some domains. Thus, we approximately take $\eta = 1$ and -1 for the case of $\rho = 0.35$ and -0.37, respectively. For the Moran model, its marginals must be normally distributed. The sample data were transformed to follow the standard normal distribution by the NQT. The correlation coefficients between the normalized sample data estimated using Eq. (4) are also listed in Table 4.

Table 4
Association parameters of the bigamma models

ρ	Association (η)			Moran (ρ_N)
	Izawa	SAT	FGM	
0.351	0.610	0.610	1.0	0.267
0.800	0.885	0.885	2.4	0.771
−0.366	−0.414	−0.414	−1.0	−0.398

3.3. The joint CDF of the generated data X and Y with correlation coefficient 0.35

Empirical joint probabilities are computed using the approach proposed by Yue (1999). A two-dimensional table was first constructed in which the samples x and y were arranged in ascending order. The element (n_{ij}) in row i and column j of the table is the number of the concurrence of these two sample values. Their joint frequency function was computed by

$$f(x_i, y_j) = \Pr(X = x_i, Y = y_j) = \frac{n_{ij}}{N+1} \quad (9)$$

where N is the total number of the sample data. The joint cumulative frequency (non-exceedance joint probability) was given as:

$$F(x_i, y_j) = \Pr(X \le x_i, Y \le y_j) = \sum_{m=1}^{i} \sum_{l=1}^{j} f(x_m, y_l) \quad (10)$$

Theoretical joint probabilities of the real occurrence combinations of x_i and y_j were computed using the reviewed different bigamma models: Izawa, Moran, SAT, and FGM models. The theoretical probabilities of the Izawa model were arranged by ascending order and were plotted in Fig. 1. Then the empirical probabilities and theoretical joint probabilities computed by the other models, corresponding to the same occurrence combination of x_i and $y_j$are displayed in Fig. 1. Fig. 1 indicates that there is no significant difference between the observed and theoretical probabilities. Fig. 1 also shows that the differences among the theoretical joint probabilities computed by the different models are subtle.

3.4. The joint CDF of the generated data X and Y with correlation coefficient 0.80

Similar to the procedure as in the preceding subsection, empirical joint probabilities as well as theoretical joint probabilities computed by Izawa, Moran, and

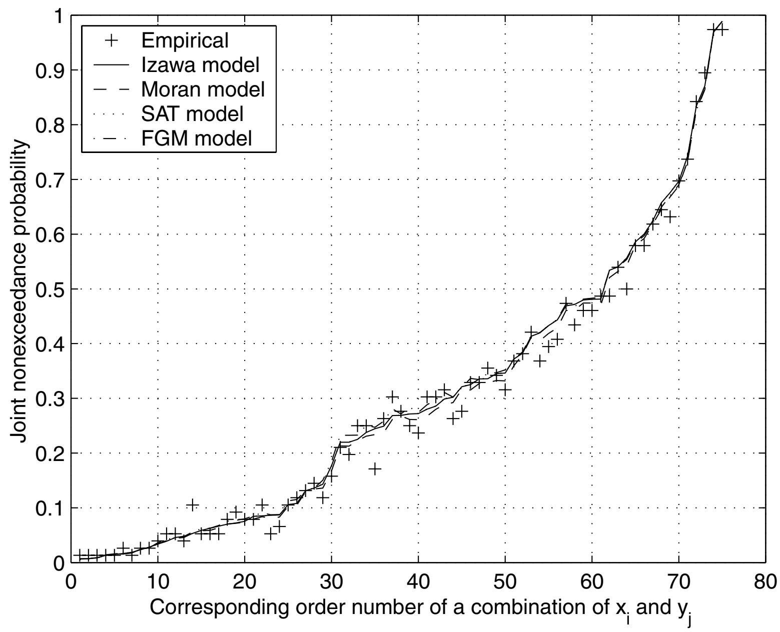


Fig. 1. Comparison of empirical and theoretical joint probabilities of X and Y ($\rho = 0.35$).

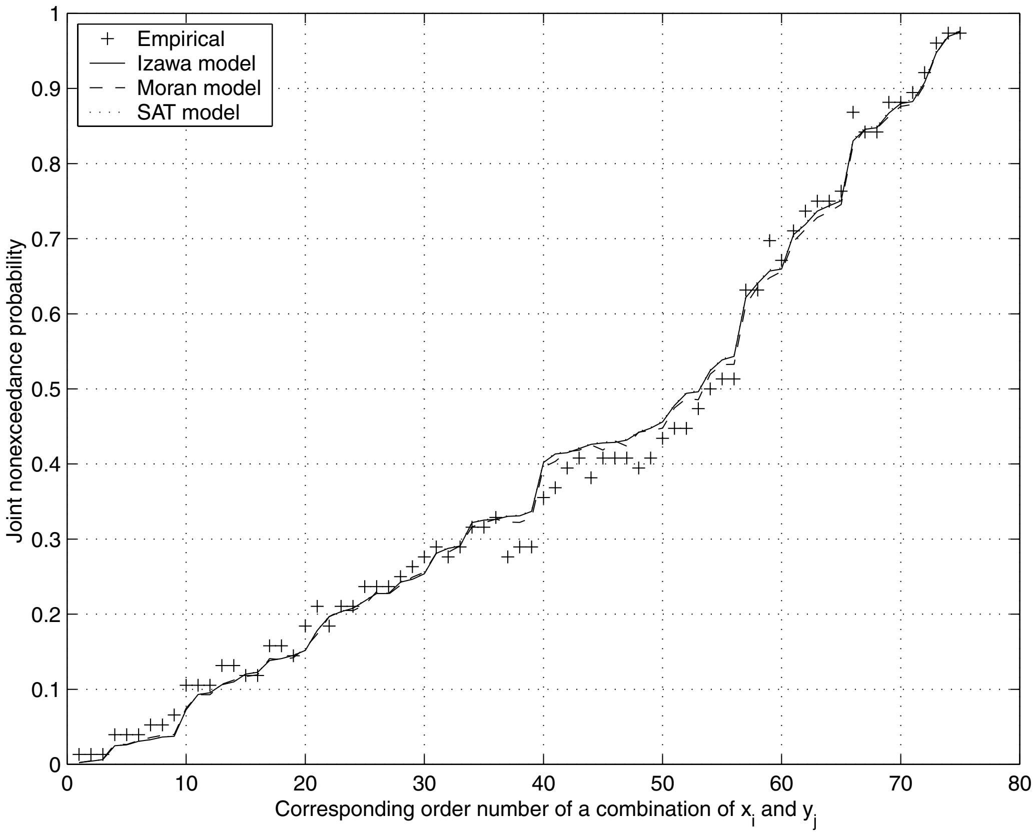


Fig. 2. Comparison of empirical and theoretical joint probabilities of X and Y ($\rho = 0.80$).

SAT model are depicted in Fig. 2. There is no significant difference between empirical and theoretical probabilities, and the theoretical probabilities computed by these models are almost identical. For the FGM model, its association parameter greatly exceeds its upper limitation ($\eta = 2.4$). Thus this model cannot be used to represent the joint distribution of these two highly correlated variables. Schucany et al. (1978) and Long and Krzysztofowicz (1992) have theoretically documented this.

3.5. The joint CDF of the generated data X and Y with correlation coefficient −0.37

For two negatively correlated random variables, among the above reviewed bivariate distributions, only the Moran and FGM models can be employed to represent their joint statistical properties. The association parameter (η) of the FGM model was approximately taken to be -1.0. Empirical joint probabilities and theoretical joint probabilities computed by these models are displayed in Fig. 3. It can be seen that the results of both models are almost identical and both can be used to represent negatively correlated random variables.

3.6. The joint distribution of multivariate flood events

In order to demonstrate the usefulness of the above reviewed bigamma models for representing multivariate hydrological events, a real flood data set from the Madawask river basin was utilized to investigate the suitability of these models. The basin has an area of 2690 km^2 and is located in Quebec, Canada. Daily streamflow data from 1919 to 1995 are available at the gauging station 01AD001 at latitude of 47:32:54N and longitude of 68:38:11W, near the outlet of the basin. Spring represents the high flow season and

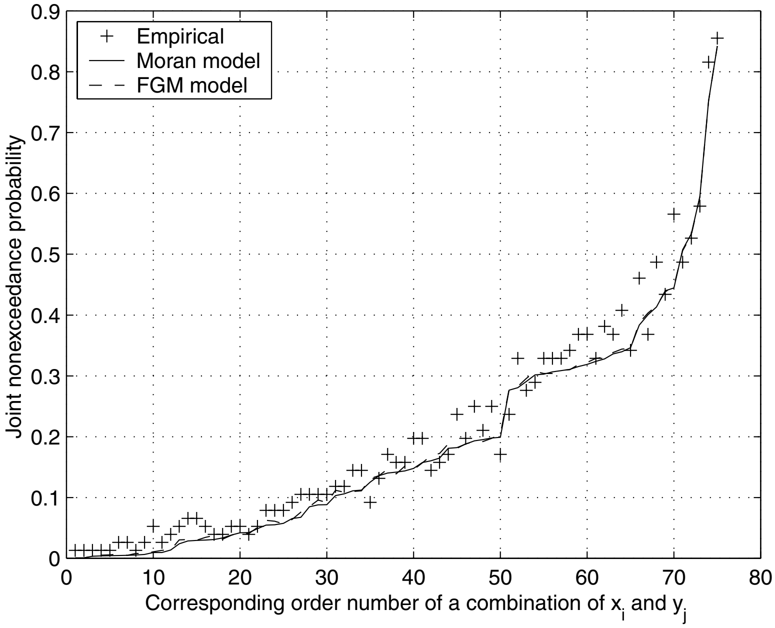


Fig. 3. Comparison of empirical and theoretical joint probabilities of X and $Y(\rho = -0.37)$.

the spring flood is the annual maximum both in peak and volume. Flood characteristics, namely Flood peak, flood volume, and flood duration of the flood hydrographs are determined based on the same procedure as in the work of Yue (1999) and Yue et al. (1999). The flood boundaries (starting and ending dates) is determined first, then the flood duration and the corresponding flood peak and flood volume can be obtained.

3.6.1. Empirical probabilities

The non-exceedance empirical probabilities of sample data were estimated using the Weibull formula (Weibull, 1939; Chow, 1953).

$$P_k = \frac{k}{N+1} \tag{11}$$

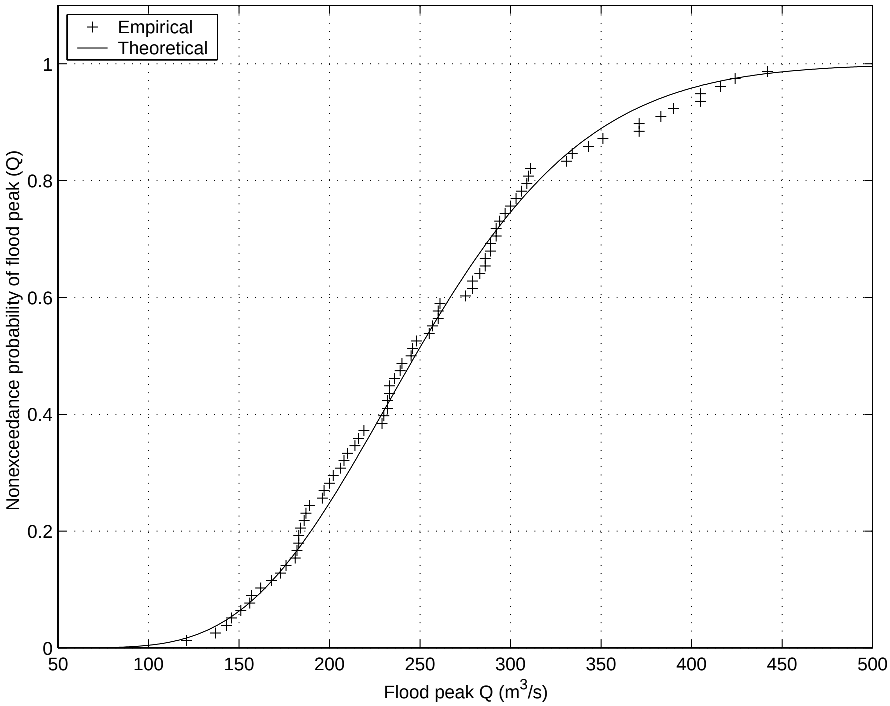
where P_k is the cumulative frequency, the probability that a given value is less than the k-th smallest observation in the data set of N observations.

Table 5
Statistics of flood peak (Q), volume (V), and duration (D) and parameters of the gamma distribution

	Statistics		Parameters of gamma distribution	
	M	S	λ	α
Q(m^3/s)	254.7	75.87	11.27	0.0443
V(day.m^3/s)	9184	2837	10.48	0.0011
D(days)	81.03	26.90	9.08	0.1120

3.6.2. Marginal distributions of the flood peak, flood volume, and flood duration

The mean and standard deviation of flood peak (Q), volume (V), and duration (D) computed from the sample data are listed in Table 5. The scale and shape parameters of the gamma distribution estimated by Eqs. (8a) and (8b) are also presented in Table 5. The Kolmogorov–Smirnov (KS) test (Kanji, 1993) was applied to examine the goodness of fit of the gamma distribution to the flood peak, volume, and duration data. The test indicates that the null hypothesis H_0 that the underlying distributions of all these three flood characteristics are the gamma distribution type is accepted at the significance level of 0.05. The empirical probabilities and fitted theoretical probabilities by the gamma distributions for the flood



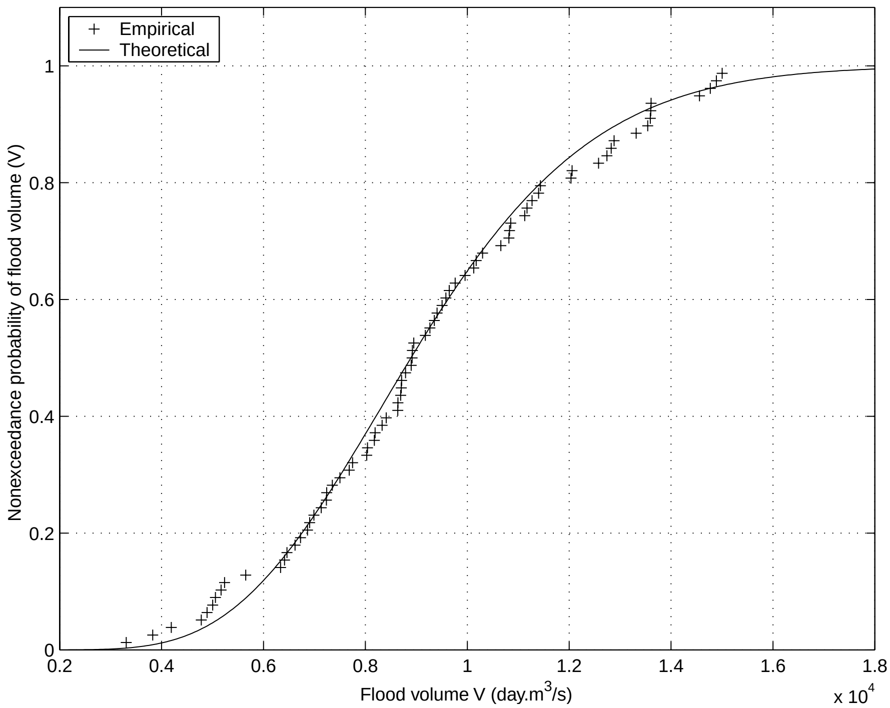


Fig. 4. (a) Marginal distribution of flood peak; (b) marginal distribution of flood volume; (c) marginal distribution of flood duration.

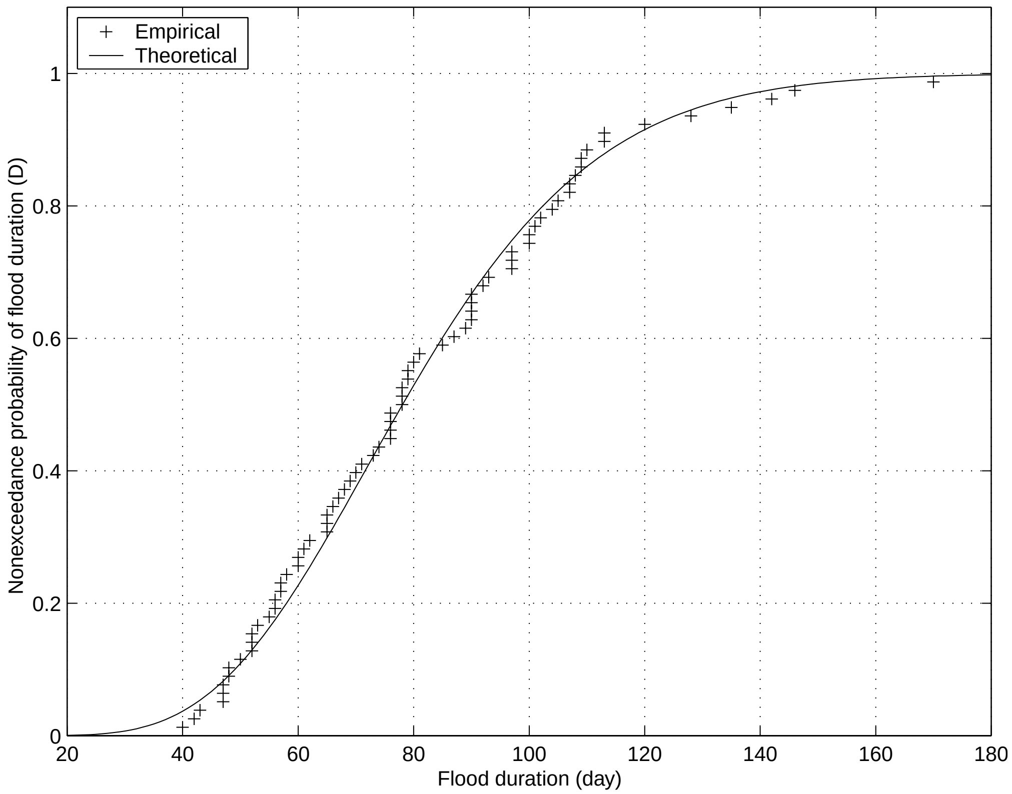


Fig. 4. (*continued*)

peak, volume, and duration are illustrated in Fig. 4(a)–(c), respectively.

The product-moment correlation coefficients between flood peaks and volumes and between flood volumes and durations were estimated using Eq. (4), and are 0.675 and 0.577, respectively. The corresponding association parameters for the Izawa, SAT, and FGM models are given in Table 6. The correlation coefficient of the normalized sample data for the Moran model is also presented in Table 6.

3.6.3. Joint PDFs of the flood peak (Q) and volume (V)

Theoretically, joint PDF, $f(q,v) \geq 0$. The reviewed models were checked by plotting their joint PDFs. The joint PDFs of the flood peak and volume with $Q = 0$ (10) 500 and $V = 0$ (400) 20000 were computed using Eqs. (3a), (5f), (6a), and (7a) for the Izawa, Moran, SAT, and FGM models, respectively. They are displayed in Fig. 5(a)–(d). Fig. 5(d) indicates that the joint PDF of the FGM model is negative in some domains (the minimum value of the joint PDF is -4.9×10^{-8}) because its association parameter $\eta = 2.025$, which is much greater than its upper limitation of one. Thus, the FGM model cannot be used to represent the joint probability distribution of the correlated flood peak and volume.

Table 6
Association parameters of the bigamma models

ρ	Association (η)		Moran(ρ_N)
	Izawa	SAT	
0.675 (Q& V)	0.700	0.700	0.685
0.577 (V& D)	0.620	0.620	0.605

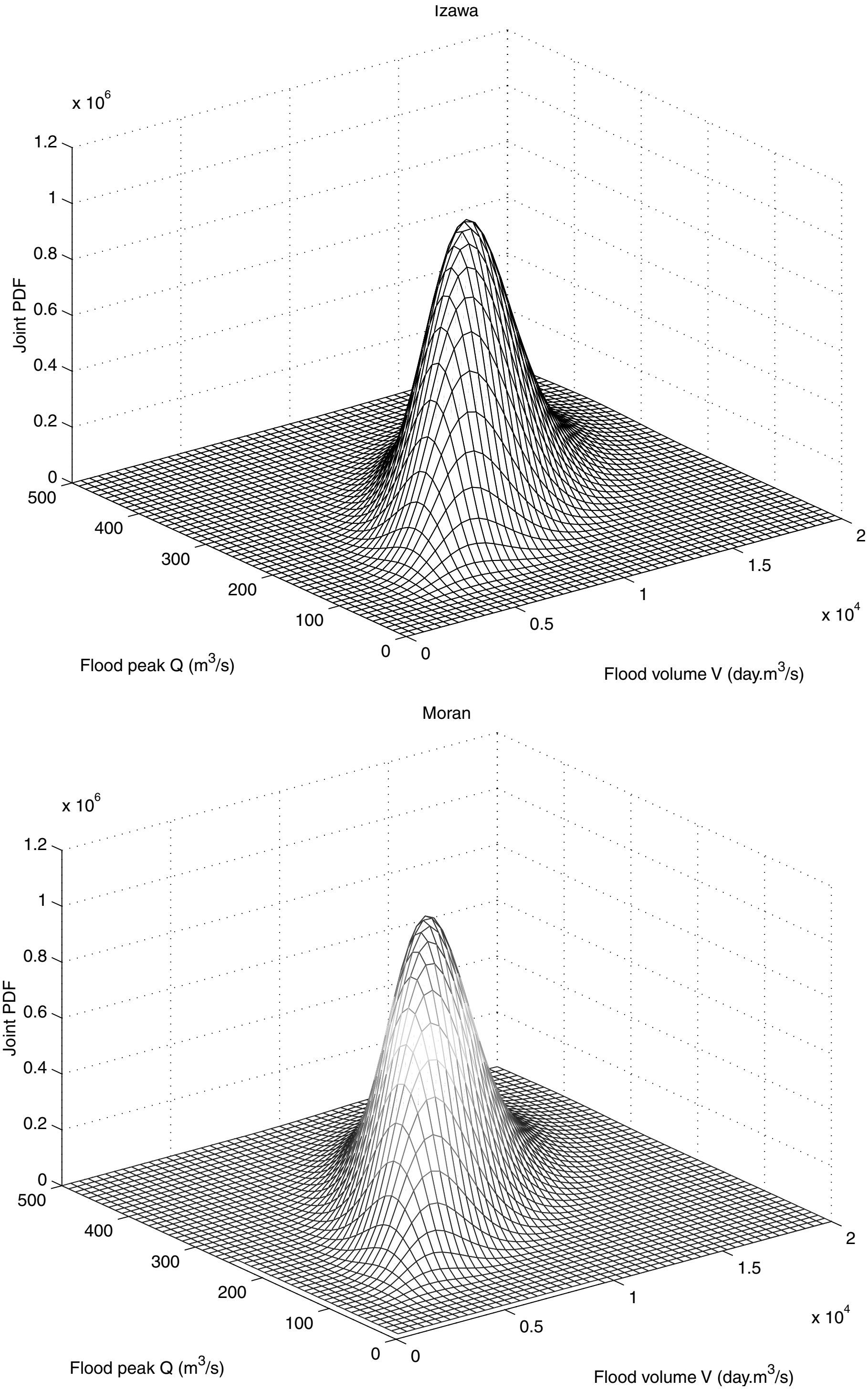


Fig. 5. (a) Joint PDF of flood peak and volume by the Izawa model; (b) joint PDF of flood peak and volume by the Moran model; (c) joint PDF of flood peak and volume by the SAT model; (d) joint PDF of flood peak and volume by the FGM model.

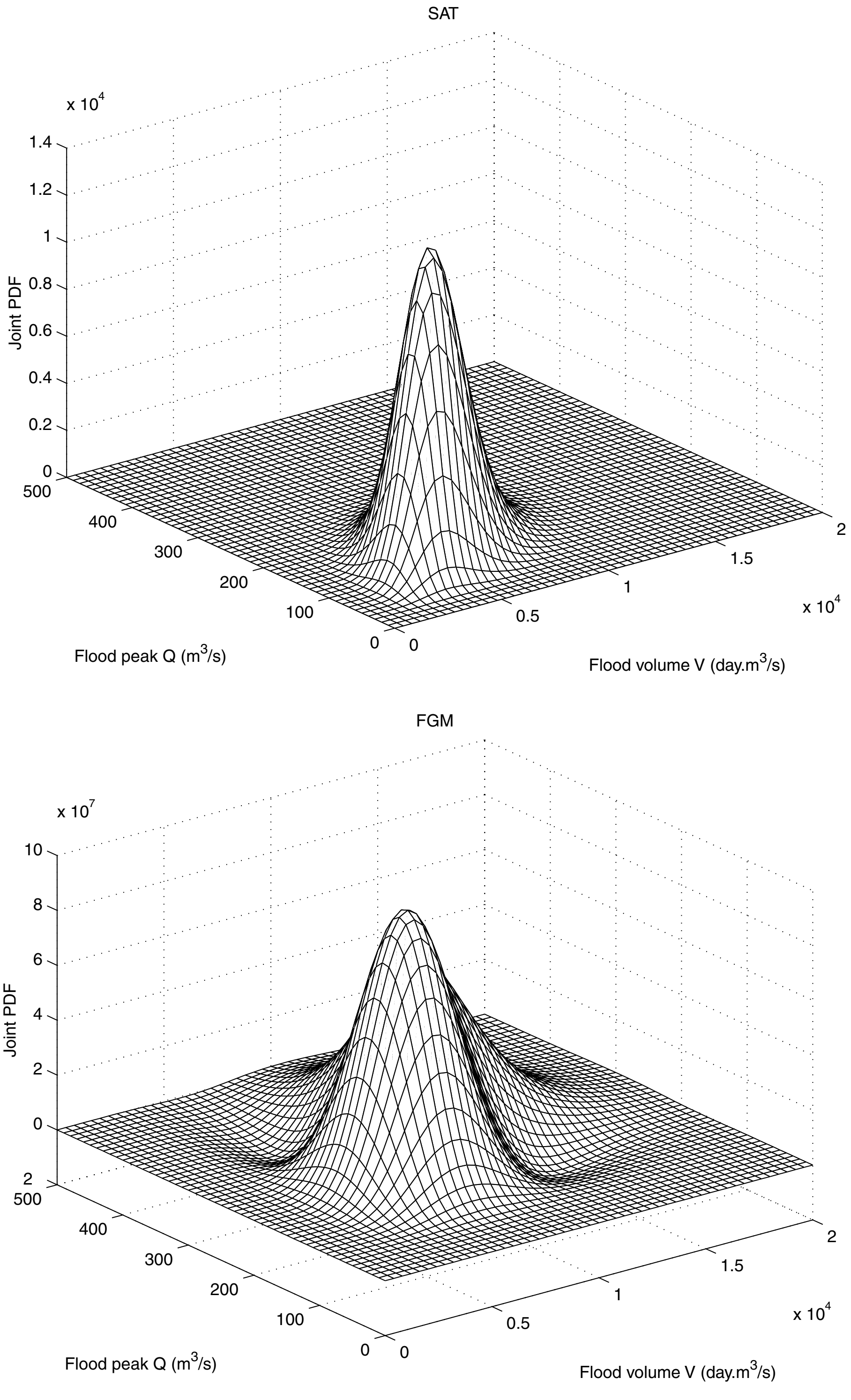


Fig. 5. (*continued*)

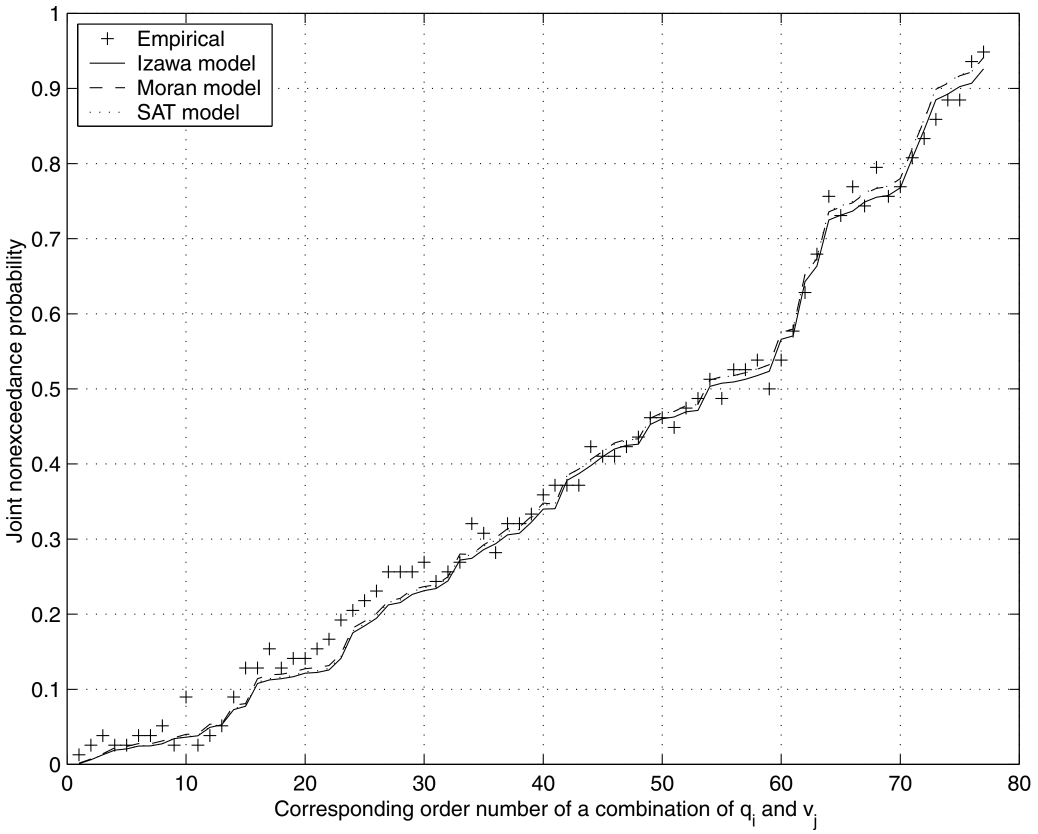


Fig. 6. Comparison of empirical and theoretical joint probabilities of flood peak and volume.

3.6.4. Joint CDF of flood peak (Q) and volume (V)

Empirical joint probabilities corresponding to the real occurrence combinations of flood peaks and volumes were computed using Eqs. (9) and (10) in which $N = 77$. Theoretical joint probabilities of the real occurrence combinations of q_i and v_j were calculated by the Izawa, Moran, and SAT models. The empirical and theoretical joint probabilities are depicted in Fig. 6. Fig. 6 demonstrates that the theoretical probabilities computed by the Izawa, Moran, and SAT models fit the empirical ones well.

3.6.5. Joint CDF of flood volume (V) and duration (D)

Similarly, empirical and theoretical joint probabilities of real occurrence combinations of the flood volume v_i and duration d_j were computed and displayed in Fig. 7. The three models (Izawa, Moran, and SAT) provide a good fit to the empirical probabilities.

From the above observations, it can be concluded that the reviewed bigamma models: Izawa, Moran and SAT models might be useful for representing the joint distribution of two gamma distributed random variables within the range of the models' association limitations. The Moran and the FGM models will be useful for describing a joint distribution of negatively correlated random variables. But the usage of the FGM is only limited to describe the joint distribution of two correlated variables with weak association ($|\rho| \leq 1/3$), as documented by Schucany et al. (1978) and Long and Krzysztofowicz (1992), and others.

4. Conclusions

This paper summarized the bivariate gamma

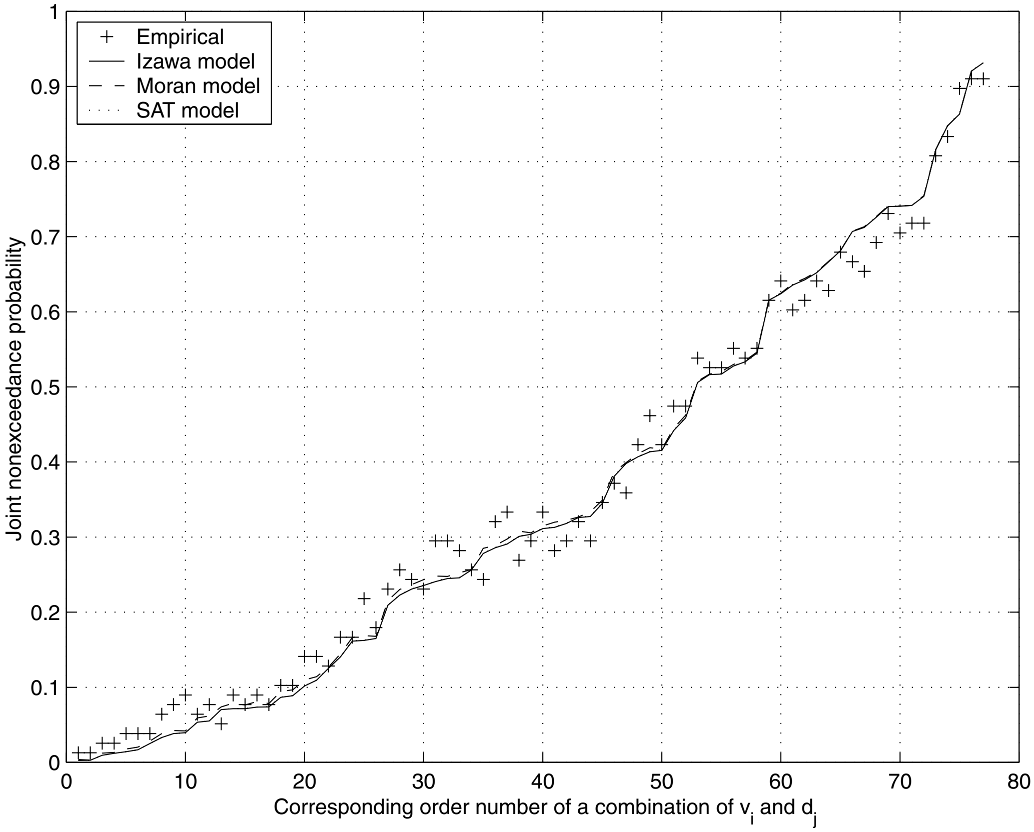


Fig. 7. Comparison of empirical and theoretical joint probabilities of flood volume and duration.

distributions that exist in the literature. These models may be useful for hydrological engineers to analyze joint statistical behavior of multivariate hydrological events such as floods and storms. The advantages and limitations of these models were pointed out. The applicability of the Izawa, Moran, and SAT bigamma models with a general form, i.e. with five-parameters was investigated using both generated and observed flood data. Results indicate that these three models can be used to represent the joint probability distribution of two positively correlated random variables with different gamma marginals.

The Moran bigamma model represents a full range of the association between two correlated variables, i.e. $|\rho| < 1$. It is a special case of the bivariate meta-Guassian model of Kelly and Krzysztofowicz (1997).

The FGM model can allow its marginals to have different distribution types. It can also represent a joint distribution of both positively and negatively correlated random variables. But the FGM has a limitation on association between two variables, i.e. $|\rho| \leq 1/3$, or $|\eta| \leq 1$.

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Appendix A

A.1. Kibble model 1 (KM1)

Kibble (1941) derived a bivariate gamma PDF by using the bivariate normal PDF of two standard normal variates W and G given by Eq. (6a). Let $X = W^2/2$ and $Y = G^2/2$, then by taking into account the fact that there are four possible pairs (w, g) corresponding to each pair (x, y) (due to the fact that w and g can be positive or negative, but x and y are always positive), the joint PDF of X and Y can be deduced as follows:

$$f(x,y) = \frac{(xy)^{-1/2}}{2\pi\sqrt{1-\rho^2}}\left[\exp\left(-\frac{x - 2\rho\sqrt{xy} + y}{1-\rho^2}\right) + \exp\left(-\frac{x + 2\rho\sqrt{xy} + y}{1-\rho^2}\right)\right] \quad (|\rho| < 1) \tag{A1}$$

where ρ is the product-moment correlation coefficient of X and Y. This form represents a bivariate gamma PDF of X and Y whose special marginal distributions are $f_X(x; 1,1/2)$ and $f_Y(y; 1,1/2)$, respectively.

A.2. Kibble model 2 (KM2)

Eq. (A1) represents a simple and special form of the BGD. It provides the basis for developing other BGD models. From this model, Kibble (1941) obtained the moment generating function and developed another bivariate gamma model as follows:

$$f(x,y) = \frac{(xy)^{(\lambda-1)/2} e^{-\frac{x+y}{1-\rho}}}{\Gamma(\lambda)(1-\rho)\rho^{(\lambda-1)/2}} I_{\lambda-1}\left(\frac{2\sqrt{\rho xy}}{1-\rho}\right) \tag{A2}$$

$(\lambda > -1, \mathrm{x} \geq 0;\ \mathrm{y} \geq 0;\ 0 \leq \rho < 1)$

where $I_s(\cdot)$ is the same as defined by Eq. (5b). The marginal PDFs of X and Y are $f_X(x; 1,\lambda)$ and $f_Y(y; 1,\lambda)$, respectively. Blokhinov and Sarmanov (1968) and Sarmanov (1971) termed the model (A2) as a symmetric bivariate gamma distribution and they also provided Eq. (A2) in the form of Laguerre polynomials.

A.3. Kibble model 3 (KM3)

Kibble (1941) also derived another canonical form bigamma distribution in terms of Laguerre polynomials as follows:

$$f(x,y) = f_X(x; 1, \lambda_x) f_Y(y; 1, \lambda_y) \times \left[1 + \sum_{k=1}^{\infty} a_k \rho^{2k} L_k^{\lambda_x - 1}(x) L_k^{\lambda_y - 1}(y)\right] \tag{A3a}$$

$(|\rho| < 1)$

where

$$a_k = \frac{\Gamma(p+k)\Gamma(\lambda_x)\Gamma(\lambda_y)}{\Gamma(p)\Gamma(\lambda_x + k)\Gamma(\lambda_y + k)} \qquad (\lambda_x, \lambda_y > p\rho^2) \tag{A3b}$$

and $L_k^{\lambda}(s)$ is the Laguerre polynomial

$$L_k^{\lambda-1}(s) = \sqrt{\left[\frac{\Gamma(\lambda)\Gamma(\lambda+k)}{k!}\right]} \sum_{r=0}^{k} \frac{(-1)^r C_k^r s^r}{\Gamma(\lambda + r)} \tag{A3c}$$

$(k = 1, 2, \ldots)$

The same formula also appears in the works by Blokhinov and Sarmanov (1968) and Sarmanov (1971), and Johnson and Kotz (1972). Blokhinov and Sarmanov (1968) discussed the use of the model (A3a) for computing long-term streamflows. This model was also extended by Gupta (1979).

The dependence of the KM1 and KM2 is fully measured by the product-moment correlation coefficient; while the dependence of the KM3 is measured via the Laguerre polynomial expressions. Although the KM3 can be readily generalized to have five-parameter bigamma distribution by using $X = \alpha_x X$ and $Y = \alpha_y Y$, the parameter p in Eq. (A3b) have no explicit expression.

A.4. Nagao and Kadoya model 1 (NKM1)

On the basis of the Izawa five-parameter bigamma model Eq. (3a) (Izawa, 1953), Nagao and Kadoya (1970) derived a four-parameter bivariate gamma model whose marginals have same shape parameter, i.e. $\lambda_x = \lambda_y = \lambda$. The joint PDF is

$$f(x,y) = \frac{(\alpha_x\alpha_y)^{(\lambda+1)/2}(xy)^{(\lambda-1)/2}e^{-\frac{\alpha_x x+\alpha_y y}{1-\rho}}}{\Gamma(\lambda)(1-\rho)\rho^{(\lambda-1)/2}} \times I_{\lambda-1}\left(\frac{2\sqrt{\alpha_x\alpha_y\rho xy}}{1-\rho}\right) \quad (0<\rho<1) \quad \text{(A4)}$$

This model was published in Japanese. The PDFs of the marginal distributions of X and Y are $f_X(x;\ \alpha_x,\lambda)$ and $f_Y(y;\ \alpha_y,\lambda)$, respectively.

A.5. Nagao and Kadoya model 2 (NKM2)

Also from the Izawa five-parameter bigamma model Eq. (3a), based on that an exponential distribution is a special case of a gamma distribution, they derived a bivariate exponential distribution (Nagao and Kadoya, 1970, 1971) as follows:

$$f(x,y) = \frac{(\alpha_x\alpha_y)e^{-\frac{\alpha_x x+\alpha_y y}{1-\rho}}}{1-\rho} I_0\left(\frac{2\sqrt{\alpha_x\alpha_y\rho xy}}{1-\rho}\right) \quad \text{(A5)}$$

$(0 \le \rho < 1)$

In fact, the KM2 is a special case of the NKM 1 in which $\lambda_x = \lambda_y = 1$. The marginal PDFs of X and Y are $f_X(x;\ \alpha_x,1$ and $f_Y(y;\ \alpha_y,1)$, respectively. These marginals are the exponential distributions. Choulakian et al. (1990) used the NKM2 to represent the joint distribution of flood peak beyond a threshold and the corresponding duration. Ashkar et al. (1998) employed the NKM2 for representing the joint probability distribution of the volume and duration of low-flow events. Goel et al. (2000) exploited the NKM2 to represent the joint distribution of rainfall intensity and duration which was used to derive flood frequency distribution. The work of Yue (2000b) explored the suitability of the model for representing two positively correlated exponentially distributed random variables.

A.6. Cherian model (1941)

Let X_1, X_2 and X_3 be the independent identically distributed (iid) variables that have gamma marginal distributions with index parameters λ_1, λ_2 and λ_3, respectively. Then $X = X_1 + X_3$ and $Y = X_2 + X_3$ have gamma marginal PDFs $f_X(x;\ 1,\lambda_1+\lambda_3)$ and $f_Y(y;\ 1,\lambda_2+\lambda_3)$, respectively. The joint PDF of X and Y is

$$f(x,y) = \frac{e^{-(x+y)}}{\prod_{i=1}^{3}\Gamma(\lambda_i)} \times \int_0^{\min(x,y)} s^{\lambda_3-1}(x-s)^{\lambda_1-1}(y-s)^{\lambda_2-1}e^{s}\,ds \quad \text{(A6a)}$$

where λ_1, λ_2 $\lambda_3 > 0$;x, $y > 0$. The dependence between X and Y is measured by

$$\eta = \frac{\lambda_3}{\sqrt{(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)}} \quad \text{(A6b)}$$

A.7. Crovelli model

Crovelli (1973)proposed the two random variables X and Y that have particular gamma PDFs $f_X(x;\alpha_x,2)$ and $f_Y(y;\alpha_y,2)$, respectively. The joint PDF of X and Y is presented by

$$f(x,y) = \begin{cases} \alpha_x\alpha_y e^{-\alpha_y y}(1-e^{-\alpha_x x}) & (0 \le \alpha_x x \le \alpha_y y) \\ \alpha_x\alpha_y e^{-\alpha_x x}(1-e^{-\alpha_y y}) & (0 \le \alpha_y y \le \alpha_x x) \end{cases} \quad \text{(A7)}$$

The dependence between X and Y is described by a linear regression. He discussed the properties of this distribution and employed it to model the joint distribution of storm depths and durations. This model is also a special case of the bivariate gamma model.

A.8. McKay model

Let $\{X_1,\ldots,X_N\}$ be a random sample from a normal population. Suppose S_N^2 is the sample variance, and let S_n^2 be the variance in a sub-sample of size n from $\{X_1,\ldots,X_N\}$. McKay (1934) deduced the joint probability density function of S_N^2 and S_n^2 as follows:

$$f(x,y) = \frac{\alpha^{\lambda_x+\lambda_y}}{\Gamma(\lambda_x)\Gamma(\lambda_y)} x^{\lambda_x-1}(y-x)^{\lambda_y-1}e^{-\alpha y} \quad \text{(A8a)}$$

where $y > x > 0$ and α, λ_x, $\lambda_y > 0$. The dependence of the model is measured by

$$\eta = \sqrt{\frac{\lambda_x}{\lambda_x + \lambda_y}} \tag{A8b}$$

The marginal PDFs of X and Y have the special forms $f_X(x;\alpha,\lambda_x)$ and $f_Y(y;\alpha,\lambda_x + \lambda_y)$, respectively. Clarke (1980) employed this model for extending annual streamflow records from precipitation data.

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