

# Generalized Richards' equation to simulate water transport in unsaturated soils

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## Abstract

Simulations of water transport in soil are ubiquitous, and the Richards' equation introduced in 1931 is the main tool for that purpose. For experiments on water transport in soil horizontal columns, Richards' equation predicts that volumetric water contents should depend solely on the ratio (distance)/(time)<sup>q</sup> where  $q = 0.5$ . Substantial experimental evidence shows that value of  $q$  is significantly less than 0.5 in some cases. Donald Nielsen and colleagues in 1962 related values of  $q < 0.5$  to 'jerky movements' of the wetting front, i.e. occurrences of rare large movements. The physical model of such transport is the transport of particles being randomly trapped and having a power law distribution of waiting periods. The corresponding mathematical model is a generalized Richards' equation in which the derivative of water content on time is a fractional one with the order equal or less than one. We solved this equation numerically and fitted the solution to data on horizontal water transport. The classical Richards' equation predicted a decrease of the soil water diffusivity for the same water content as infiltration progressed whereas the generalized Richards' equation described all observations well with a single diffusivity function. Validity of the generalized Richards' equation indicates presence of memory effects in soil water transport phenomena and may help to explain scale-dependence and variability in soil hydraulic conductivity encountered by researchers who applied classical Richards' equation.

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**Keywords:** Soil water transport; Scaling; Fractional derivative; Horizontal infiltration

## 1. Introduction

Transport of soil water affects heat and solute transport in soils, defines rates of biological processes in soil and water supply to plants, governs transpiration and ground water replenishment, controls runoff, and has many other important functions in the environment. Therefore,

simulations of water transport in soil have many applications in hydrology, meteorology, agronomy, environmental protection, and other soil-related disciplines. Success of a multitude of projects depends on the correctness of the model of soil water transport.

The Richards' equation is the most often used model. It has been introduced by Richards (1931) who has suggested that the Darcy's law originally devised for saturated flow in porous media is also applicable to unsaturated flow in porous media. One-dimensional horizontal soil columns present

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the simplest systems to assess the validity of the Richards' equation. For such systems, Richards' equation reduces to

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x} \left[ D(\theta) \frac{\partial \theta}{\partial x} \right]. \quad (1)$$

Here  $\theta$  is the volumetric soil water content ( $\text{m}^3 \text{m}^{-3}$ ),  $D$  is the soil water diffusivity ( $\text{m}^2 \text{s}^{-1}$ ),  $x$  is the distance from one of the ends of the column (m),  $t$  is time (s). Soil bulk density changes and soil water hysteresis are ignored in this formulation (Miller and Miller, 1956). Introduction of the Boltzmann variable

$$\lambda = \frac{x}{t^{0.5}} \quad (2)$$

transforms Eq. (1) into an ordinary differential equation

$$-\frac{\lambda}{2} \frac{d\theta}{d\lambda} = \frac{d}{d\lambda} \left[ D(\theta) \frac{d\theta}{d\lambda} \right] \quad (3)$$

which has been used to find analytical solutions for soil water flow problems and also to find the dependence of the diffusivity  $D$  on soil water content  $\theta$  (Hillel, 1980). If Eq. (3) is applicable then soil water content is a function of the Boltzmann variable  $\lambda$ , and, for the same values of soil water content, one should expect the same values of the Boltzmann variable.

Validity of Eq. (3) can be tested with experimental data consisting of observed soil moisture changes during infiltration in horizontal soil columns with initially uniform soil water content as shown in Fig. 1. Distances and times at which the same values of water content have been observed must obey equations

$$\frac{x_1}{t_1^{0.5}} = \frac{x_2}{t_2^{0.5}} = \frac{x_3}{t_3^{0.5}} = \dots,$$

and, in general,

$$x = At^{0.5} \quad (4)$$

where the multiplier  $A$  depends only on water content. This equation means that the dependence between  $\lg(x)$  and  $\lg(t)$  plotted in log–log coordinates is linear and the slope of this dependence is 0.5 whereas the intercept depends on the water content.

Significant deviations from Eq. (4) have been observed in many published experiments. Gardner

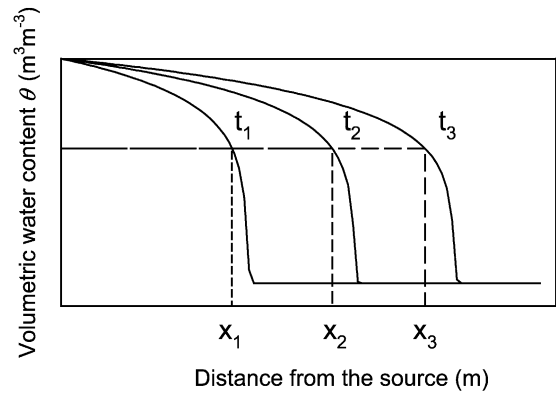


Fig. 1. Sequential observations of soil water contents in an experiment on infiltration in horizontal soil columns.

and Widtsoe (1921) and Nielsen et al. (1962) recorded the progress of the wetting front in air-dry soil uniformly packed in horizontal columns. A negative pressure head was held at one end of the columns. The largest distance where the wetting front was observed was 50 cm. Relationships between wetting front positions and time are shown in Fig. 2. A linear dependence

$$\lg x = \lg A + q \lg t \quad (5)$$

can be traced in those figures. The companion Table 1 contains values of the slope  $q$ ; all are significantly less than the value of 0.5 predicted by Eq. (4). Rawlins and Gardner (1963) and Ferguson and Gardner (1963) studied movement of water in horizontal soil columns using gamma ray attenuation. The furthest distance where water content was measured was 40 cm. Fig. 3 shows their data on the distance at which a particular water content had been reached and time when this water content was reached. Slopes of the regression lines at this figure are given in Table 1. Those slopes are also significantly less than 0.5. Similar data were published by Biggar and Taylor (1959) and Guerrini and Swartzendruber (1992, 1994). On the other hand, Selim et al. (1970) and Whisler et al. (1968) reported data from experiments on horizontal infiltration in soil columns in which no significant deviations from Eq. (4) were found.

Results of this review show that the Richards' equation is not general enough to simulate water

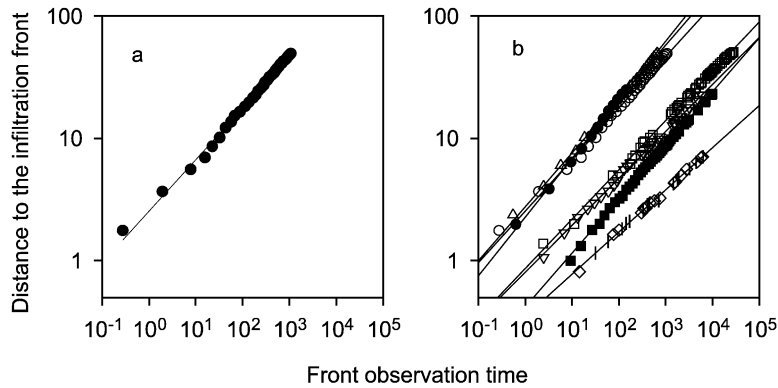


Fig. 2. Relationships between positions and observation times of the wetting front in experiments on infiltration in horizontal soil column. (a) Data of Gardner and Widtsoe (1921), dimensions of distance and time not reported; (b) data of Nielsen et al. (1962), distance in cm, time in min, symbols explained in the companion Table 1.

transport in various soils. There were attempts to generalize Richards' equation by introducing an empirical dependence of the diffusivity on time or on distance (Guerrini and Swartzendruber, 1994; Pachepsky and Timlin, 1998). Although a formal

agreement with experiments could be reached, a physical interpretation of such empirical dependence was never presented.

The objective of this work was to develop a physics-based model of water transport in unsaturated

Table 1  
Values of the parameter  $q$  found from data on horizontal movement of water to soil columns

Data source	Soil	$q$ (mean $\pm$ standard error)	Graph, symbol
Gardner and Widtsoe (1921)	Name and texture not reported	$0.417 \pm 0.006$	Fig. 2a
Nielsen et al. (1962)	Columbia silt loam wet at - 50 mb	$0.402 \pm 0.003$	Fig. 2b, $\circ$
	Columbia silt loam wet at - 100 mb	$0.425 \pm 0.006$	Fig. 2b, $\square$
	Columbia silt loam wet with oil at - 2 mb	$0.480 \pm 0.008$	Fig. 2b, $\bullet$
	Columbia silt loam wet with oil at - 38 mb	$0.440 \pm 0.003$	Fig. 2b, $\blacksquare$
	Hesperia sandy loam at - 2 mb	$0.440 \pm 0.004$	Fig. 2b, $\triangle$
	Hesperia sandy loam at - 50 mb	$0.384 \pm 0.002$	Fig. 2b, $\nabla$
	Hesperia sandy loam wet at - 100 mb	$0.344 \pm 0.003$	Fig. 2b, $\diamond$
Rawlins and Gardner (1963)	Salkum silty clay loam, $\theta = 0.51$	$0.439 \pm 0.007$	Fig. 3a
	Salkum silty clay loam, $\theta = 0.50$	$0.430 \pm 0.008$	Fig. 3a
	Salkum silty clay loam, $\theta = 0.48$	$0.437 \pm 0.011$	Fig. 3a
	Salkum silty clay loam, $\theta = 0.45$	$0.467 \pm 0.009$	Fig. 3a
	Salkum silty clay loam, $\theta = 0.40$	$0.479 \pm 0.003$	Fig. 3a
	Salkum silty clay loam, $\theta = 0.05$	$0.461 \pm 0.002$	Fig. 3a
Ferguson and Gardner (1963)	Salkum silty clay loam, $\theta = 0.05$	$0.454 \pm 0.002$	Fig. 3b
	Salkum silty clay loam, $\theta = 0.10$	$0.453 \pm 0.002$	Fig. 3b
	Salkum silty clay loam, $\theta = 0.15$	$0.452 \pm 0.003$	Fig. 3b
	Salkum silty clay loam, $\theta = 0.20$	$0.452 \pm 0.003$	Fig. 3b
	Salkum silty clay loam, $\theta = 0.25$	$0.452 \pm 0.003$	Fig. 3b
	Salkum silty clay loam, $\theta = 0.30$	$0.454 \pm 0.003$	Fig. 3b
	Salkum silty clay loam, $\theta = 0.35$	$0.458 \pm 0.004$	Fig. 3b
	Salkum silty clay loam, $\theta = 0.40$	$0.465 \pm 0.006$	Fig. 3b

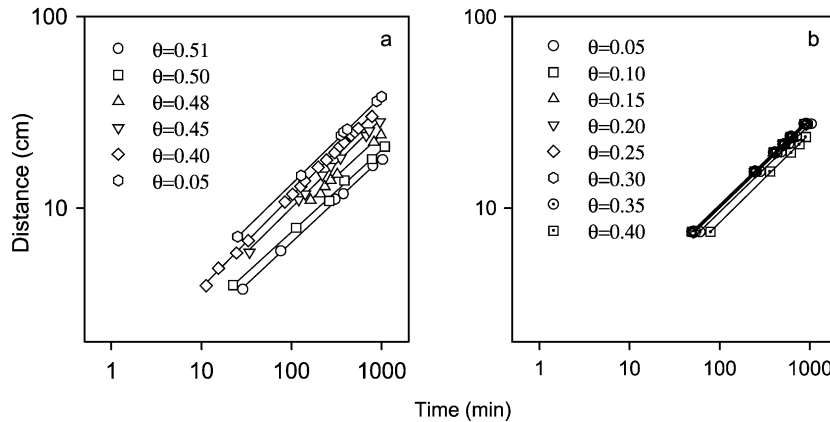


Fig. 3. Dependencies between observation time and distances for the same water contents in experiments on infiltration in horizontal soil column. (a) Data of Ferguson and Gardner (1963); (b) data of Rawlins and Gardner (1963).

soils that would explain and simulate both deviations from and agreeing to the Boltzmann scaling.

## 2. The generalized Richards' equation

### 2.1. Physical model

For a long time, the Richards' equation (1) was considered to be a purely empirical flow equation, the result of combining the equation of continuity with the experimentally based 'Buckingham–Darcy' flux law (Swartzendruber, 1968). Bhattacharya et al. (1976) showed that this equation can be derived from physically based molecular assumption. These authors had to assume that water moves in a Brownian motion in the form of quasi-molecules. Assumption that particles perform a Brownian motion has been also instrumental in deriving the diffusion equation and a convective–dispersive equation (Bhattacharya and Gupta, 1990). The Boltzmann scaling is applicable to both horizontal water transport equation and to the solute diffusion equation (Hillel, 1980).

Solute transport in porous media has been shown to exhibit deviations from Boltzmann scaling and follow a more general scaling law (5) with both  $q > 0.5$  and  $q < 0.5$  (Neuman, 1990; Hatano and Hatano, 1998; Haggerty et al., 2000; Pachepsky et al., 2001). The physical model for such transport is the movement of particles that do not perform a Brownian motion

because of constraints imposed either by structure of porous media or by the solute–surface interactions (Metzler and Klafter, 2000). The case of  $q > 0.5$  was interpreted resulting from Lévy motion of solute particles, which is similar to the Brownian motion except that the relatively large transitions occur relatively more often. This may happen because of the presence of highly conductive fractures, channels, or macropores. The case  $q < 0.5$  is also interpreted as non-Brownian transport of particles that remain motionless for extended periods of time, for example, when waiting periods have a power law distribution. Such a physical model was envisaged by Donald Nielsen and colleagues in 1962 who had suggested that the exponent  $q < 0.5$  might occur because the infiltration front underwent 'jerky movements', i.e. immobility of the wetting front occurred for substantially extended time periods. This physical model of water particles being randomly trapped and having a power law distribution of waiting periods is used in this work to generalize the Richards' equation.

### 2.2. Mathematical model

Particles with a power law distribution of waiting periods have an infinite mean waiting time, and it has been suggested to simulate the transport of an ensemble of such particles using fractional derivative on time (Meerschaert et al., 2002). The transport equation appears to be similar to Eq. (1) except that

fractional derivative of water content on time is used:

$$\frac{\partial^\gamma \theta}{\partial t^\gamma} = \frac{\partial}{\partial x} \left[ D_\gamma(\theta) \frac{\partial \theta}{\partial x} \right]. \quad (6)$$

Here  $\gamma$  is the order of the fractional derivative, and  $D_\gamma$  is fractional diffusivity. The order of the fractional derivative is less or equal to 1. Eq. (6) transforms into the classical Richards' equation when  $\gamma = 1$ . The fractional diffusivity  $D_\gamma$  depends on water content. The water flux  $Q$  is governed by the Darcy law

$$Q = -D_\gamma(\theta) \frac{\partial \theta}{\partial x}$$

as in the classical Richards' equation.

The meaning of a fractional derivative may be perceived from its finite-difference approximation. If the function  $\theta(t)$  is defined at time moments  $t_0 = 0, t_1 = \Delta t, t_2 = 2\Delta t, \dots, t_n = n\Delta t$ , and it is smooth at  $t = 0$ , then the following approximation can be used (Gorenflo, 1997):

$$\begin{aligned} \frac{\partial^\gamma \theta}{\partial t^\gamma} \Big|_{t=t_n} &\approx \frac{\theta^n - c_1 \theta^{n-1} - c_2 \theta^{n-2} - c_3 \theta^{n-3} - \dots - c_{n-1} \theta^1 - c_n \theta^0}{(\Delta t)^\gamma}. \end{aligned} \quad (7)$$

Here  $\theta^n$  is the value of  $\theta$  at time  $t_n$ ,  $\theta^{n-1}$  is the value of  $\theta$  at time  $t_{n-1}$ , etc. coefficients  $c_j, j=1,2,3,\dots$ , depend on the order of the fractional differentiation as:

$$c_j = (-1)^{j-1} \binom{\gamma}{j} = \frac{\gamma(\gamma-1)\dots(\gamma-j)}{1 \cdot 2 \dots j}. \quad (8)$$

The important feature of the fractional derivative approximation (7) is the incorporation of the values of the dependent variable  $\theta$  not only at the current and the previous time moments  $t_n$  and  $t_{n-1}$ , as it is usually done with the first-order derivatives, but at all previous incremental time moments  $t_n, t_{n-1}, t_{n-2}, \dots, t_1, t_0$ .

Coefficients  $c_j$  are decaying functions of  $j$  as shown in Fig. 4. Beginning from  $j = 10$ , dependencies of  $c_j$  on  $j$  can be approximated by power laws  $c = c_1 j^{-m}$  where  $m = 1 + \gamma$ .

By introducing an analog  $\xi$  of the Boltzmann variable as

$$\xi = \frac{x}{t^{\gamma/2}}, \quad (9)$$

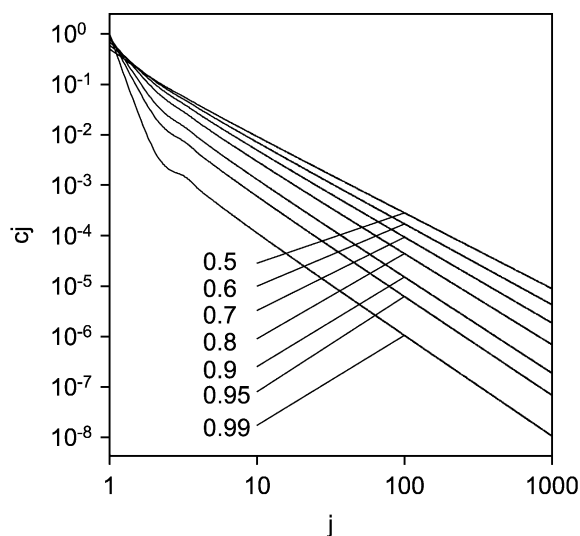


Fig. 4. Coefficients  $c_j$  to compute the fractional derivatives for several values of the order of the fractional derivatives.

Eq. (6) can be easily transformed into

$$\frac{\Gamma(1 - \gamma/2)}{\Gamma(1 - 3\gamma/2)} \xi \frac{d\theta}{d\xi} = \frac{d}{d\xi} \left[ D_\gamma \frac{d\theta}{d\xi} \right], \quad (10)$$

which reduces to Eq. (3) when  $\gamma = 1$  (the derivation is in Appendix A,  $\Gamma$  is the gamma-function). Eq. (10) shows that soil water content is a function of the variable  $\xi$ , and, for the same values of soil water content, one should expect the same values of the variable  $\xi$ . Therefore, the equation

$$x = A(\theta)t^{\gamma/2} \quad (11)$$

or its analog

$$\lg x = \lg A + \frac{\gamma}{2} \lg t \quad (12)$$

has to be valid in experiments with horizontal infiltration in a soil column. Comparison of Eqs. (12) and (5) leads to the conclusion that Eq. (12) is indeed valid in experiments with horizontal infiltration and the empirical parameter  $q$  in this equation is equal to  $\gamma/2$ . Values of  $q$  in Table 1 are all less than 0.5 which means that Eq. (6) is valid in this experimental conditions where  $\gamma < 1$ .

Eq. (10) can be rearranged to compute the fractional diffusivity  $D_\gamma$  from experimentally defined

dependence of  $\xi$  on  $\theta$

$$D(\theta) = \frac{\Gamma(1 - \gamma/2)}{\Gamma(1 - 3\gamma/2)} \frac{d\xi}{d\theta} \int_{\theta_i}^{\theta} \xi(\theta) d\theta \quad (13)$$

where  $\theta_i$  is the initial water content, when  $\gamma = 1$ , this equation reduces to Philip's (1955) equation to compute the hydraulic diffusivity:

$$D(\theta) = -\frac{1}{2} \frac{d\lambda}{d\theta} \int_{\theta_i}^{\theta} \lambda(\theta) d\theta. \quad (14)$$

Water transport equations have to be solved numerically when boundary conditions are variable, the initial water content is not homogeneous, and the flow domain cannot be assumed semi-infinite. Using Eq. (7), one derives an implicit finite difference approximation of Eq. (6) as

$$\frac{\theta_i^n - \sum_{j=1}^n c_j \theta_i^{n-j}}{(\Delta t)^\gamma} = \frac{\frac{1}{2} [D(\theta_{i+1}^n) + D(\theta_i^n)] \frac{\theta_{i+1}^n - \theta_i^n}{\Delta x} - \frac{1}{2} [D(\theta_i^n) + D(\theta_{i-1}^n)] \frac{\theta_i^n - \theta_{i-1}^n}{\Delta x}}{\Delta x} \quad (15)$$

where the subscript  $i$  denotes values of  $\theta$  at  $x = x_i = i\Delta x$ ,  $i = 1, 2, \dots, M - 1$ . Simple transformation converts this system of equations into a tridiagonal system on non-linear equations with respect to  $\theta_i^n$ ,  $i = 1, 1, 2, \dots, M - 1$

$$\begin{aligned} & \frac{D(\theta_{i-1}^n) + D(\theta_i^n)}{2(\Delta x)^2} \theta_{i-1}^n \\ & - \left[ \frac{D(\theta_{i-1}^n) + 2D(\theta_i^n) + D(\theta_{i+1}^n)}{2(\Delta x)^2} + \frac{1}{(\Delta t)^\gamma} \right] \theta_{i+1}^n \\ & + \frac{D(\theta_{i+1}^n) + D(\theta_i^n)}{2(\Delta x)^2} \theta_{i+1}^n = - \frac{\sum_{j=1}^n c_j \theta_i^{n-j}}{(\Delta t)^\gamma} \end{aligned} \quad (16)$$

for  $i = 1, 2, 3, \dots, M$ . Two more equations are derived from boundary conditions, after that the resulting system of  $M + 1$  equations can be solved using iterations with Gauss elimination (Press et al., 1992). The code to implement this algorithm for numerical solution of Eq. (6) has been written in FORTRAN and is available from the corresponding author upon request. It was tested to see how accurately Eq. (12) would apply to the numerical solution. Function  $D_\gamma(\theta)$  was taken as  $\exp(5(\theta - 1.5))$ ,

boundary conditions were  $\theta_0^n = 1$  and  $\theta_{M-1}^n = \theta_M^n$  for  $n = 0, 1, 2, \dots$ , the initial condition was  $\theta_i^0 = 0$  for  $i = 1, 2, \dots, M$ , intervals for  $x$  and  $t$  were  $0 \leq x \leq 50$  cm,  $0 \leq t \leq 2000$  min, the discretization was made with  $M = 100$ ,  $N = 1000$ . The scaling (12) was applicable to the numerical solution with 2% difference between  $\gamma$  values from the numerical solution and  $\gamma$  values assumed to obtain this solution for  $0 \leq x \leq 30$  cm.

### 3. Apparent scale effects on hydraulic diffusivity

The dependencies of Boltzmann variable  $\lambda$  and the fractional scaling variable  $\xi$  on volumetric water content  $\theta$  were found from data of Ferguson

and Gardner (1963) collected at 3.5, 7.5, 15.5 and 27.8 cm from the source. The experimental dependencies of the Boltzmann variable on  $\theta$  for those distances are shown in Fig. 5a. Dependencies are distinctly different for different depths.

To compute the hydraulic diffusivity from the Philips' equation (14), we had to approximate the dependencies shown in Fig. 5a because the equation includes derivatives of the dependence of  $\lambda$  on  $\theta$ . Data for each depth were fitted with the empirical logistic equation

$$\begin{aligned} \lambda &= \lambda_0 \frac{1 + u \exp(v\theta_0)}{1 + u \exp(v\theta)} \exp[v(\theta - \theta_0)] \\ &+ \lambda_{\max} \frac{1 - \exp[v(\theta - \theta_0)]}{1 + u \exp(v\theta)} \end{aligned} \quad (17)$$

where  $\theta_0$  is the maximum water content close to porosity where observation are available,  $\lambda_0$  is the observed value of  $\lambda$  at  $\theta = \theta_0$ ,  $\lambda_{\max}$ ,  $u$  and  $v$  are fitting parameters. Lines in Fig. 5a show results of this fitting. Dependencies of hydraulic diffusivity on water contents computed according to Eq. (14) are shown in Fig. 6a. Differences among dependencies of  $\lambda$  on  $\theta$  at different depths cause

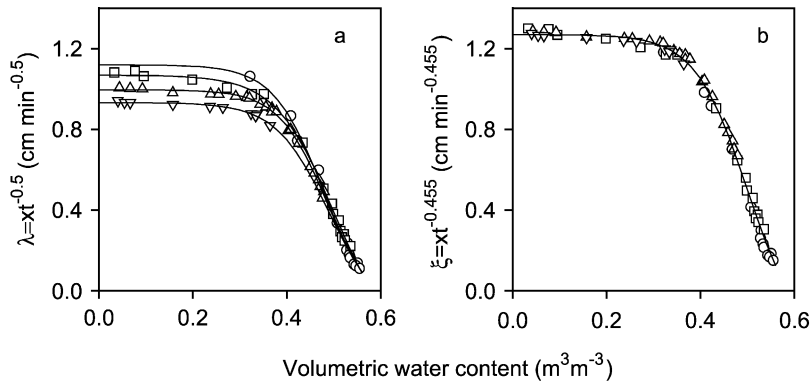


Fig. 5. Boltzmann scaling variable  $\lambda$  and fractional scaling variable  $\xi$  as functions of the volumetric water contents from observations of Ferguson and Gardner (1963) at several distances from the infiltration source;  $\circ$ —3.5 cm,  $\square$ —7.5 cm,  $\triangle$ —15.5 cm,  $\nabla$ —27.5 cm.

variability in diffusivity values. The average range of variations is about half an order of magnitude for any given  $\theta$  value.

Dependencies of the fractional scaling variable  $\xi$  on  $\theta$  are shown in Fig. 5b. The order of the fractional derivative  $\gamma$  was taken as  $\bar{q}/2$  where  $\bar{q} = 0.455$  is the average value for this data set in Table 1. Data for different depths coalesce, and it is possible to derive a unique, depth-independent function  $\xi(\theta)$ . This function could be used to generate a unique dependence of the fractional diffusivity on  $\theta$  as shown in Fig. 6b.

We carried out numerical experiments consisting of water transport simulations with the generalized Richards' equation and calculating classic Richards

diffusivity from water content profiles at several times. Results of one such experiment are shown in Fig. 7. The order of the fractional derivative was  $\gamma = 0.8$ , the initial water content and the porosity were 0.04 and  $0.55 \text{ m}^3 \text{ m}^{-3}$ , respectively, the diffusivity was  $D_\gamma = 40\theta^{(1.8-2.5 \ln \theta)} \text{ cm min}^{-0.8}$ , the distances and time ranged from 0 to 40 cm and from 0 to 1200 min, respectively. Fig. 7a shows dependencies of the Boltzmann variable on water content obtained from water content profiles at different times. The decrease in values of  $\lambda$  as time progresses can be observed. This decrease causes an apparent decrease in diffusivity values obtained for the same water contents as time increases (Fig. 7b). When the water

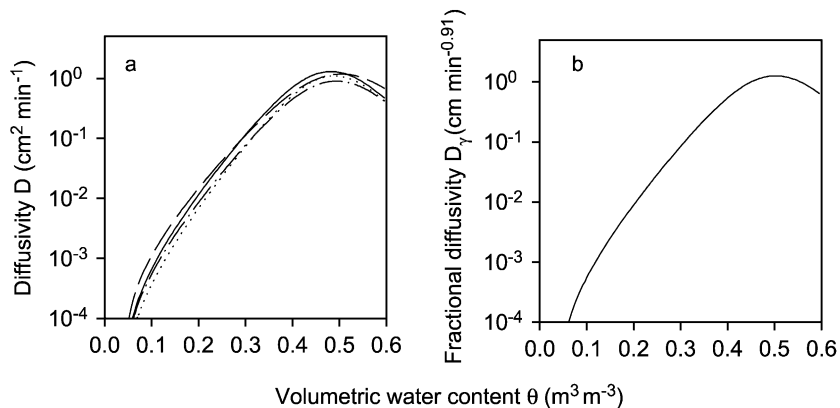


Fig. 6. Diffusivity for the classic Richards equation (a) and for the generalized Richards' equation (b) from observations of Ferguson and Gardner (1963) at several distances from the infiltration source; — 3.5 cm, --- 7.5 cm, ... 15.5 cm, -.-.- 27.5 cm.

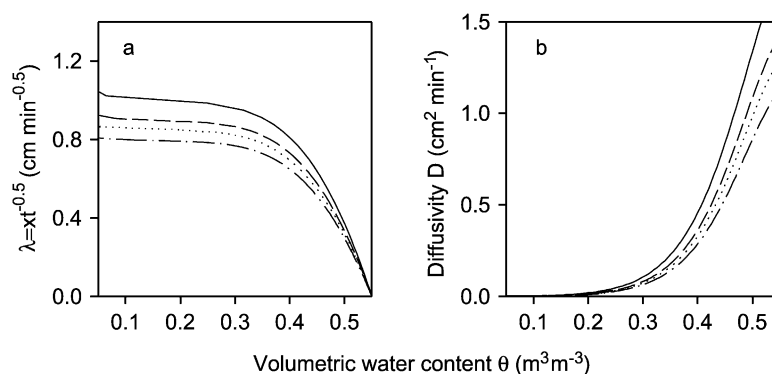


Fig. 7. Boltzmann scaling variable  $\lambda$  and diffusivity from the numerical solution of the generalized Richards' equation for several simulated times; — 120 min, - - - 360 min, - · - · 600 min, · · · · 1200 min.

contents at different distances rather than times were used to compute  $\lambda$ , the diffusivity decreased with the distance (data not shown).

#### 4. Discussion

The generalized Richards' equation provided a good description of the experimentally observed scaling of soil water contents during the horizontal infiltration. The difference between assumptions leading to the classical Richards' equation and to the generalized Richards' equation consists in the type of movements that water particles perform. The assumption that long waiting periods without movement occur relatively more often than in homogeneous medium may be more applicable to soils which are aggregated media. Water within aggregates may have difficulty moving to intraaggregate space. The assumption of long waiting periods should be especially true in the experiments considered in this work because the soil columns were packed with sieved soil.

It seems to be important that the generalized Richards' equation includes the classical Richards' equation as a specific case when the parameter  $\gamma$  does not differ significantly from one. Some authors did not find tangible deviations from the Boltzmann scaling in experiments with infiltration into horizontal columns (Reichardt et al., 1972; Selim et al., 1970; Whisler et al., 1968). Eq. (6) encompasses both such cases and cases when the Boltzmann scaling is violated. It is worth noting that

the presence of the fractional derivative in the generalized Richards' equation does not result in violations of the mass balance (Metzler and Klafter, 2000).

The fractional derivative is a quite old mathematical notion, and, for example, the approximation (7) has been proposed in 1868 (Letnikov, 1868). For a specific distance, this approximation includes water contents at all times from the beginning of the transport. This can be interpreted as incorporating the history of the process in the model. A review of applications of fractional derivatives to systems with history-dependent behavior can be found in paper of Mainardi (1997). Note that water contents at earlier times enter the approximation with smaller coefficients. Nevertheless, retaining only a small number of coefficients in the expansion (7) leads to an error accumulation, and scaling with  $q = \gamma/2$  appears to be violated. With respect to the numerical solution of Eq. (12), this means that the algorithm will require much more storage memory than the algorithm for numerical solution of the classical Richards' equation. Water contents at all previous time steps must be stored to assure correct computation of the fractional derivative. Computer memory was not an issue in our simulations done on PC-333; however, it may be desirable in future to carry out an error analysis related to retaining water contents at a limited number of foregoing time steps.

Scanning through Table 1 shows an interesting feature of the data from Nielsen et al. (1962) for Hesperia sandy loam. The lower the soil water potential, the lower is the value of  $q$ . The same



authors did not observe that for the Columbia silt loam. We do not have other data to decide whether it is a common feature or just an artifact caused by using different soil columns. Since the connectivity of water-filled pores may be different at different water contents, there is no reason why the value of  $q$  should be constant over the range of water contents. However, the value of  $q$  was approximately constant over a range of water contents in other experiments listed in Table 1.

Applying the classical Richards' equation to soil in which the generalized Richards' equation is actually valid led to both variability and scale-dependence in the hydraulic diffusivity function. The scale dependence shown in Fig. 7 is very similar to the scale dependence in dispersivity observed by Rawlins and Gardner (1963) in experiments on infiltration into horizontal soil columns (Fig. 8). Both data of numerical simulations used to derive diffusivities in Fig. 7 and experimental data used to derive diffusivities in Fig. 8 were of high spatial and temporal density. Experimental data of Ferguson and Gardner (1963) used to derive diffusivities in Fig. 6a were not dense in time and space (Fig. 5a). Because Eq. (4) contains a derivative of the very steep dependence of  $\lambda$  on  $\theta$ , an inaccuracy in computing the derivatives from sparse data results in the diffusivity

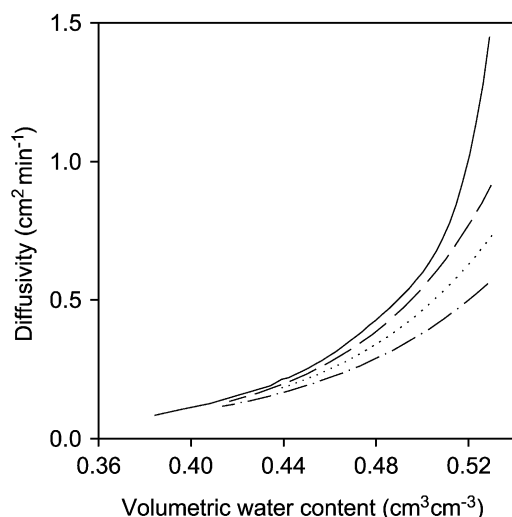


Fig. 8. Diffusivity computed by Rawlins and Gardner (1963) from observations of horizontal infiltration made at various times. — 16 min, - - - 144 min, . . . 400 min, - · - · 1440 min.

variability shown in Fig. 6b rather than in scale dependence of the type shown in Figs. 7b and 8. The data density depends on experimental setup and may not be high in some types of experiments. However, when the correct scaling of water contents is applied (Fig. 5b), the source of the inaccuracy disappears.

Several reasons for violation of the Boltzmann scaling were pondered by the authors who encountered it. Biggar and Taylor (1959) indicated that soil particles might change as the infiltration front advanced, and variations in packing might affect the infiltration rate. Nielsen et al. (1962) invoked a dependence of contact angle on the rate of movement. Rawlins and Gardner (1963) discussed heat evolving as water wets soil, and changes in soil solution composition during wetting as other possible cases of non-Boltzmann scaling. We note that the magnitude of the aforementioned effects alleged to cause non-Boltzmann scaling was not actually observed in experiments. Neither was their absence shown. Therefore, one cannot exclude a role of those effects. We stress, however, that the structure of soil pore space preventing purely Brownian motion may be a probable cause of the non-Boltzmann scaling.

Physical models other than the one in this paper were also shown to lead to the fractional derivative on time in diffusion-type transport equations. For example, fractional Brownian motion of particles and random walks on fractals were suggested (Jumarie, 1992; Roman and Alemany, 1994). Soil structure is a primary candidate to be an underlying cause of non-Brownian transport leading to non-Boltzmann scaling.

Soil columns in experiments on horizontal infiltration were filled with sieved aggregates. Soil in situ has pronounced structural hierarchy. Effects of this hierarchy on scaling in water transport present an interesting issue to explore. No extrapolations of the generalized Richards' equation can be made at this point for the case of vertical infiltration when an external force of gravity affects water transport. It remains to be seen whether a gravitational term can be added to Eq. (6) to use this equation to explain empirical infiltration equations that do not follow from the classical Richards' equation.

## 5. Conclusions

1. There is evidence in literature that Richards' equation cannot explain observations of water transport in horizontal soil columns.
2. A physical model of non-Brownian transport of particles that remain motionless for extended periods conforms to the observations of jerky movements of the infiltration front made by Nielsen and colleagues. This model constitutes a basis for the generalized Richards' equation that explains experimental data and includes the classical Richards' equation as a specific case.
3. The generalized Richards' equation includes a fractional derivative on time. It can be solved numerically with an algorithm similar to the one used for the classical Richards' equation. The fractional diffusivity can be determined from experimental data with the method analogous the one developed by J. Philip.
4. Both apparent variability in soil hydraulic properties and apparent scale effects in soil hydraulic properties may arise from using the classical Richards' equation where the generalized Richards equation is actually valid.

## Appendix A

Eq. (10) is derived using the property of the fractional derivative (Gorenflo, 1997; Mainardi, 1997):

$$\frac{d^\alpha t^\beta}{dt^\alpha} = \frac{\Gamma(1 + \beta)}{\Gamma(1 + \beta - \alpha)} t^{\beta - \alpha} \quad (\text{A1})$$

Assuming that  $\theta$  depends only on  $\xi$  given by Eq. (9), one obtains

$$\begin{aligned} \frac{\partial^\gamma \theta}{\partial t^\gamma} &= \frac{d\theta}{d\xi} \frac{\partial \xi}{\partial t^\gamma} = \frac{d\theta}{d\xi} x \frac{d^\gamma (t^{-\gamma/2})}{dt^\gamma} \\ &= \frac{d\theta}{d\xi} x \frac{\Gamma(1 - \gamma/2)}{\Gamma(1 - 3\gamma/2)} t^{-\gamma/2 - \gamma} \\ &= \frac{d\theta}{d\xi} \frac{\Gamma(1 - \gamma/2)}{\Gamma(1 - 3\gamma/2)} \frac{x}{t^{3\gamma/2}} \end{aligned} \quad (\text{A2})$$

$$\frac{\partial \theta}{\partial x} = \frac{d\theta}{d\xi} \frac{\partial \xi}{\partial x} = \frac{d\theta}{d\xi} \frac{1}{t^{\gamma/2}} \quad (\text{A3})$$

and

$$\frac{\partial}{\partial x} \left[ D_\gamma(\theta) \frac{\partial \theta}{\partial x} \right] = \frac{1}{t^\gamma} \frac{d}{dx} \left[ D_\gamma(\theta) \frac{d\theta}{dx} \right]. \quad (\text{A4})$$

Substitution of derivatives in Eq. (6) for their expressions from Eqs. (A2) and (A4) yields Eq. (10).

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