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The width of a bankfull channel; Lacey's formula explained

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Abstract

According to Lacey's formula, the width of a natural channel at bankfull flow is proportional to the root of the discharge. It is a very simple formula that has been confirmed by many authors and which, until now, has had no physical explanation. It appears that Lacey's equation is composed of physical and measurable parameters which agree with field observations. The equation hinges on the fact that the velocity at bankfull discharge is a sole function of the bed material. At bankfull discharge the average velocity is no longer a function of the discharge, as is assumed in regime theory. At discharges below bankfull level the stream velocity is a function of the discharge to the power 1/6. However at bankfull discharge a singularity occurs where the water slope is forced on the slope of the natural levees. The velocity of flow associated with this slope is fully determined by the bed material, and is independent of the discharge. An analytical expression for Lacey's coefficient of proportionality is presented.

Keywords: Width predictor; Stable channel; Regime theory; Alluvial rivers

1. Introduction

What are the dimensions of a stable channel? Engineers concerned with the design of such channels have asked themselves this question for many years. Kennedy (1894), Lindley (1919), Lacey (1930), Blench (1952), and Lane (1955) have done pioneering work in what has become known as the regime concept. In the 1960s intensive debates were held on regime concepts among famous hydraulic engineers including Lacey et al. (1963). As Stevens and Nordin (1987) and Stevens (1989) pointed out, there is redundancy in the equations used in regime theory. Most of the equations are better replaced by the two physical laws of conservation of mass (one for water and one for sediment), the two equations for conservation of momentum (one for water and one for sediment) and the two geometric functions for cross-sectional area and discharge. These provide a set of six equations, which is sufficient to solve the one-dimensional problem, provided the width is given. If the width is left free to vary, however, there are seven dependent variables, and an additional equation is required to describe the lateral morphological process. Lacey's equation which states that the wetted perimeter P is proportional to the square root of the bankfull discharge Q_b could serve as the seventh equation since its applicability still stands

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unchallenged. The only problem is that until now it lacks a physical explanation. Lacey's equation reads:

$$P = 4.8Q_b^{0.5} \tag{1}$$

In alluvial rivers, the wetted perimeter P is only slightly larger than the surface width B. About a century after the debate on regime theory started, there appears to be a convergence of opinion, which is presented by Cao and Knight (1996). Hence it is now widely held that width B, average depth h and average flow velocity over the cross-section U are power functions of the discharge with the following exponents: 0.5, 0.33 and 0.17. The first corresponds with Lacey's equation (and with that of many others, see Table 1); the second follows from the combination of Lacey's equation with Chezy's equation (stating that the flow velocity U is proportional to the root of the depth of flow h); and the third follows by definition ($Q = BhU$). Cao and Knight (1996) present the following relation between the width and the depth:

$$h = \frac{1}{\eta}B^{\theta} \tag{2}$$

where η is a constant and θ is an exponent equal to 0.66, i.e. the ratio of the first two exponents (0.33/0.5). These exponents and the proportionality of Eq. (2) were presented by Blench (1952) and are considered to be common knowledge in regime theory.

It is recognised that Lacey's equation is related to processes of self-organisation occurring in nature. One can relate the hydraulic geometry and the channel pattern to minimum stream power (Chang, 1979) or to optimal energy expenditure (Molnar and Ramirez, 2002). Rodrigues-Iturbe et al. (1992), based on the principle of minimum energy expenditure in river links and constant energy expenditure per unit channel bed area, determined the exponents of B, h, and U as being: 0.5, 0.5 and 0. Interestingly, in this paper, the same exponents are arrived at, but through a different route. The explanation presented here

Table 1
Values of the exponents belonging to B, h and U_b as power functions of Q. Data are based on Table 1 of Cao and Knight (1996)

Author	Year	Exponent B	Exponent h	Exponent U_b
Lacey	1929	0.50	0.33	0.17
Bose	1936	0.50	0.33	0.17
Glover et al.	1951	0.46	0.46	0.08
Leopold et al.	1953	0.45–0.56	0.37–0.45	0.17–0.00
Leopold et al.	1956	0.50	0.28	0.22
Blench, sand	1957	0.50	0.33	0.17
Blench, gravel	1957	0.50	0.40	0.10
Inglis	1957	0.50	0.33	0.17
Nash	1959	0.54	0.27	0.19
Nixon	1959	0.50	0.38	0.12
Simons et al.	1960	0.51	0.36	0.13
Ackers, cohesive	1964	0.53	0.35	0.12
Ackers, experiments	1964	0.42	0.43	0.15
Kellerhals	1967	0.50	0.40	0.10
Lapturev	1969	0.50	0.33	0.17
Bray, regression	1982	0.53	0.33	0.14
Bray, threshold	1982	0.50	0.48	0.02
Hey	1982	0.54	0.41	0.05
Hey and Thorne	1983	0.50	0.39	0.11
Ghosh	1983	0.46	0.46	0.08
Chang	1988	0.50	0.30	0.20
Yalin, sand	1991	0.50	0.33	0.17
Yalin, gravel	1991	0.50	0.43	0.07
Cao and Knight	1995	0.50	0.33	0.17
Savenije, bankfull	2001	0.50	0.50	0.00

makes use of the theory of stable sections developed by Lane (1955). These exponents are different from the exponents: 0.5, 0.33 and 0.17 mentioned earlier. In this paper the explanation for this difference lies in the singularity that occurs at bankfull flow. Although it is not always recognised, Lacey's formula applies to bankfull flow, a situation where the water slope is forced by the slope of the natural levees. This slope is not necessarily equal to the bottom slope along a meandering channel. Above bankfull flow, the valley slope determines the water slope, which can be significantly larger if the main channel is a meandering channel. Hence at bankfull level a singularity occurs where the power functions mentioned earlier no longer apply. It will be demonstrated that at bankfull level: $\theta = 1$.

This paper does not aim at providing a final answer to the questions raised by Lacey's equation. Rather it aims at opening the discussion from a different perspective. The regime of an alluvial river is dynamic and never completely stable. However, bankfull flow is a dominant bed-shaping situation which a river experiences at regular times and which is essential for maintaining the river channel.

The following analysis applies to the middle and lower reaches of meandering alluvial rivers. It excludes the upper reaches where the river is primarily eroding or where it has a braided pattern of channels. Lacey's width does not apply to braided channels, or channels with more than one main channel. Dade (2000) shows that a wide range of width to depth ratios are possible in alluvial rivers with similar sediment characteristics. The analysis below refers to a stable cross-section of minimum width.

2. The need for a seventh equation

The geometry and the hydraulic characteristics of alluvial streams (including estuaries) are governed by the dynamics of water and sediment. The movement of water and sediment is generally described by a set of four one-dimensional equations: the conservation of momentum and mass for water, the conservation of mass for sediment, and an empirical formula that relates sediment transport to flow parameters

$$\frac{\partial Q}{\partial t} + \alpha' \frac{\partial (Q^2/A)}{\partial x} + gA\frac{\partial h}{\partial x} + gA\frac{\partial z_b}{\partial x} + gA\frac{U|U|}{C^2 h} = 0 \quad (3)$$

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0 \quad (4)$$

$$B\frac{\partial z_b}{\partial t} + \frac{\partial Q_s}{\partial x} = 0 \quad (5)$$

$$Q_s = B d_s U^n \quad (6)$$

where:

- $Q = Q(x,t)$ is the discharge in m^3/s;
- α' is a shape factor (assumed constant) to account for the spatial variation of the flow velocity over the cross-section;
- $A = A(x,t)$ is the cross-sectional area in m^2;
- $h = h(x,t)$ is the mean cross-sectional depth of flow in m;
- $z_b = z_b(x,t)$ is the mean cross-sectional bottom elevation in m;
- g is the acceleration due to gravity in m/s^2;
- $U = U(x,t)$ is the mean cross-sectional flow velocity in m/s;
- $C = C(x)$ is the coefficient of Chezy in $m^{0.5}/s$;
- $B = B(x,t)$ is the channel width in m;
- $Q_s = Q_s(x,t)$ is the sediment discharge in m^3/s;
- n is an exponent;
- $d_s = d_s(x)$ is a parameter with the dimension $m^{(2-n)}\,s^{(n-1)}$ that depends on sediment characteristics and channel roughness.

The sediment discharge is defined as the transported volume of bed material, including pores, per unit time.

In the last term of Eq. (3) the depth h is used instead of the hydraulic radius. This assumption is justified if the channel is wide in relation to its depth ($B \gg h$). In alluvial channels this is generally the case. The coefficient α' is larger than unity. The more irregular a cross-section and the more the variation in flow velocity over the cross-section, the larger α'. In a regularly shaped, single channel, alluvial stream, α' is usually close to unity (Jansen et al., 1979, p. 43).

The fourth equation appears in several forms in the literature. The most widespread formula, which is highly appreciated for its applicability in alluvial rivers and for its simplicity, is the formula of Engelund and Hansen (1967), where the exponent n equals 5 and the parameter d_s is defined by:

$$d_s = \frac{0.05}{D_{50}\Delta^2 C^3 \sqrt{g}} \tag{7}$$

where D_{50} is the diameter in m of the bed material that is exceeded by 50% of the sample by weight, and Δ is the relative density of submerged sediment (generally $\Delta = (2600 - 1000)/1000 = 1.6$). In addition, the following geometric functions define A and Q as:

$$A = hB \tag{8}$$

$$Q = UA \tag{9}$$

Assuming that α', C, g, n, Δ, D_{50} and hence d_s are known, the list of dependent variables consists of the following seven parameters:

- the mean cross-sectional flow velocity $U(x,t)$
- the mean cross-sectional depth of flow $h(x,t)$
- the mean cross-sectional bottom elevation $z_b(x,t)$
- the channel width $B(x,t)$
- the cross-sectional area $A(x,t)$
- the discharge $Q(x,t)$
- the sediment discharge $Q_s(x,t)$.

Hence, there are six equations (Eqs. (3)–(6), (8) and (9)) with seven dependent variables. Consequently, one more equation is required to solve the set of equations for the seven dependent variables. In general, in computational hydraulics, the seventh equation used is one that fixes the width as a function of distance x and the water level elevation $(z_b + h)$. For a freely varying width, however, a 'seventh equation' is needed. In stable channel design, Lacey's formula is often proposed as the seventh equation.

2.1. The seventh equation

Although several efforts have been made to relate the width B to flow parameters, to the disappointment of many researchers, no unequivocal physically based method has, as yet, been developed. For alluvial channels, Lacey, in 1930, formulated a theory based on earlier work by Kennedy (1894) and Lindley (1919) which became known as 'regime' theory and which was based on the assumption that an alluvial channel adjusts its width, depth and slope in accordance to the amount of water and the amount and kind of sediment supplied (Stevens and Nordin, 1987). Lacey's theory is almost entirely empirical and supplies simple power expressions that relate stream depth, width, slope and velocity to the discharge. Regime theory has been relatively successful in India and Pakistan in the design of stable irrigation channels under natural regime. On the other hand, regime theory has been widely criticised, mainly because of its lack of physical basis, its empirical character and the scanty and incomplete database used for its derivation (Stevens and Nordin, 1987). Investigations by Stevens (1989) on stream width, however, indicated that, although there still is no satisfactory physical backing, there is also no reason to reject the empirical relation between stream width and discharge.

For his stream width formula, Lacey made use of the wetted perimeter P instead of the width B. The wetted perimeter is somewhat larger than the width: in a rectangular profile $P = B + 2h$, and in alluvial streams, where the width is generally much larger than the depth $(B \gg h)$ the wetted perimeter is approximately equal to the stream width $(P \cong B)$. Hence:

$$P \approx B = 4.8 Q_b^{0.5} \tag{10}$$

The bankfull discharge is the discharge at which the river starts spilling over the natural levees. It is the discharge above which the river can deposit sediments on its banks. Regular overtopping is necessary for the river to maintain its bed.

Leopold and Maddock (1953), who extended the regime concept to American rivers, confirmed that the width is proportional to the square root of the bankfull discharge. Blench (1952) arrived at the same conclusion and gave an expression for Lacey's coefficient k_s, which he related to the bed material and tractive force acting on the sides of the river bed. Later studies in American streams by Simons and Albertson (1963) showed similar results:

$$B = k_s Q_b^{0.51} \tag{11}$$

albeit that the exponent was slightly increased. The coefficient of proportionality k_s appeared to vary with the soil properties of the banks. The value of k_s varied between 3.1 for banks with coarse non-cohesive material and 6.3 for sandy banks (in metric units). The former value is lower than the latter because sandy banks are easier to erode. Lacey (1963), in the following discussion of the paper, maintained that an exponent of 0.5 is correct.

3. Lacey's formula explained

A first intuitive explanation of Lacey's equation can be obtained if we reformulate the equation as: the discharge is proportional to the square of the width. Because $Q = UhB$, this situation occurs if the velocity at bankfull discharge is not a function of Q and if the depth is proportional to the width. The latter indeed follows from Lane's analysis of stable cross-sections. The derivation is presented below. The former requires further explanation. This is done in Section 4 of this paper.

The theory of stable cross-sections was initially developed by Lane (1955) and duly described in the textbooks of Raudkivi (1967, 1976) and Graf (1971, 1998). It is not considered useful to go into the derivation of this formula; it follows from the equilibrium of forces acting on a particle in a channel cross-section, subject to gravity and the drag force of the flow. The resulting shape of a stable cross-section is a sinusoid, given by:

$$\frac{h(y)}{h_m} = \cos\left(\frac{\tan\varphi}{h_m}y\right) \tag{12}$$

where $h(y)$ is the depth as a function of the lateral ordinate y; h_m is the maximum depth at the centre of the channel, and φ is the natural angle of repose of the sediment material in still water, which is a function of sediment characteristics (see Fig. 1). It follows from integration of Eq. (12) that:

$$A = 2h_m^2/\tan\varphi \tag{13}$$

and:

$$B = \pi h_m/\tan\varphi \tag{14}$$

These are two straightforward expressions that one comes across in several textbooks. What is surprising, however, is that the obvious next step: the derivation of the average depth h from Eqs. (13) and (14) is not encountered in the literature. Combination of Eqs. (13) and (14) yields:

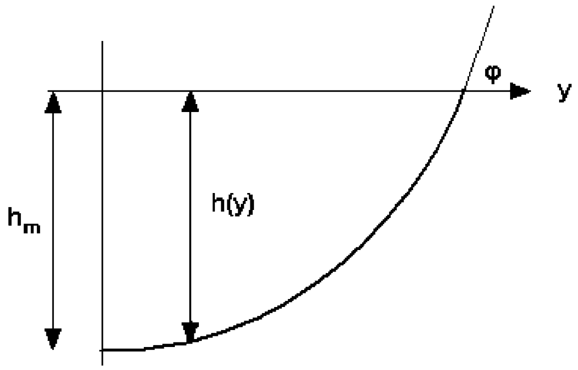

Fig. 1. Definition sketch for a stable cross-section.

$$h = \frac{A}{B} = \frac{2h_m^2}{\tan\varphi}\frac{\tan\varphi}{\pi h_m} = \frac{2h_m}{\pi} \tag{15}$$

By simple elimination of h_m from Eqs. (14) and (15) the required relation between h and B is obtained:

$$h = \frac{2\tan\varphi}{\pi^2}B \tag{16}$$

This equation is straightforward and is a linear relation between the average depth of flow, the surface width and the natural angle of repose of the sediment. Maybe the reason why this derivation has not been presented in the literature is that it is in conflict with Eq. (2) which states that, according to the current regime theory, the depth is proportional to the width to the power 0.66.

4. The singularity of bankfull flow

If however Eq. (16) is correct, then Lacey's equation for bankfull flow follows directly from the geometric functions (Eqs. (8) and (9)). Substitution of Eq. (16) in Eq. (8) and subsequently in Eq. (9) yields:

$$Q_b = \frac{2\tan\varphi}{\pi^2}B^2U_b \tag{17}$$

where Q_b and U_b are the discharge and velocity belonging to the design situation. These are the bankfull discharge and the corresponding cross-sectional average velocity, which is the velocity that occurs in a channel that is just overtopping its banks.

This velocity is in lowland alluvial channels in the order of 1–2 m/s.

Although in the regular regime theory the velocity is considered to be a function of the discharge to the power 0.17, this is no longer the case at the bankfull situation. At bankfull level, the water slope is forced on the slope of the natural levees, which is not necessarily equal to the equilibrium slope of the Chezy equation.

Alluvial rivers need to spill over the banks regularly to maintain the dynamic equilibrium of sediment accumulation and erosion. The longitudinal profile of the river bottom has a convex shape since the bottom slope reduces along the river axis. Hence the sediment transport capacity reduces as the water flows downstream, leading to continuous deposition. For the dynamic equilibrium of the riverbed it is required that, at regular times, bed material is lifted from the bottom and deposited on the banks. As a rule of thumb this occurs on average once every 1.5 years. The water which spills over the banks deposits the coarsest sediments directly on the natural levee, the size of which is related to the lifting capacity of the water flow at bankfull flow. As a result, we see, as we move with the water along the river, a decreasing slope and a sieving process of bottom sediments, which is determined by U_b. Hence U_b is not a function of Q_b but of the median sediment size D_{50}.

Lacey's equation follows directly from Eq. (17) provided the bankfull velocity is a constant that merely depends on the bed material characteristics

$$B = \sqrt{\frac{\pi^2}{2U_b \tan \varphi}} \sqrt{Q_b} \tag{18}$$

This is Eq. (11) of Simons and Albertson (1963), providing a physical and dimensionally sound expression for k_s

$$k_s = \sqrt{\frac{\pi^2}{2U_b \tan \varphi}} \tag{19}$$

Comparison of Eq. (19) with the documented values of k_s shows that for sandy banks $U_b \tan \varphi$ should equal 0.12, whereas in coarse material it should equal 0.51 (see Table 2).

In the Dutch estuaries, observations of natural angles of repose at various land-fill projects in still water yielded a relation between the angle of repose and the average grain size (CUR, 1992; Fig. 56 on p. 86), which can be described by an approximate regression line (derived for grain sizes between 100 and 500 μm):

$$\tan \varphi = 5.4 \times 10^5 D_{50}^{1.88} \tag{20}$$

With a velocity in the order of 1–2 m/s this would mean that the coefficients of Simons and Albertson relate to grain sizes of 620–430 μm for coarse sand and 290–200 μm for fine sand respectively (see Table 2). Similarly, Lacey's coefficient of 4.8 yields an average grain size of 390–270 μm. These are very acceptable results.

Hence Lacey's relation is nothing more than a combination of the lateral stability in a cross-section and the observation that the average velocity over the cross-section at bankfull discharge is solely a function of the bed material (in the order of 1–2 m/s in lowland rivers).

Combination of Lacey's equation with Eq. (16) leads to the following exponents of the power functions for B, h, and U_b, respectively: 0.5, 0.5 and 0, instead of: 0.5, 0.33 and 0.17. One can see in Table 1 that some researchers arrived at a similar result (Bray,

Table 2
Values of the constants of proportionality (k_s) and the related angle of repose ϕ and median grain size D_{50}

Author	Observation	k_s	$U_b \tan \varphi$[a]	D_{50}[b] ($U = 1–2$ m/s) (μm)
Simons and Albertson	Sand	6.3	0.12	290–200
Simons and Albertson	Coarse sand	3.1	0.51	620–430
Lacey		4.8	0.21	390–270

[a] Derived from Eq. (18).
[b] Median grain size derived from Eq. (19) assuming $U_b = 1–2$ m/s.

Ghosh, and Glover). The author suspects that these authors indeed looked at bankfull flow, whereas the others may have looked at lower discharges.

The question remains why the theory for a stable cross-section of Lane, which has been derived for a one-dimensional stable canal, appears to apply to the bankfull discharge of a naturally meandering channel. Clearly the derivation is not applicable to the erosion and deposition process in the bends of a meandering channel. However, the theory does apply to the straight stretches between meanders, and that is where the relation should be apparent. Also one may observe that the system of river channel, levees and overbank storage is never regular and not in agreement with a uniform channel that overtops its banks at a given flow velocity. But that is not necessary. In a natural channel, the river may overtop the bank at a certain location in one year, and at another location in a subsequent year, after sediments have been deposited on the levee. In fact the river requires this dynamics to maintain its levees.

5. Conclusions

Lacey's equation has been explained and the resulting physical description for Lacey's constant has been derived. In fact, Lacey's constant appears to be a function of the bed material, in line with the conclusions of Simons and Albertson (1963). As a result, the applicability of Lacey's equation hinges on the assumption that the bankfull cross-sectional average flow velocity is a sole function of the bed material. At bankfull discharge, a singularity occurs, whereby the water slope equals the longitudinal slope of the natural levee. The latter is a function of the bed material, and not the discharge. There is a bed-shaping flow velocity that has just sufficient power to lift the bottom material to the natural levee. This velocity is a function of the bed material. Above bankfull level a new situation occurs, with a different slope (the valley slope), a different width (the valley width) and a new stage discharge relation.

For stages below the bankfull level the equilibrium cross-section, where B and h are directly proportional to each other, does not occur. It is possible that within the river channel different (combinations of) temporarily stable cross-sections occur with a total stream width smaller than the bankfull width, i.e. the width that corresponds with the bed-shaping discharge.

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