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Magnetic field annihilators: invisible magnetization at the magnetic equator

S. Maus* and V. Haak

GeoForschungszentrum Potsdam, Germany. E-mail: smaus@gfz-potsdam.de

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SUMMARY

Some distributions of magnetization give rise to magnetic fields that vanish everywhere above the surface, rendering these distributions of magnetization completely invisible. They are the annihilators of the magnetic inverse problem. Known examples are the infinite sheet with constant magnetization and the spherical shell of constant susceptibility magnetized by an arbitrary internal field. Here, we show that remarkably more interesting annihilators exist for the Earth's dipole-dominated inducing field. Indeed, any susceptibility profile along the magnetic equator can be extended north/south into an annihilator. Consequently, the induced magnetization along the magnetic equator is entirely undetermined by the visible magnetic field. In contrast to the Backus effect, this ambiguity persists even if the full magnetic vector field is known.

Key words: crustal magnetization, lithospheric field, magnetic interpretation, magnetic inverse problem.

INTRODUCTION

Following 20 yr without near-Earth satellite magnetic coverage, recent high-resolution maps from the CHAMP satellite (Maus *et al.* 2002) have stimulated interest in the interpretation of long-wavelength lithospheric magnetic anomalies. Visual interpretation of the maps can immediately yield interesting structural, tectonic and geological information. However, any such interpretation remains incomplete without actually modelling the source magnetization. Some examples of recent modelling of Magsat anomalies are Arkani-Hamed & Dyment (1996), Whaler & Langel (1996), Purucker *et al.* (1998) and Purucker *et al.* (2002). A comprehensive overview of global lithospheric field mapping and interpretation is given by Langel & Hinze (1998).

Unfortunately, solutions to the magnetic inverse problem are generally non-unique, due to several factors.

(1) Rocks carry induced magnetization proportional to the present main field, as well as remanent magnetization independent of the present field. However, any magnetic anomaly above a half-space can be represented solely by induced magnetization from a constant inducing field in a thin sheet with only laterally varying susceptibility (Blakely 1995, eq. 11.35). Thus, without further knowledge concerning the subsurface, it is neither possible to separate remanent from induced magnetization, nor is it possible to infer the vertical distribution of magnetization.

(2) Since spherical harmonic (SH) degrees 1–14 of the internal geomagnetic field are dominated by the much stronger main field, degrees <14 of the lithospheric field remain unknown. Thus, large-scale features such as the continent–ocean boundaries or large Archaean shields only show up, if they are seen at all, as edge effects in the observable lithospheric field. Such edge effects are easily misinterpreted as the signatures of small local bodies.

However, there is an additional source of ambiguity, the implications of which are not yet fully appreciated in magnetic interpretation: even if the entire lithospheric field were known and all magnetization were induced, it would still not be possible to invert unambiguously for an equivalent susceptibility $\chi(\vartheta, \varphi)$. Here, the term 'equivalent susceptibility' refers to the laterally varying susceptibility in a thin shell, representing the entire induced magnetization of the lithosphere. It can therefore be expressed as a spherical harmonic expansion. This further ambiguity can be described in terms of annihilators (Parker 1994). These are distributions of magnetization that do not give rise to any magnetic field outside of the Earth. Such behaviour is known for an infinite sheet of constant magnetization (Affleck 1958). Furthermore, Runcorn (1975) showed that the magnetic field of a shell with constant susceptibility, magnetized from the inside, completely vanishes outside of the shell. Earth ellipticity only slightly modifies Runcorn's theorem (Jackson *et al.* 1999; Lesur & Jackson 2000). This constant susceptibility shell corresponds to the degree-0 SH coefficient of equivalent susceptibility.

Here, we show that Runcorn's homogeneous shell is not the only unconstrained degree of the equivalent susceptibility. An indication that the non-uniqueness could extend further than degree 0 was given by Arkani-Hamed & Strangway (1985), who found that the sectorial harmonics of the equivalent susceptibility were not constrained by global scalar anomaly maps. Knowing that the magnetic

*Corresponding author: Telegrafenberg, 14473, Germany.

potential was not uniquely determined by scalar data (Backus 1970) the authors attributed the problem to the shortcomings of scalar versus vector data. Indeed, in a later paper, Arkani-Hamed & Dyment (1996, Appendix B) claimed that no relevant non-uniqueness existed apart from that pointed out by Runcorn if the SH coefficients of the magnetic potential were known. This appears to be the generally accepted view and the homogeneous shell is often referred to as *the* annihilator (Harrison *et al.* 1994; Langel & Hinze 1998). In particular, Arkani-Hamed & Dyment (1996) and Langel & Hinze (1998) suggest recursive algorithms to estimate the higher-order SH coefficients of the equivalent susceptibility from the Gauss coefficients of the internal field, assuming that a unique solution exists. The equivalent susceptibility is required, in particular, when the field is to be reduced to the pole (Blakely 1995), meaning that the anomalies are transformed to those that would be observed in a vertical inducing field. In trying to implement such an algorithm we noticed a whole range of further annihilators for a dipole-dominated inducing field. Every SH degree spawns two series of coefficients, which add up to annihilators of the field. Starting with a sectorial coefficient, these series resemble the Backus series describing magnetic fields that do not contribute to the scalar anomaly at a given altitude. In contrast to the latter, though, the magnetic fields of the magnetization annihilators vanish in the entire space external to the sphere.

ANNIHILATORS

In the context of the global magnetic inverse problem, an annihilator is a distribution of magnetization which gives rise to a magnetic field that vanishes everywhere outside of the Earth (Parker 1994). Here, we shall take a narrower view and seek distributions of equivalent susceptibility $\chi(\vartheta, \varphi)$ in a thin shell of radius a, where a is the mean radius of the Earth (usually taken as 6371.2 km), which do not generate an induced magnetic anomaly with the present geomagnetic field as an inducing field. An annihilator is then characterized by a potential $V(r, \vartheta, \varphi)$ of the induced anomaly field that vanishes (or is constant) everywhere outside of the Earth.

To find these annihilators one has to look at the relation between the equivalent susceptibility and the corresponding induced magnetic anomaly. This relation was expanded into spherical harmonics by Nolte & Siebert (1987): the equivalent susceptibility $\chi(\vartheta, \varphi)$ can be written in terms of Schmidt normalized surface spherical harmonics Y_ℓ^m as

$$\chi(\vartheta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \chi_\ell^m Y_\ell^m(\vartheta, \varphi), \tag{1}$$

with colatitude ϑ and longitude φ. Following the notation of Backus *et al.* (1996), orders m are in the range of $-\ell \leq m \leq \ell$. The χ_ℓ^m are the coefficients of the equivalent susceptibility. These coefficients must then be related to the Gauss coefficients g_ℓ^m of the corresponding induced magnetic anomaly potential $V(r, \vartheta, \varphi)$,

$$V(r, \vartheta, \varphi) = a \sum_{\ell=1}^{\infty} \left(\frac{a}{r}\right)^{\ell+1} \sum_{m=-\ell}^{\ell} g_\ell^m Y_\ell^m(\vartheta, \varphi). \tag{2}$$

Of course, the relationship between χ_ℓ^m and g_ℓ^m depends on the inducing (main) field, which can also be expanded into spherical harmonics. If only the dipolar component of the inducing field is considered, the relationship can be written in a convenient analytical form (Nolte & Siebert 1987, eq. 41). If we choose the z-axis of our coordinate system in the direction of the geomagnetic dipole axis,

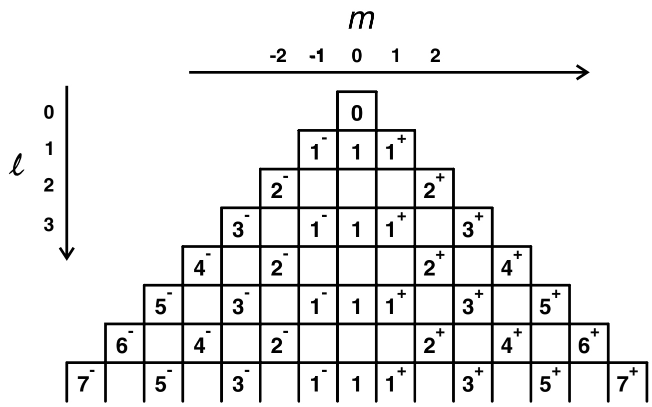

Figure 1. Every χ_ℓ^ℓ and $\chi_\ell^{-\ell}$ (and also χ_1^0) is the starting point for a series of spherical harmonics adding up to an annihilator for a dipolar inducing field, when the dipole moment is aligned with the z-axis. The coefficients belonging to a particular basic annihilator share a common label in the above sketch, e.g. 1^- starts with the coefficient of Y_1^{-1}. Since the series decays by a factor of approximately 3 from one coefficient to the next, only the first few coefficients actually make a significant contribution.

this relation reduces to the simple form

$$g_\ell^m = \frac{G_1^0}{a(2\ell+1)} \left[\frac{\ell-1}{2\ell-1} \sqrt{(\ell-m)(\ell+m)}\, \chi_{\ell-1}^m + \frac{3\ell}{2\ell+3} \sqrt{(\ell-m+1)(\ell+m+1)}\, \chi_{\ell+1}^m \right], \tag{3}$$

where G_1^0 is the SH coefficient of the dipolar inducing field. The Gauss coefficients g_ℓ^m of the induced anomaly have to vanish if χ_ℓ^m are to represent an annihilator. By definition, coefficients χ_ℓ^m with $|m| > \ell$ are zero. Here we can already see that a constant susceptibility is an annihilator, since χ_0^0 does not contribute to any g_ℓ^m. Demanding that all coefficients g_ℓ^m be zero yields the iterative relation

$$\frac{\ell-1}{2\ell-1} \sqrt{(\ell-m)(\ell+m)}\, \chi_{\ell-1}^m = -\frac{3\ell}{2\ell+3} \sqrt{(\ell-m+1)(\ell+m+1)}\, \chi_{\ell+1}^m, \tag{4}$$

which can be rewritten as

$$\chi_{\ell+2}^m = -\frac{\ell(2\ell+5)}{3(\ell+1)(2\ell+1)} \sqrt{\frac{(\ell+1-m)(\ell+1+m)}{(\ell-m+2)(\ell+m+2)}}\, \chi_\ell^m. \tag{5}$$

From relation (3) the annihilators can be directly inferred. The idea is illustrated in Fig. 1. Every magnetic field coefficient g_ℓ^m is influenced by two source coefficients $\chi_{\ell-1}^m$ and $\chi_{\ell+1}^m$. However, all g_ℓ^m and χ_ℓ^m with $|m| > \ell$ are zero by definition. This opens up the possibility that a larger number of source coefficients determine a smaller number of field coefficients, giving room for annihilators.

Starting with g_ℓ^ℓ

If g_ℓ^ℓ is to vanish then $\chi_{\ell+1}^\ell = 0$ because $\chi_{\ell-1}^\ell = 0$ by definition. Consequently, all further coefficients $\chi_{\ell+3}^\ell, \chi_{\ell+5}^\ell, \ldots$ must be zero. This does not yield an annihilator.

Starting with $g_\ell^{\ell-1}$

In contrast, coefficients $g_\ell^{\ell-1}$ are generated by the two source coefficients $\chi_{\ell-1}^{\ell-1}$ and $\chi_{\ell+1}^{\ell-1}$. Of these, $\chi_{\ell-1}^{\ell-1}$ does not contribute to any

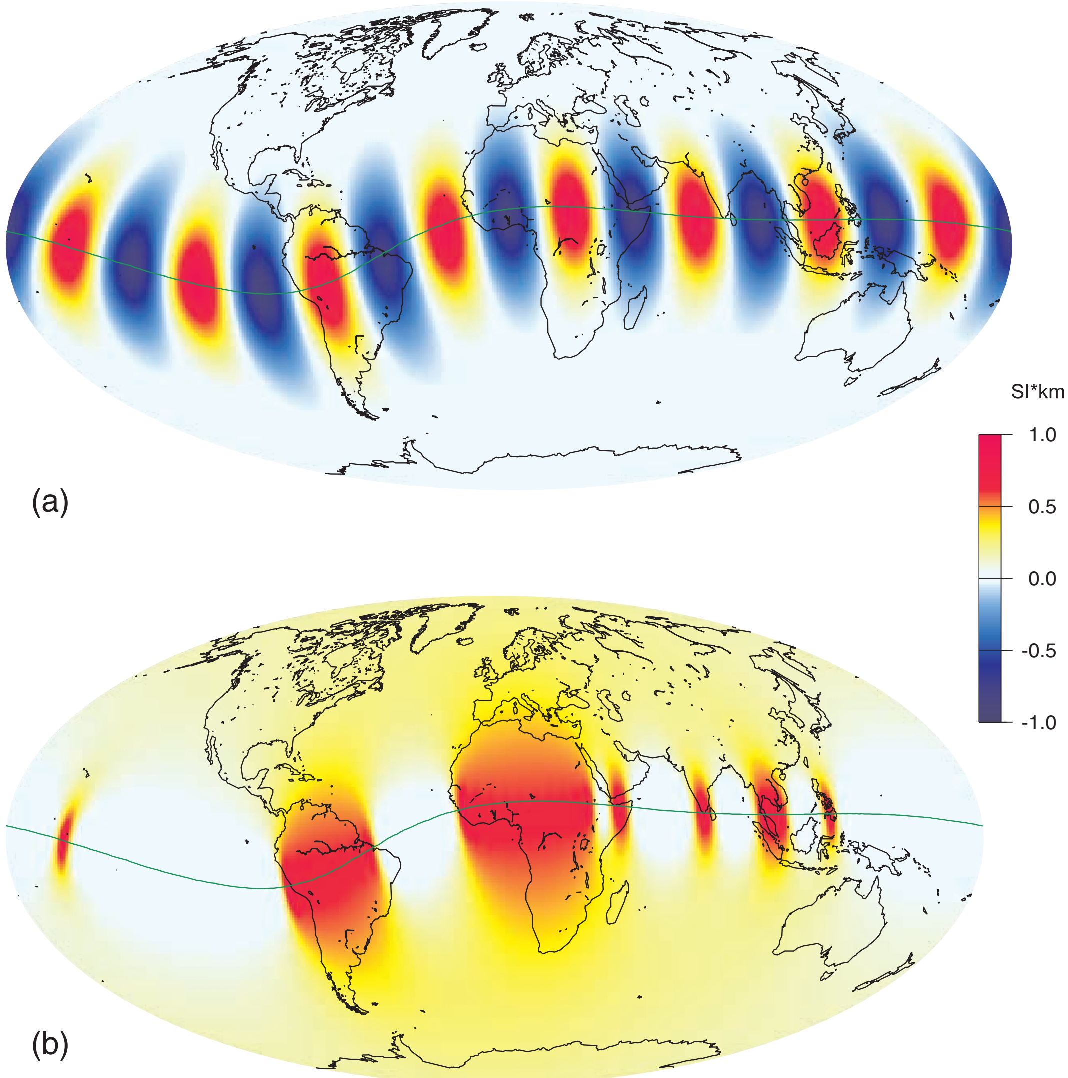

Figure 2. The susceptibility distributions displayed here do not produce any magnetic anomaly, anywhere outside of the Earth, with the present geomagnetic field (taken from model Ørsted-10b (Olsen 2002)). A 'basic' annihilator for $\ell = 8$, behaving like sin (8φ) with longitude φ along the dip equator, is shown in (a). An annihilator closer to the true structure of the Earth's crust is displayed in (b). It results from the superposition of basic annihilators. To construct it we assigned a vertically integrated susceptibility of 1 SI km to land and 0 to the oceans and extracted a profile along the magnetic dip equator. It was then 1-D Fourier transformed and the values of the Fourier coefficients were assigned to the sectorial harmonics of susceptibility. The exact annihilators of the present geomagnetic field were then found in a linear inversion by adjusting the non-sectorial harmonics to minimize the anomaly potential. The present magnetic dip equator is indicated in green. Part (b) illustrates that the continent of South America may be approximated fairly well by an annihilator. Hence, as a single body, South America will not generate a continental-scale anomaly. Only the deviations of the continental outline from the shape of an annihilator will produce smaller-scale anomalies. The same holds for Africa.

other g_ℓ^m, rendering it a free parameter. Then, following eq. (5), $\chi_{\ell+1}^{\ell-1}$ can be chosen in such a way that $g_\ell^{\ell-1} = 0$. Subsequently, $g_{\ell+2}^{\ell-1}$ can be made to vanish by the choice of $\chi_{\ell+3}^{\ell-1}$, and so forth. Thus, every χ_ℓ^ℓ (and $\chi_\ell^{-\ell}$) is the starting point of a series of coefficients that describe an annihilator. While each annihilator is made up of an infinite series, the SH coefficients within a particular series given by eq. (5) rapidly decrease in amplitude with increasing degree. Thus, only the first few coefficients of a series actually make a significant contribution to the annihilator.

Starting with g_0^0

Since the second term in (3) vanishes for $\ell = 0$, sensibly, χ_1^0 does not contribute to the monopole moment g_0^0. Thus, χ_1^0 is the starting point of an additional annihilator series.

Contributions from higher moments of the inducing field

The above reasoning supplies the complete set of annihilators for a dipolar inducing field, where the dipole is aligned with the *z*-axis. Since the coordinate system can always be chosen in this way, the equations are applicable—after suitable coordinate transforms—to any dipolar inducing field. However, extending eq. (5) to the higher moments of the inducing field is not feasible analytically. Instead, the annihilators have to be sought by a numerical procedure. We find the annihilators of the present geomagnetic field by first fixing the sectorial susceptibility harmonics $\chi_\ell^{|m|=\ell}$ (and also the exception χ_1^0). For a given inducing field, these give rise to a magnetic anomaly potential $g_{\ell'}^{m'}$ $(\chi_\ell^{|m|=\ell})$. If χ_ℓ^m are to represent an annihilator, the non-sectorial harmonics must exactly cancel the field of the sectorial susceptibility harmonics. In a linear inversion we therefore determine the non-sectorial harmonics $\chi_\ell^{|m|<\ell}$ in such a way that $g_{\ell'}^{m'}(\chi_\ell^{|m|<\ell}) = -g_{\ell'}^{m'}(\chi_\ell^{|m|=\ell})$. Here, for a general inducing field, each anomaly potential coefficient $g_{\ell'}^{m'}$ is a linear combination of all χ_ℓ^m.

The above derivations show that this procedure must yield a unique solution for a dipolar inducing field. Whether such a unique solution also exists in the presence of higher moments of the inducing field is not immediately apparent from analytical considerations. It turns out, however, that the inversion is well conditioned and a unique solution with a vanishing potential does exist. Convincingly, the annihilators—which are symmetric about the geographic equator for an axial dipole field—become symmetric about the magnetic dip equator for the present geomagnetic field. Two examples are given in Fig. 2 and are further discussed below.

Connection with the Backus effect

These annihilator series starting with χ_ℓ^ℓ and $\chi_\ell^{-\ell}$ are reminiscent of the Backus series, which starts with g_ℓ^ℓ and $g_\ell^{-\ell}$ and describe magnetic fields perpendicular to the main field at a given altitude. To first order, these perpendicular fields do not contribute to the scalar anomaly of the magnetic field measured by a satellite. However, while both kinds of series describe annihilators and start with orders $m = \ell$ and $-\ell$, there is no obvious physical connection between the two types of phenomena. Our annihilators describe distributions of magnetization that do not generate a magnetic field anywhere outside of the Earth, while the Backus annihilators describe magnetic fields that do not contribute to the scalar anomaly measured on a spherical surface at a given altitude.

DISCUSSION

Let us now focus on the practical relevance of these annihilators. A typical example of a basic annihilator is illustrated in Fig. 2(a). Such a ringing distribution of susceptibility is highly unlikely for the real Earth and at first sight the existence of such annihilators could appear irrelevant. Their significance, however, stems from possible combinations of the basic annihilators. Since a spherical harmonic has a longitude φ dependence given by $\cos m\varphi$ for order m, and $\sin m\varphi$ for order $-m$ (in Backus' notation), the annihilator series χ_ℓ^ℓ and $\chi_\ell^{-\ell}$ define a complete basis for a Fourier representation of the susceptibility along a parallel of latitude. Hence, any single east/west profile of the susceptibility along a given magnetic latitude can be represented as a sum of annihilators. Consequently, given such a susceptibility profile, we can always extend it in the north/south direction in such a way that it produces no magnetic anomaly at all. However, all annihilators are symmetric about the magnetic equator (except for the one starting with χ_1^0 named '1' in Fig. 1) and have their peak amplitude there. For practical purposes, it is therefore more appropriate to state the ambiguity as 'any specified vertically integrated susceptibility distribution *along the magnetic equator* can be extended north/south into an annihilator for a given dipole-dominated inducing field'.

As an example, consider a vertically integrated susceptibility profile along the magnetic dip equator with a value of 1 SI km for land and 0 SI km for ocean. Taking the Fourier transform of the profile, we can use the Fourier coefficients of the ℓth harmonic as the starting values χ_ℓ^ℓ and $\chi_\ell^{-\ell}$ of the annihilator series and then use the numerical procedure described above under *contributions from higher moments of the inducing field* to find the rest of the annihilator series coefficients. Thus, we construct an annihilator that has a given susceptibility on the equator. In substituting spherical harmonic coefficients with Fourier coefficients one has to take into account that the amplitude of spherical harmonics Y_ℓ^ℓ along the equator changes with ℓ (depending on the value $P_\ell^\ell(0)$ of the associated Legendre function and on the chosen normalization). Finding the annihilator for a given susceptibility profile along the equator can be regarded as extending the equatorial profile in the magnetic N/S direction into an annihilator. The example, illustrated in Fig. 2(b), explains the well-known difficulties faced in interpreting magnetic anomalies in the vicinity of the magnetic equator (Blakely 1995). It also shows that the shapes of South America and Africa can be approximated fairly well by annihilators. Consequently, these continents as a whole do not generate significant magnetic anomalies. Only departures from the overall shape of an annihilator generate smaller-scale anomalies.

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