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Statistical properties of reflection traveltimes in 3-D randomly inhomogeneous and anisomeric media

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SUMMARY

We study the statistical properties of seismic reflection traveltimes in the presence of the double-passage effect (DPE) in order to characterize the inhomogeneities of the reflector overburden. A detailed analysis of the traveltime covariance function is presented for more general conditions than in the pioneering works by Touati, Iooss and Gaerets *et al.*, namely 3-D geometry, quasi-homogeneous fluctuations of the medium parameters, anisomeric (i.e. statistically anisotropic fluctuations), longitudinal and transversal positions of receivers, and curved rays. The behaviour of the traveltime covariance function is elucidated by some numerical examples. It is shown, in particular, that at large offsets the effective longitudinal correlation scale associated with the ray trajectory is much larger than the transverse correlation scale and their product equals the product of the correlation lengths of the fluctuations of the medium.

Key words: double-passage effect, random media, regular refraction, seismicity, tomography, traveltime.

1 INTRODUCTION

Statistical characterization of rocks is of significant interest for many purposes in global and exploration seismology. First, it has been recognized that large parts of the lithosphere show spatial heterogeneities on several length-scales (Sato & Fehler 1998; Margerin *et al.* 1999) and, therefore, deterministic Earth models should be supplemented with statistical information on rock heterogeneity in order to describe correctly the propagation of seismic waves. Secondly, statistical information on rock heterogeneity can be helpful for petrophysical interpretations. For example, when estimating the quality factor of rocks from seismic data, it is of interest to know whether seismic attenuation has been caused either by lithological contrasts, leading to scattering attenuation (Frankel & Clayton 1986), or by viscoelastic properties of rocks, leading to intrinsic attenuation. Thirdly, in combination with usual macromodel-based imaging techniques, statistical characterization of small-scale heterogeneities can be used in order to retrieve 'true' reflection coefficients of large-scale heterogeneities from seismic data. Moreover, the statistical characterization of rocks can contribute to geostatistical modelling of reservoirs, to estimates of uncertainties of seismic images, and to a better understanding of different features of structures and geoprocesses. Statistical characterization implies estimates for two main parameters of rocks: the correlation scale of inhomogeneities and the degree of variation of the elastic wave velocity v (or its inverse value, slowness $\mu = 1/v$).

A promising method for estimating the characteristic horizontal length of inhomogeneities $l_{\rm hor}$ from traveltime fluctuations $\tilde{t}$ of a pulse signal reflected backwards to the source was suggested by Touati (1996) and analysed by Iooss (1998) and Iooss *et al.* (2000). The method is based on the comparison of traveltime variance var$[\tilde{t}(X = 0)]$, measured near the source (zero offset, $X = 0$), with the variance at large offsets, $X \gg l_{\rm hor}$, which we denote by var$[\tilde{t}(\infty)]$. As shown by Iooss (1998), var$[\tilde{t}(0)]$ is about twice as large compared with var$[\tilde{t}(\infty)]$:

$$\frac{\mathrm{var}[\tilde{t}(0)]}{\mathrm{var}[\tilde{t}(\infty)]} \approx 2 \tag{1}$$

and is a manifestation of the so-called double-passage effect (DPE). The transition from zero offset, $X = 0$, to a sufficiently large offset, $X \gg l_{\rm hor}$, is associated with a spatial scale X that is comparable with $l_{\rm hor}$. Therefore, measurements of var$[\tilde{t}(X)]$ as a function of X might be helpful for recovering the horizontal scale $l_{\rm hor}$ of the inhomogeneities from experimental data. A further numerical analysis of the method of Touati and Iooss was presented in Gaerets *et al.* (2001).

The doubling of traveltime variance at $X = 0$ occurs when the wave passes twice through randomly inhomogeneous media with large-scale (as compared with wavelength λ) inhomogeneities. The

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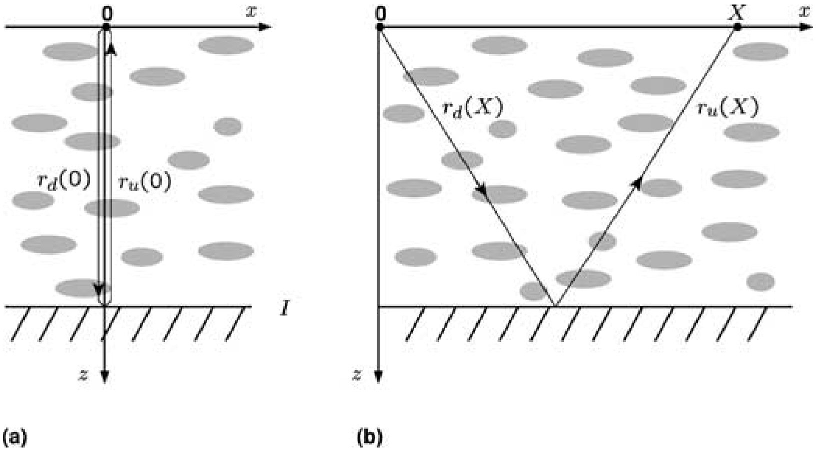

Figure 1. Geometry of the double-passage effect. (a) Vertical ray intersects one and the same inhomogeneities. (b) Oblique ray, corresponding to large offset $X \gg b$, passes mostly through different inhomogeneities.

DPE was revealed earlier for phase-path fluctuations in other physical situations: for light waves reflected from a mirror in a turbulent atmosphere and for radio waves reflected from the ionosphere (see Denison & Erukhimov 1962; Kravtsov & Saichev 1982, 1985, and references therein and also exercise I.7.6 from the textbook by Rytov *et al.* 1989; Barabanenkov *et al*. 1991).

The DPE can be easily explained within the framework of geometrical optics (GO). At zero offset, $X = 0$, the incident (downgoing) and reflected (up-going) rays, $\mathbf{r}_d(0)$ and $\mathbf{r}_u(0)$, respectively, pass through the very same inhomogeneities of the random medium (Fig. 1a). Therefore, the fluctuations of the traveltimes $\tilde{t}_d(0)$ and $\tilde{t}_u(0)$ are equal, $\tilde{t}_d(0) = \tilde{t}_u(0)$, and the variance of the total traveltime $\tilde{t}(0) = \tilde{t}_d(0) + \tilde{t}_u(0) = 2\tilde{t}_d(0)$ is four times larger than the variance for one-way passage $\tilde{t}_d(0)$:

$$\mathrm{var}[\tilde{t}(0)] = 4\mathrm{var}[\tilde{t}_d(0)]. \quad (2)$$

At the same time, at a sufficiently large offset X, $X \gg l_{hor}$, downgoing and up-going rays $\mathbf{r}_d(X)$ and $\mathbf{r}_u(X)$ pass through different inhomogeneities (Fig. 1b). Therefore, the cross-product of $\tilde{t}_d(X)$ and $\tilde{t}_u(X)$ on average is close to zero and as a result at $X \gg l_{hor}$

$$\mathrm{var}[\tilde{t}(\infty)] = \mathrm{var}[\tilde{t}_d(X)] + \mathrm{var}[\tilde{t}_u(X)] + 2\mathrm{covar}[\tilde{t}_d(X)\tilde{t}_u(X)] \approx 2\mathrm{var}[\tilde{t}_d(X)], \quad (3)$$

which is roughly half of $\mathrm{var}[\tilde{t}(0)]$. The relationship between these equations and the DPE was not clearly emphasized in the works of Touati (1996) and Iooss (1998) or Gaerets *et al.* (2001).

This paper is devoted to the further analysis of DPE manifestations in seismics and to the generalization of the above-mentioned results of Touati (1996), Iooss (1998) and Gaerets *et al.* (2001) in several directions. First of all, we extend the DPE formulation from 2-D (which was used in the mentioned works) to 3-D geometry. Secondly, we consider fluctuations of medium parameters with quasi-homogeneous statistics. This model allows one, in general, to take into account the depth dependence of the slowness variance. Thirdly, changes of the mean slowness with depth are taken into account, so rays in our considerations can be curved. Fourthly, horizontally anisomeric (i.e. statistically anisotropic) fluctuations are considered. Fifthly, our considerations also deal with traveltime covariance functions in a more general treatment than before.

The outline of the paper is the following. In Section 2.1 we formulate the problem of describing traveltime fluctuations in random media within the geometrical optics method. In Sections 2.2 and 2.3 we derive an expression for the traveltime covariance function in a 3-D random medium with quasi-homogeneous fluctuations. Assuming small offsets, this general formula is analysed in detail for several measurement configurations and geometries. In particular, a formula for the total traveltime covariance assuming a horizontal reflector and a medium with statistical properties depending only on depth is derived in Sections 3.1–3.3. The results are specified for both mid-point geometry (Section 3.4) and a two-sources two-receiver configuration (Section 3.5). Section 4 is devoted to the analysis of the traveltime variance for small offsets, which can be derived using the results of the previous sections. The DPE for the special cases of horizontally anisomeric inhomogeneities and tilted inhomogeneities are discussed in Sections 4.2 and 4.3, respectively. For larger offsets the ray trajectories may be curved due to regular refraction. The behaviour of the traveltime covariance in such situations is analysed in Section 5, starting with the asymptotic behaviour of the traveltime variance in the small-offset formulation (Section 5.1). The incorporation of refraction effects in the formulation of the traveltime covariance function is performed in Section 5.2. Finally, in Section 5.3 the case of very large offsets (oblique propagation) is considered. It is shown that in this case the DPE disappears. The longitudinal and transversal (relative to the ray trajectory) traveltime covariance function is formulated and analysed for a random medium characterized by the Gaussian correlation function.

2 STATISTICAL PROPERTIES OF REFLECTION TRAVELTIMES—BASIC RELATIONS

2.1 Traveltime fluctuations in the frame of geometrical optics method

We aim to establish a relation between fluctuations of seismic reflection traveltimes and fluctuations of the propagation medium parameters. To do so, let us consider a point source placed at $X = 0$ (see Figs 1a and b) radiating a pulse signal that propagates through a random medium. The signal is reflected from a horizontal or slightly inclined plane interface I and is recorded by a receiver placed in the vicinity of the source.

Assuming the inhomogeneities of the medium are large in size compared with the typical wavelength of a seismic pulse, we apply the geometrical optics approximation for traveltime calculations. In this approximation the traveltime t in non-dispersive media obeys the formula

$$t = \int \frac{ds}{v[\mathbf{r}(s)]} = \int \mu[\mathbf{r}(s)]\, ds, \quad (4)$$

where $v(\mathbf{r})$ is the wave velocity, $\mu(\mathbf{r}) = 1/v(\mathbf{r})$ is the slowness and ds is an element of the ray trajectory $\mathbf{r}(s)$. The ray trajectory obeys the ray equations (Kravtsov & Orlov 1990; Born & Wolf 1999)

$$\frac{d\mathbf{r}}{ds} = \mathbf{t}, \qquad \frac{d\mathbf{t}}{ds} = \nabla_\perp n \equiv \nabla n - \mathbf{t}(\mathbf{t}\nabla n), \quad (5)$$

where $\mathbf{t}$ is a unit vector tangent to the ray,

$$n(\mathbf{r}) = \frac{v_0}{v(\mathbf{r})} = v_0 \mu(\mathbf{r}) \quad (6)$$

is the refractive index, v_0 is the wave velocity near the source and $\nabla_\perp n = \nabla n - \mathbf{t}(\mathbf{t}\,\nabla\, n)$ is a transverse (relative to the ray) gradient of the refractive index. In non-dispersive media the traveltime (4) is proportional to the eikonal ('optical path') $\Psi = \int n[\mathbf{r}(s)]\, ds$:

$$t = \int \mu[\mathbf{r}(s)]\, ds = \frac{1}{v_0} \int n[\mathbf{r}(s)]\, ds = \frac{\Psi}{v_0}. \quad (7)$$

Therefore, all the results obtained earlier for the optical path fluctuations (Chernov 1960; Tatarskii 1971; Ishimaru 1978; Rytov *et al.* 1989) can be immediately used for traveltime calculations.

In a random medium slowness can be presented as a sum of regular (average), $\bar{\mu}(\mathbf{r})$ and random, $\tilde{\mu}(\mathbf{r})$, parts:

$$\mu(\mathbf{r}) = \bar{\mu}(\mathbf{r}) + \tilde{\mu}(\mathbf{r}), \quad (8)$$

where the mean value of $\tilde{\mu}$ is zero ($\langle\tilde{\mu}\rangle = 0$). The same is true for the ray trajectory $\mathbf{r}(s)$, traveltime $t(s)$ and the refractive index $n(s)$:

$$\mathbf{r}(s) = \bar{\mathbf{r}}(s) + \tilde{\mathbf{r}}(s), \quad t(s) = \bar{t}(s) + \tilde{t}(s), \quad n(s) = \bar{n}(s) + \tilde{n}(s) \quad (9)$$

For sufficiently weak fluctuations $\tilde{\mu}$, when

$$\sigma_\mu^2 \equiv \mathrm{var}[\tilde{\mu}(\mathbf{r})] \equiv \langle\tilde{\mu}^2(\mathbf{r})\rangle \ll 1/v_0^2 \quad (10)$$

and

$$\sigma_n^2 \equiv \mathrm{var}[\tilde{n}(\mathbf{r})] \ll 1 \quad (11)$$

the ray trajectory deviates only slightly from a regular trajectory $\bar{\mathbf{r}}(s)$ (here and henceforth both upper bar $\bar{(\cdot)}$ and angular brackets $\langle(\cdot)\rangle$ are used for statistical averaging). Therefore, first-order fluctuations of traveltime $\tilde{t}$ can be calculated by integrating the perturbation $\tilde{\mu}(\mathbf{r})$ along the unperturbed ray $\bar{\mathbf{r}}(s)$ (Chernov 1960; Tatarskii 1971; Ishimaru 1978; Rytov *et al.* 1989; Snieder & Sambridge 1992):

$$\tilde{t}(s) \cong \int \tilde{\mu}[\bar{\mathbf{r}}(s)]\, ds. \quad (12)$$

In the following, we omit the bar over regular ray trajectory for brevity. Eq. (12) provides the necessary link between fluctuations of the reflection traveltimes and fluctuations of the propagation medium properties. Thus, in order to study the statistical properties of reflection traveltimes we also need to know the statistics of the fluctuations of the medium.

2.2 Medium fluctuations with quasi-homogeneous statistics

Let us now specify the statistical properties of $\tilde{\mu}$. The model of quasi-homogeneous fluctuations seems to be sufficiently general and flexible for many seismological applications. Quasi-homogeneous fluctuations are described by the covariance of the form (Rytov *et al.* 1989):

$$\begin{aligned} C_\mu(\mathbf{r}_1, \mathbf{r}_2) &\equiv \mathrm{covar}[\tilde{\mu}(\mathbf{r}_1), \tilde{\mu}(\mathbf{r}_2)] \equiv \langle\tilde{\mu}(\mathbf{r}_1)\tilde{\mu}(\mathbf{r}_2)\rangle \\ &= \sigma_\mu^2(\mathbf{r}_+) K(\mathbf{r}_1 - \mathbf{r}_2; \mathbf{r}_+), \end{aligned} \quad (13)$$

where $\mathbf{r}_+ = (\mathbf{r}_1 + \mathbf{r}_2)/2$ is the radius vector of the 'centre of gravity' of vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, and K is a normalized correlation function ('coefficient of correlation'), which turns out to be unity at $\mathbf{r}_1 - \mathbf{r}_2 = 0$: $K(0; \mathbf{r}_+) = 1$. This quantity is supposed to decrease rapidly with difference $\mathbf{r}_1 - \mathbf{r}_2$ with a small characteristic scale l_c, but it can also depend slowly (along with σ_μ^2) on $\mathbf{r}_+$ with a large characteristic scale $l_+ \gg l_c$. Note that the model of quasi-homogeneous fluctuations is more general than that of statistically homogeneous media, where the covariance function C_μ is the same for the entire medium. In the quasi-homogeneous model (13) the variance and the correlation coefficient are allowed to change slowly within the medium.

On the basis of this model we shall consider traveltime statistics for depth-dependent fluctuations of medium parameters (Section 3.1–3.5), for statistically anisotropic fluctuations (Section 4.2) and for a medium with tilted layers (Section 4.3).

2.3 Traveltime covariance function

The unperturbed ray trajectory $\mathbf{r}(s)$ of a ray reflected from interface I (Fig. 2) consists of a down-going and an up-going part:

$$\mathbf{r}(s) = \begin{cases} \mathbf{r}_\mathrm{d}(s_\mathrm{d}; \mathbf{R}), & 0 < s_\mathrm{d} < S_\mathrm{d}(\mathbf{R}) \\ \mathbf{r}_\mathrm{u}(s_\mathrm{u}; \mathbf{R}), & 0 < s_\mathrm{u} < S_\mathrm{u}(\mathbf{R}), \end{cases} \quad (14)$$

where s_d and s_u are current arclengths along down-going and up-going sections, respectively, and $S_\mathrm{d}(\mathbf{R})$ and $S_\mathrm{u}(\mathbf{R})$ are the corresponding total arclengths, which depend on the receiver position $\mathbf{R} = (X, Y, 0)$. In eq. (14) the final point $\mathbf{r}_\mathrm{d}(S_\mathrm{d};\mathbf{R})$ of the down-going ray serves as the starting point of the up-going ray: $\mathbf{r}_\mathrm{u}(s_\mathrm{u} = 0; \mathbf{R}) = \mathbf{r}_\mathrm{d}(S_\mathrm{d}; \mathbf{R})$. According to eq. (14), the traveltime $\tilde{t}(\mathbf{R})$ can be expressed as a sum

$$\tilde{t}(\mathbf{R}) = \tilde{t}_\mathrm{d}(\mathbf{R}) + \tilde{t}_\mathrm{u}(\mathbf{R}), \quad (15)$$

where

$$\tilde{t}_\mathrm{d}(\mathbf{R}) = \int_0^{S_\mathrm{d}(\mathbf{R})} \tilde{\mu}[\mathbf{r}_\mathrm{d}(s_\mathrm{d})]\, ds_\mathrm{d} \quad (16)$$

and

$$\tilde{t}_\mathrm{u}(\mathbf{R}) = \int_0^{S_\mathrm{u}(\mathbf{R})} \tilde{\mu}[\mathbf{r}_\mathrm{u}(s_\mathrm{u})]\, ds_\mathrm{u} \quad (17)$$

are time-delay fluctuations along the down-going and up-going sections of the ray trajectory.

The covariance of traveltime, recorded by two receivers placed at points $\mathbf{R}_1$ and $\mathbf{R}_2$, as shown in Fig. 2, has the form

$$\begin{aligned} C_\mathrm{t}(\mathbf{R}_1, \mathbf{R}_2) &= \langle\tilde{t}(\mathbf{R}_1)\tilde{t}(\mathbf{R}_2)\rangle \\ &= \langle[\tilde{t}_\mathrm{d}(\mathbf{R}_1) + \tilde{t}_\mathrm{u}(\mathbf{R}_1)][\tilde{t}_\mathrm{d}(\mathbf{R}_2) + \tilde{t}_\mathrm{u}(\mathbf{R}_2)]\rangle \\ &= C_\mathrm{dd}(\mathbf{R}_1, \mathbf{R}_2) + C_\mathrm{du}(\mathbf{R}_1, \mathbf{R}_2) \\ &\quad + C_\mathrm{ud}(\mathbf{R}_1, \mathbf{R}_2) + C_\mathrm{uu}(\mathbf{R}_1, \mathbf{R}_2), \end{aligned} \quad (18)$$

where

$$C_\mathrm{dd}(\mathbf{R}_1, \mathbf{R}_2) = \langle\tilde{t}_\mathrm{d}(\mathbf{R}_1)\tilde{t}_\mathrm{d}(\mathbf{R}_2)\rangle, \quad C_\mathrm{ud}(\mathbf{R}_1, \mathbf{R}_2) = \langle\tilde{t}_\mathrm{u}(\mathbf{R}_1)\tilde{t}_\mathrm{d}(\mathbf{R}_2)\rangle,$$

$$C_\mathrm{du}(\mathbf{R}_1, \mathbf{R}_2) = \langle\tilde{t}_\mathrm{d}(\mathbf{R}_1)\tilde{t}_\mathrm{u}(\mathbf{R}_2)\rangle, \quad C_\mathrm{uu}(\mathbf{R}_1, \mathbf{R}_2) = \langle\tilde{t}_\mathrm{u}(\mathbf{R}_1)\tilde{t}_\mathrm{u}(\mathbf{R}_2)\rangle. \quad (19)$$

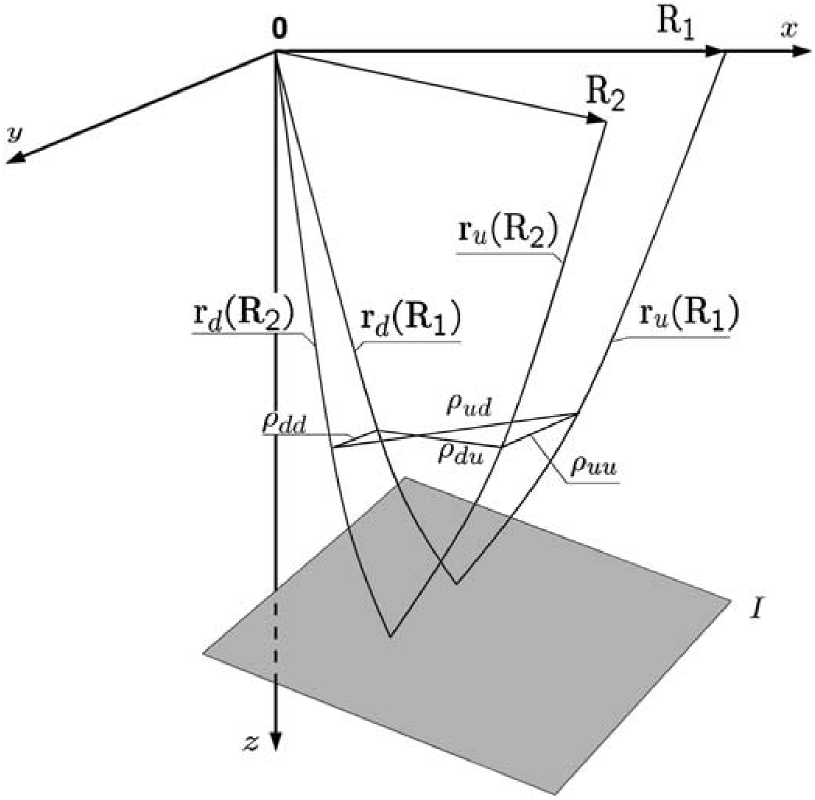

Figure 2. Geometry for two up- and down-going ray trajectories in 3-D space.

According to eqs (16), (17) and (13), all values (19) can be expressed in the form of double integrals over the correlation function C_μ of slowness fluctuations along the non-perturbed ray trajectories. For example, the covariance $C_{\rm dd}$ has the form

$$C_{\rm dd}(\mathbf{R}_1, \mathbf{R}_2) = \int_0^{S_{\rm d}(\mathbf{R}_1)} ds'_{\rm d} \int_0^{S_{\rm d}(\mathbf{R}_2)} ds''_{\rm d} C_\mu[\mathbf{r}_{\rm d}(s'_{\rm d}; \mathbf{R}_1), \mathbf{r}_{\rm d}(s''_{\rm d}; \mathbf{R}_2)]$$
$$= \int_0^{S_{\rm d}(\mathbf{R}_1)} ds'_{\rm d} \int_0^{S_{\rm d}(\mathbf{R}_2)} ds''_{\rm d} \sigma^2_\mu(\mathbf{r}_{dd+}) K[\mathbf{r}_{\rm d}(s'_{\rm d}, \mathbf{R}_1) - \mathbf{r}_{\rm d}(s''_{\rm d}, \mathbf{R}_2); \mathbf{r}_{dd+}], \quad (20)$$

where $\mathbf{r}_{dd+} = [\mathbf{r}_{\rm d}(s', \mathbf{R}_1) + \mathbf{r}_{\rm d}(s'', \mathbf{R}_2)\,]/2$. These expressions generalize the results of Touati (1996), Iooss (1998) and Gaerets *et al.* (2001) for 3-D geometry and quasi-homogeneous statistics of the medium properties. In what follows we analyse and simplify the expression (20) as well as the analogous expressions for $C_{\rm du}$, $C_{\rm ud}$ and $C_{\rm uu}$. In particular, we analyse the traveltime covariance for seismic reflection surveys including small and large offsets.

3 TRAVELTIME COVARIANCE FUNCTION FOR SMALL OFFSETS

3.1 Ray trajectories for small offsets

We assume that the interface I is horizontal and that the regular and statistical properties of a medium depend only on depth z. In the simplest case when the horizontal offset vectors $\mathbf{R}_1$ and $\mathbf{R}_2$ are small as compared with the depth D of the interface I,

$$|\mathbf{R}_{1,2}| \ll D, \quad (21)$$

the angles between the rays and the vertical axis z are also sufficiently small. In this case the rays are only slightly curved, so that the ray trajectories can be approximated by straight lines and the ray lengths $S_{\rm d}(\mathbf{R}_1)$ and $S_{\rm u}(\mathbf{R}_2)$ are equal and differ from depth D only by second-order terms in the small parameters $|\mathbf{R}_{1,2}|/D$:

$$S(\mathbf{R}_{1,2}) = \sqrt{D^2 + (R_{1,2}/2)^2} \approx D\left(1 + R^2_{1,2}/8D^2\right) \approx D. \quad (22)$$

Under these conditions the ray trajectory is

$$\mathbf{r}_{\rm d}(s_{\rm d}; \mathbf{R}) = s_{\rm d}\mathbf{t}_{\rm d}(\mathbf{R}), \quad 0 < s_{\rm d} < D,$$
$$\mathbf{r}_{\rm u}(s_{\rm u}; \mathbf{R}) = D\mathbf{t}_{\rm d}(\mathbf{R}) + s_{\rm u}\mathbf{t}_{\rm u}(\mathbf{R}), \quad 0 < s_{\rm u} < D, \quad (23)$$

Here, $\mathbf{t}_{\rm d}(\mathbf{R})$ and $\mathbf{t}_{\rm u}(\mathbf{R})$ are unit vectors tangent to the down-going and up-going rays, respectively. In the small-offset (angle) approximation

$$\mathbf{t}_{\rm d}(\mathbf{R}) \cong \frac{\mathbf{R}}{2D} + \mathbf{i}_z, \qquad \mathbf{t}_{\rm u}(\mathbf{R}) \cong \frac{\mathbf{R}}{2D} - \mathbf{i}_z, \quad (24)$$

where $\mathbf{i}_z$ is the unit vector in the z-direction.

3.2 Covariances and cross-covariances for down- and up-going rays

Let us introduce new variables into eq. (20)

$$\xi = s'_{\rm d} - s''_{\rm d}, \qquad \zeta = (s'_{\rm d} + s''_{\rm d})/2, \quad (25)$$

and expand trajectories $\mathbf{r}_{\rm d}(s_{\rm d}'; \mathbf{R}_1)$ and $\mathbf{r}_{\rm d}(s_{\rm d}''; \mathbf{R}_2)$ into power series in the difference variable ξ, retaining only the first-order term in ξ in a difference $\mathbf{r}_{\rm d}(s_{\rm d}'; \mathbf{R}_1) - \mathbf{r}_{\rm d}(s_{\rm d}''; \mathbf{R}_2)$ and only the zeroth-order term in $\mathbf{r}_{dd+}$, which in fact happens to be $\mathbf{i}_z\zeta$.

As a result of the fast decrease of the correlation coefficient K with $\mathbf{r}_1 - \mathbf{r}_2$, one can extend the limits of integration in ξ to infinity and take the smaller value of $S(\mathbf{R}_1)$ and $S(\mathbf{R}_2)$ as the upper limit in the ζ variable, as is commonly done in the statistical theory of wave propagation in random media (Chernov 1960; Rytov *et al.* 1989). In view of eq. (22) min $[S(\mathbf{R}_1), S(\mathbf{R}_2)] \approx D$. As a result

$$C_{\rm dd}(\mathbf{R}_1, \mathbf{R}_2) = 2\int_0^D d\zeta\,\sigma_\mu^2(\mathbf{i}_z\zeta)\int_0^\infty d\xi\, K[\mathbf{i}_z\xi + \boldsymbol{\rho}_{\rm dd}(\zeta); \mathbf{i}_z\zeta]. \quad (26)$$

Here

$$\boldsymbol{\rho}_{\rm dd}(\zeta) = \frac{\boldsymbol{\rho}\zeta}{2D} \quad (27)$$

is a horizontal distance between rays along their down-going sections (with $\boldsymbol{\rho} = \mathbf{R}_1 - \mathbf{R}_2$) and $\mathbf{i}_z\zeta$ denotes the centre of gravity radius vector $\mathbf{r}_{dd+}$. The factor of 2 in eq. (26) arises because the integral over ξ from $-\infty$ to ∞ of the even function $K(\mathbf{r}_1 - \mathbf{r}_2)$ can be presented as a double integral of K from 0 to ∞.

A formula, similar to eq. (26), can also be derived for the traveltime covariance $C_{\rm uu}(\mathbf{R}_1, \mathbf{R}_2)$ of the up-going part of the ray trajectory. If we use the same variable ζ, which in fact is the distance from the surface, the expression for $C_{\rm uu}(\mathbf{R}_1, \mathbf{R}_2)$ can be presented in the form

$$C_{\rm uu}(\mathbf{R}_1, \mathbf{R}_2) = 2\int_0^D d\zeta\,\sigma_\mu^2(\mathbf{i}_z\zeta)\int_0^\infty d\xi\, K[\mathbf{i}_z\xi + \boldsymbol{\rho}_{\rm uu}(\zeta); \mathbf{i}_z\zeta], \quad (28)$$

where

$$\boldsymbol{\rho}_{\rm uu}(\zeta) = \boldsymbol{\rho}(1 - \zeta/2D) \quad (29)$$

denotes the horizontal distance between up-going rays at depth ζ. This distance equals $\boldsymbol{\rho} = \mathbf{R}_1 - \mathbf{R}_2$ at the surface of observation, where $\zeta = 0$ and reduces to $\boldsymbol{\rho}/2$ at the reflector depth $\zeta = D$.

Somewhat more complex relationships are found for cross-covariances $C_{\rm du}$ and $C_{\rm ud}$. In fact, they are given by formulae similar to eqs (26) and (28) only with other horizontal distances between rays:

$$\boldsymbol{\rho}_{\rm du}(\zeta) = \mathbf{R}_2 + (\boldsymbol{\rho}/2 - \mathbf{R}_2)\frac{\zeta}{D} = -\mathbf{R}_2 + \frac{\zeta}{D}\mathbf{R}_+, \quad (30)$$

$$\boldsymbol{\rho}_{\rm ud}(\zeta) = \mathbf{R}_1 - (\boldsymbol{\rho}/2 + \mathbf{R}_1)\frac{\zeta}{D} = \mathbf{R}_1 - \frac{\zeta}{D}\mathbf{R}_+, \quad (31)$$

where $\mathbf{R}_+ = (\mathbf{R}_1 + \mathbf{R}_2)/2$. All the differences between the rays $\boldsymbol{\rho}_{\rm dd}$, $\boldsymbol{\rho}_{\rm uu}$, $\boldsymbol{\rho}_{\rm du}$ and $\boldsymbol{\rho}_{\rm ud}$ are presented schematically in Fig. 2.

3.3 Total traveltime covariance

The total traveltime covariance (18) can be constructed using the covariances $C_{\rm dd}$, $C_{\rm uu}$, and cross-covariances $C_{\rm du}$, $C_{\rm ud}$. We obtain

$$\begin{aligned} C_{\rm t}(\mathbf{R}_1, \mathbf{R}_2) = 2\int_0^D d\zeta\,\sigma_\mu^2(\mathbf{i}_z\zeta)\int_0^\infty d\xi\,\big\{&K[\mathbf{i}_z\xi + \boldsymbol{\rho}_{\rm dd}(\zeta); \mathbf{i}_z\zeta] \\ &+ K[\mathbf{i}_z\xi + \boldsymbol{\rho}_{\rm uu}(\zeta); \mathbf{i}_z\zeta] \\ &+ K[\mathbf{i}_z\xi + \boldsymbol{\rho}_{\rm du}(\zeta); \mathbf{i}_z\zeta] + K[\mathbf{i}_z\xi + \boldsymbol{\rho}_{\rm ud}(\zeta); \mathbf{i}_z\zeta]\big\}. \end{aligned} \quad (32)$$

Eq. (32) is the final result for the total traveltime covariance in the small-offset approximation for random media with quasi-homogeneous statistics.

Let us now illustrate the behaviour of $C_{\rm t}$ for a specific case. Fig. 3 presents behaviour of the normalized covariance function

$$K_{\rm t}(\mathbf{R}_1, \mathbf{R}_1 + \boldsymbol{\rho}) = \frac{C_{\rm t}(\mathbf{R}_1, \mathbf{R}_1 + \boldsymbol{\rho})}{C_{\rm t}(\mathbf{R}_1, \mathbf{R}_1)} \quad (33)$$

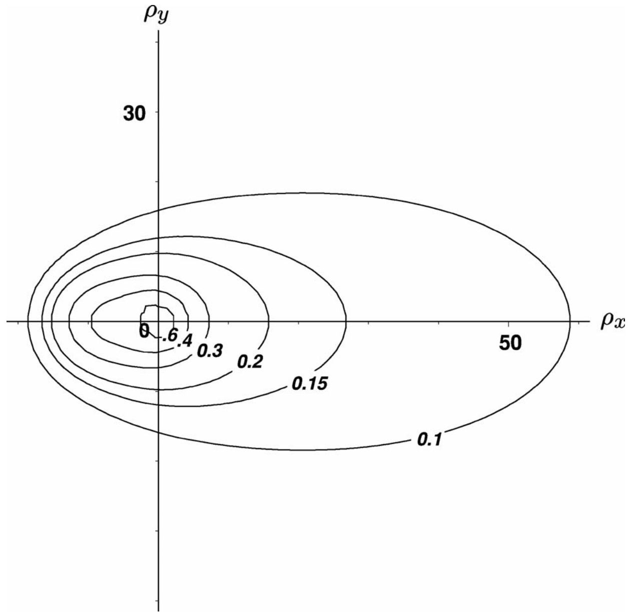

Figure 3. The contour plot displays the asymmetric behaviour of the normalized covariance function (33) in the plane $\boldsymbol{\rho} = (\rho_x, \rho_y)$. The number near each contour denotes the value of $K_{\rm t}$. In this numerical example we choose a Gaussian correlation function with $l_x = l_y = l_{\rm hor}$, $l_{\rm hor} = 2\, l_z =$ 6 m and a constant variance of the inhomogeneities ($\sigma_\mu^2 = 1$). One receiver is placed at $\mathbf{R}_1 = (2\, l_{\rm hor,}\ 0)$. The unit of the components of $\boldsymbol{\rho}$ is metres.

in a plane $\boldsymbol{\rho} = (\rho_{\rm x}, \rho_{\rm y})$, where $\boldsymbol{\rho} = \mathbf{R}_2 - \mathbf{R}_1$ is the radius vector between two receivers. Calculations were performed on the basis of expression (32) for $\mathbf{R}_1 = (2l_x, 0)$, for isomeric (statistically isotropic) fluctuations in the horizontal plane with $l_x = l_y = l_{\rm hor}$ and for $l_{\rm hor} = 2l_z$, assuming a Gaussian correlation function for the slowness fluctuations:

$$K(\mathbf{r}_1 - \mathbf{r}_2) = \exp(-g^2), \quad (34)$$

$$g^2 = \left[\frac{(x_1 - x_2)^2}{l_x^2} + \frac{(y_1 - y_2)^2}{l_y^2} + \frac{(z_1 - z_2)^2}{l_z^2}\right]. \quad (35)$$

Fig. 3 demonstrates the asymmetric character of traveltime fluctuations: the correlation length in the longitudinal direction (along the x-axis) is somewhat larger than in the transverse direction. This result admits a simple 'geometrical' explanation: for longitudinal geometry, when receivers are placed along the x-axis, the rays on average are closer to each other (Fig. 4a) than for the transverse geometry (Fig. 4b), when receiver no 2 is displaced in the y-direction relative to receiver no 1. It can also be observed that there is an asymmetry of $K_{\rm t}$ with respect to the ρ_y-axis: for small values of $|\boldsymbol{\rho}|$ ($|\boldsymbol{\rho}| < 2l_x$, which is the x-coordinate of receiver no 1) $K_{\rm t}$ is larger for negative values of ρ_x. This behaviour becomes reversed for large $|\boldsymbol{\rho}|$. The explanation for this effect is again the relative proximity of the rays. For radio waves propagating in a layered random ionosphere a similar effect was described in the textbook by Rytov *et al.* (1989).

3.4 Covariance for mid-point geometry

The total traveltime covariance function (32) can be further simplified, assuming typical recording geometries. Let the receivers be placed symmetrically relative to the source (mid-point geometry).

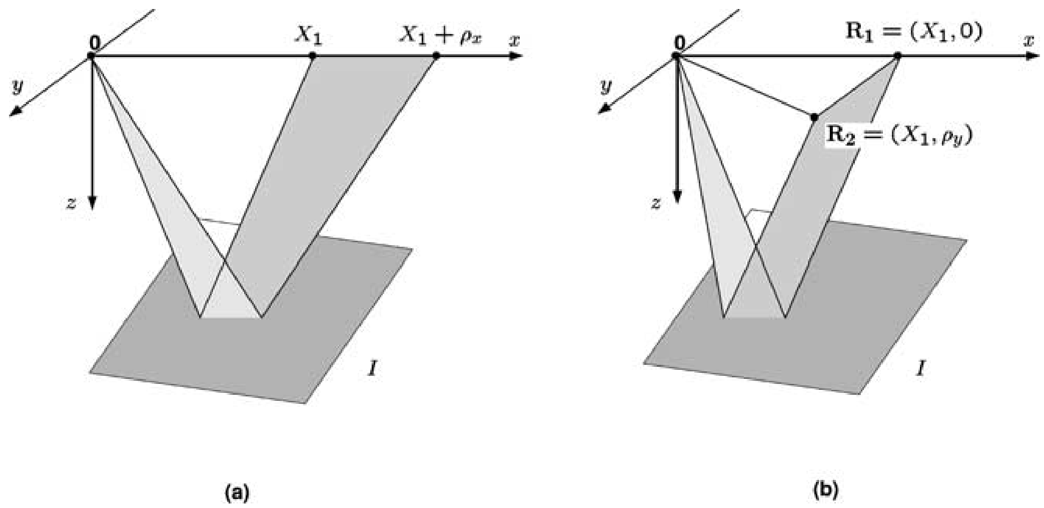

Figure 4. Explanation of the anisomeric character of the normalized covariance function (33) as shown in Fig. 3. When the receivers are placed along the x-axis (a), the rays are closer in average as compared with the rays within the recording geometry displayed in (b). This effect causes larger correlation in x-direction as compared with the transverse direction.

In this case $\mathbf{R}_2 = -\mathbf{R}_1$ and $\boldsymbol{\rho} = \mathbf{R}_1 - \mathbf{R}_2 = 2\mathbf{R}_1$ so that the horizontal distances between rays are of the form

$$\boldsymbol{\rho}_{\rm dd} = \frac{\zeta}{D}\mathbf{R}_1, \quad \boldsymbol{\rho}_{\rm uu} = \left(2 - \frac{\zeta}{D}\right)\mathbf{R}_1, \quad \boldsymbol{\rho}_{\rm du} = -\boldsymbol{\rho}_{\rm ud} = -\mathbf{R}_1. \tag{36}$$

For statistically homogeneous media (σ_μ^2 = constant) and for the Gaussian correlation function (34) the total traveltime covariance (32) can be presented as

$$C_{\rm t}(\mathbf{R}_1, -\mathbf{R}_1) = \sqrt{\pi}\sigma_\mu^2 l_z \left[\int_0^D d\zeta \left\{\exp\left(-R_1^2\zeta^2/D^2 l_x^2\right) + \exp\left[-R_1^2(2-\zeta/D)^2/l_x^2\right]\right\} + 2\exp\left(-R_1^2/l_x^2\right) D\right]. \tag{37}$$

The resulting covariance (37) in fact looks like a combination of the correlation function

$$C_{t_{\rm sph}}(2\mathbf{R}_1, 2D) = \sqrt{\pi}\sigma_\mu^2 l_z \int_0^D d\zeta \left\{\exp\left(-R_1^2\zeta^2/D^2 l_x^2\right) + \exp\left[-R_1^2(2-\zeta/D)^2/l_x^2\right]\right\} \tag{38}$$

for spherical waves, which travelled the distance 2 D in statistically homogeneous random media, and the correlation function

$$C_{t_{\rm pl}}(\mathbf{R}_1, 2D) = 2\sqrt{\pi}\sigma_\mu^2 l_z D \exp\left(-R_1^2/l_x^2\right), \tag{39}$$

for a plane wave, so that the total covariance can be rewritten as

$$C_{\rm t}(\mathbf{R}_1, -\mathbf{R}_1) = C_{t_{\rm sph}}(2\mathbf{R}_1; 2D) + C_{t_{\rm pl}}(\mathbf{R}_1; 2D). \tag{40}$$

3.5 Traveltime covariance for the two-sources–two-receivers recording scheme

The total traveltime covariance function (32) can be easily generalized for a measurement scheme involving two sources and two receivers (Fig. 5). Let $\mathbf{R}_{01}$ and $\mathbf{R}_{02}$ be positions of the sources, and $\boldsymbol{\rho}_0 = \mathbf{R}_{01} - \mathbf{R}_{02}$ the difference between them. Then, for the traveltime covariance $C_t(\mathbf{R}_1, \mathbf{R}_2; \mathbf{R}_{01}, \mathbf{R}_{02})$, where $\mathbf{R}_1$ and $\mathbf{R}_2$ denote the points of observation, the formula (32) still holds, if the horizontal distances $\boldsymbol{\rho}_{\rm dd}$, $\boldsymbol{\rho}_{\rm du}$, $\boldsymbol{\rho}_{\rm ud}$ and $\boldsymbol{\rho}_{\rm uu}$ are treated in the following way:

$$\boldsymbol{\rho}_{\rm dd} = \boldsymbol{\rho}_0 + (\boldsymbol{\rho} - \boldsymbol{\rho}_0)\zeta/2D \tag{41}$$

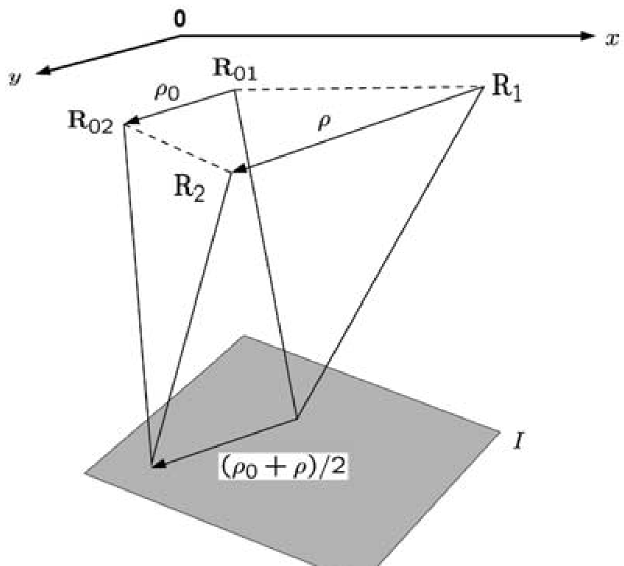

Figure 5. Geometry and notation for the two-sources and two-receivers recording scheme.

$$\boldsymbol{\rho}_{\rm uu} = \boldsymbol{\rho} + (\boldsymbol{\rho}_0 - \boldsymbol{\rho})\zeta/2D \tag{42}$$

$$\boldsymbol{\rho}_{\rm du} = \mathbf{R}_2 - \mathbf{R}_{01} + \left[\frac{\boldsymbol{\rho}_0 + \boldsymbol{\rho}}{2} - (\mathbf{R}_2 - \mathbf{R}_{01})\right]\zeta/D \tag{43}$$

$$\boldsymbol{\rho}_{\rm ud} = \mathbf{R}_1 - \mathbf{R}_{02} + \left[-\frac{\boldsymbol{\rho}_0 + \boldsymbol{\rho}}{2} - (\mathbf{R}_1 - \mathbf{R}_{02})\right]\zeta/D. \tag{44}$$

At $\mathbf{R}_{01} = \mathbf{R}_{02}$ (i.e. the single-source scheme), when $\boldsymbol{\rho}_0 = 0$, these expressions coincide with eqs (30) and (31), respectively. The suggested formulation of the problem generalizes the results of Gaerets *et al.* (2001) for the 2-D problem, where all the sources and receivers are placed along the x-axis.

4 TRAVELTIME VARIANCE FOR SMALL OFFSETS

4.1 Relation between the traveltime variance and the DPE

The DPE can be observed from the behaviour of the traveltime variance function, which itself can be obtained from the total traveltime covariance function (32) for coinciding receiver positions. Assuming $\mathbf{R}_1 = \mathbf{R}_2 = \mathbf{R}$ in eq. (32) and taking into account that in this case $\boldsymbol{\rho}$, $\boldsymbol{\rho}_{\rm dd}$ and $\boldsymbol{\rho}_{\rm uu}$ vanish and that

$$\boldsymbol{\rho}_{\rm du}(\zeta) = -\boldsymbol{\rho}_{\rm ud}(\zeta) = \mathbf{R}(1 - \zeta/D), \tag{45}$$

one obtains

$$\begin{aligned}\mathrm{var}[\tilde{t}(\mathbf{R})] \equiv \sigma_{\rm t}^2(\mathbf{R}) = C_{\rm t}(\mathbf{R}, \mathbf{R}) \\ = 4\int_0^D d\zeta\sigma_\mu^2(\mathbf{i}_z\zeta)\int_0^\infty d\xi[K(\mathbf{i}_z\xi;\mathbf{i}_z\zeta) \\ + K(\mathbf{i}_z\xi + \mathbf{R}(1-\zeta/D);\mathbf{i}_z\zeta)].\end{aligned} \tag{46}$$

In the case of statistically homogeneous media (constant variance σ_μ^2), this expression is equivalent to the result of Iooss (1998) for the 2-D problem and is analogous to the formulae for eikonal fluctuations under double-passage phenomena (Kravtsov & Saichev 1985). At $R = 0$, when the traveltime is measured directly at the source, the traveltime variance equals

$$\mathrm{var}[\tilde{t}(0)] = 8\int_0^D d\zeta\int_0^\infty d\xi\sigma_\mu^2(\mathbf{i}_z\zeta)K(\mathbf{i}_z\xi;\mathbf{i}_z\zeta), \tag{47}$$

or, in the case where $\sigma_\mu^2 = $ constant,

$$\mathrm{var}[\tilde{t}(0)] = 8D\sigma_\mu^2 l_z. \tag{48}$$

Here l_z is a vertical radius of correlation, defined as

$$l_z = \int_0^\infty d\xi K(\mathbf{i}_z\xi). \tag{49}$$

It is worth reminding the reader that the variance, eq. (47), is four times larger than the one-way traveltime variance:

$$\mathrm{var}[\tilde{t}(0)] = 4\mathrm{var}[\tilde{t}_{\rm d}(0)], \tag{50}$$

and twice as large the variance at a large offset $|\mathbf{R}| \gg l_{\rm hor}$:

$$\mathrm{var}[\tilde{t}(0)] \cong 2\mathrm{var}[\tilde{t}_{\rm d}(|\mathbf{R}| \gg l_{\rm hor})]. \tag{51}$$

In the following two sections we analyse the behaviour of the traveltime variance (46) and the DPE for different statistical models of the slowness fluctuations and recording geometries.

4.2 Horizontally anisomeric inhomogeneities

Expression (46) enables one to consider anisomeric fluctuations, which are characterized by different correlation lengths, say l_x and l_y, for different horizontal directions. Traditionally such fluctuations are referred to as being anisotropic, though spatial scales l_x and l_y are not connected with the real anisotropy of an elastic medium. Here, the terminology 'anisomeric' fluctuation might be more convenient and not confusing.

Let the statistical properties of the elastic random medium be described by the Gaussian correlation function (34). Fig. 6 illustrates the normalized cross-variance $\gamma(\mathbf{R}) = C_{\rm du}(\mathbf{R})/C_{\rm dd}(0)$ as a function of the 2-D vector $\mathbf{R} = (X, Y)$ for the case $l_x = 2l_y$. The function $\gamma(\mathbf{R})$ has different spatial scales in the X and Y directions, which are proportional to the correlation lengths l_x and l_y. The characteristic scale in a cross-section $Y = kX$ (where k is a real number) is a value between l_x and l_y.

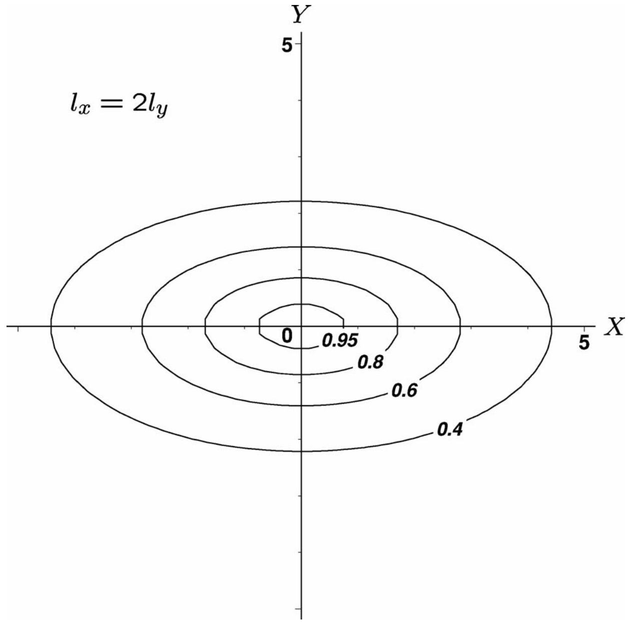

Figure 6. The normalized cross-variance $C_{\rm du}(\mathbf{R})/C_{\rm dd}(0)$ as a function of vector $\mathbf{R}$ for a medium characterized by anisomeric fluctuations. In this numerical example we choose a Gaussian correlation function with $l_x = 2l_y = 2m$. The quantities X, Y are in metres.

4.3 Tilted inhomogeneities and tilted reflecting interface

Let us imagine a horizontally isomeric layered medium that is tilted in the (X, Z) plane at angle Θ (Fig. 7a). In this situation the Gaussian correlation function is

$$K(\mathbf{r}) = \exp\left[-\frac{(x\cos\Theta - z\sin\Theta)^2}{l_x^2} - \frac{y^2}{l_y^2} - \frac{(x\sin\Theta + z\cos\Theta)^2}{l_z^2}\right]. \tag{52}$$

The distribution of traveltime variance $\sigma_{\rm t}^2(\mathbf{R})$ over the plane $\mathbf{R} = (X, Y)$ for a correlation function (52) for the case when horizontal scales l_x and l_y are equal to each other (isomeric fluctuations in a tilted plane P) yields an asymmetry in the (X, Y) plane. The characteristic scale in the X direction turns out to be less than in the Y direction, because fluctuations, which are isomeric in the P plane, look anisomeric in the (X, Y) plane. In other words, using only the information on traveltime variance $\sigma_{\rm t}^2(\mathbf{R})$, it is not possible to distinguish slightly anisomeric fluctuations from isomeric, but tilted inhomogeneities.

The majority of the relations for the traveltime (co-)variance derived above continue to be valid in the case of a tilted interface (Fig. 7b). In contrast to a strictly horizontal interface the lengths $S_{\rm d}$ and $S_{\rm u}$ of down-going and up-going sections of the ray are different, but at small angles of inclination α the difference between $S_{\rm d}$ and $S_{\rm u}$ is not large: $S_{\rm d} - S_{\rm u} \approx R\alpha \ll R$. What is more important, the first arrival curve $t(X)$ is now asymmetric.

5 TRAVELTIME COVARIANCE FOR MODERATE AND LARGE OFFSETS

5.1 Asymptotic behaviour of traveltime variance

Let us consider the behaviour of the traveltime variance for larger offsets. We start with the case when the small-angle reflection theory developed in Sections 3 and 4 is still applicable.

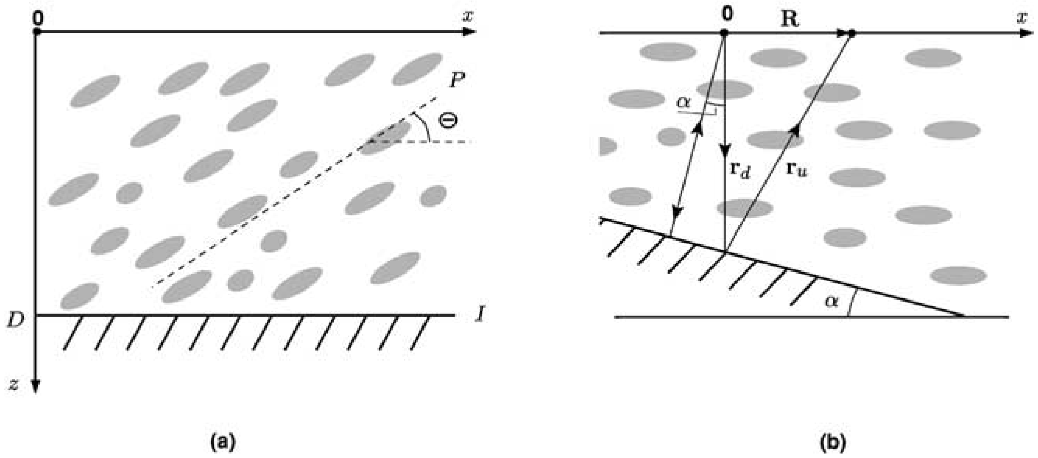

Figure 7. Geometry of a random medium containing tilted inhomogeneities (inclined by angle θ in respect to the x-axis) above a plane reflector at depth D (a). Geometry of a dipping reflector containing horizontally isomeric inhomogeneities in its overburden (b).

As long as the offset R is small compared with the horizontal correlation radius $l_{\rm hor}$, the second term in eq. (46), which is $2\,C_{\rm du}\,(\mathbf{R})$, is close to the first one (that is to $2\,C_{\rm dd}\,(0)$). In the opposite case, when $R > l_{\rm hor}$, the value $K(\mathbf{i}_z\xi + \mathbf{R}\,(1-\zeta/D))$ in eq. (46) can be neglected at $\zeta = 0$, but at the same time it becomes comparable to $K(\mathbf{i}_z\xi)$ when the distance $\boldsymbol{\rho}_{\rm du} = \mathbf{R}\,(1-\zeta/D)$ between down- and up-going rays is less than $l_{\rm hor}$. This occurs at a critical distance $(D-\zeta)_{\rm c} = l_{\rm hor}\,D/R$ from the reflecting surface at $z = D$. Therefore, the ratio $\gamma(\mathbf{R}) = C_{\rm du}(\mathbf{R})/C_{\rm dd}\,(0)$ can be estimated as the ratio of this critical distance $(D-\zeta)_{\rm c}$ to the total depth D:

$$\gamma(\mathbf{R}) \equiv \frac{C_{\rm du}(\mathbf{R})}{C_{\rm dd}(0)} \approx \frac{(D-\zeta)_{\rm c}}{D} \approx \frac{l_{\rm hor}}{R}. \tag{53}$$

These qualitative estimates are supported by numerical calculations for the Gaussian correlation function (34). Asymptotics of the ratio $\gamma(\mathbf{R}) = C_{\rm du}(\mathbf{R})/C_{\rm dd}\,(0)$ in this case take the form $\gamma(\mathbf{R}) \approx \frac{\sqrt{\pi}}{2} l_{\rm hor}/R$. A plot of the ratio $\gamma(\mathbf{R})$ for the case $l_x = l_y = l_{\rm hor}$ is presented by the black curve in Fig. 8. A grey line on the same figure corresponds to the asymptotic behaviour of $\gamma(\mathbf{R})$. It is worth noting that the analysis of the asymptotics of the ratio $\gamma(\mathbf{R})$ at $R \gg l_{\rm hor}$ can serve as another method to estimate the horizontal correlation length $l_{\rm hor}$ from experimental data, additional to the straightforward estimate of $l_{\rm hor}$ from the plot of $\sigma^2{}_{\rm t}(R)$ as used by Iooss (1998) and Gaerets *et al.* (2001).

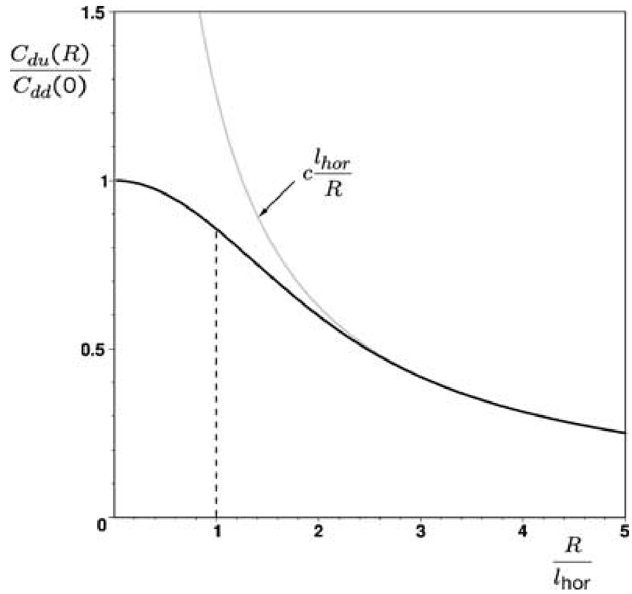

Figure 8. Asymptotic behaviour of traveltime variance for sufficiently large offsets $R \gg l_x$. The ratio $C_{\rm du}(\mathbf{R})/C_{\rm dd}\,(0)$ and its asymptotic behaviour (53) are presented for a Gaussian correlated random medium ($c = \sqrt{\pi}/2$).

5.2 Account for regular refraction—general formulation using GO

For a sufficiently large offset $|\mathbf{R}| \gg l_{\rm hor}$, comparable with the depth of the reflector D, the small-angle approximation becomes insufficient and expressions for covariances such as eq. (20) should be modified in order to take into account the phenomenon of refraction, caused by the gradient of regular sound velocity.

A standard procedure developed by Rytov *et al.* (1989) is based on the assumption that rays corresponding to two points of observation $\mathbf{R}_1$ and $\mathbf{R}_2$ are nearly parallel within an area of radius $l_{\rm hor}$. Then, one can present the difference between the two rays in eq. (20) as

$$\mathbf{r}_{\rm d}(s';\mathbf{R}_1) - \mathbf{r}_{\rm d}(s'';\mathbf{R}_2) \simeq \mathbf{t}_{\rm d}(s_{d+})s_{d-} + \boldsymbol{\rho}_{\rm dd}(s_{d+}), \tag{54}$$

where $\mathbf{t}_{\rm d} = d\mathbf{r}_{\rm d}/ds$ is a unit vector tangent to the ray and new variables $s_{d-} = s_{\rm d}{}' - s_{\rm d}{}''$ and $s_{d+} = (s_{\rm d}{}' + s_{\rm d}{}'')/2$ similar to eq. (25) are involved instead of s' and s''. Here s_{d+} is an arclength along the 'mean' ray $\mathbf{r}_{dd+} \equiv \mathbf{r}_{\rm d}\,(s_{d+})$, which is placed between $\mathbf{r}_{\rm d}(s';\mathbf{R}_1)$ and $\mathbf{r}_{\rm d}\,(s'';\mathbf{R}_2)$, and $\boldsymbol{\rho}_{\rm dd}$ is the current distance between two down-going rays (Fig. 9a). Following Rytov *et al.* (1989), we extend the limits of integration in s_{d-} to infinity, and the integration over s_{d+} is performed from $s_{d+} = 0$ to $S_{d<} = \min[S_{\rm d}(\mathbf{R}_1), S_{\rm d}(\mathbf{R}_2)]$. The final result for $C_{\rm dd}$ is

$$C_{\rm dd}(\mathbf{R}_1,\mathbf{R}_2) = 2\int_0^{S_{d<}} ds_{d+}\sigma_\mu^2(\mathbf{r}_{\rm d}(s_{d+})) \times \int_0^{\infty} ds_{d-} K[\mathbf{t}_{\rm d}(s_{d+})s_{d-} + \boldsymbol{\rho}_{\rm dd}(s_{d+})]. \tag{55}$$

An analogous expression can easily be obtained for $C_{\rm uu}$ with $S_{u<}$, s_{u-}, s_{u+} and $\rho_{\rm uu}(s_{u+})$ instead of $S_{d<}$, s_{d-}, s_{d+} and $\rho_{\rm uu}({\rm s}_{d+})$, respectively.

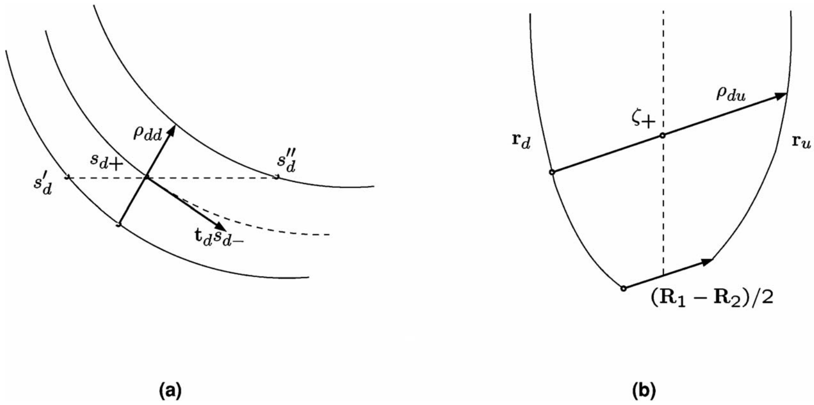

Figure 9. Notations for two down-going rays (a) and down- and up-going rays (b).

The cross-terms C_{du} and C_{ud} require a somewhat different approach. In this case the 'mean' ray is a vertical line (Fig. 9b), which can be parametrized by a vertical coordinate ζ_+ (distance from the surface). As a result we obtain

$$C_{\mathrm{du}}(\mathbf{R}_1, \mathbf{R}_2) = C_{\mathrm{ud}}(\mathbf{R}_1, \mathbf{R}_2) = 2\int_0^D d\zeta_+ \sigma_\mu^2(\zeta_+) \times \int_0^\infty d\zeta_- K[\mathbf{i}_z\zeta_- + \boldsymbol{\rho}_{\mathrm{du}}(\zeta_+)], \qquad (56)$$

where $|\boldsymbol{\rho}_{\mathrm{du}}(\zeta_+)| = |\boldsymbol{\rho}_{\mathrm{ud}}(\zeta_+)|$ and ζ_- is the current difference between points along the 'mean' ray.

The formulae obtained are in agreement with the results of the small-angle approximation (Section 3) and generalize the results of Gaerets *et al.* (2001). When the offset R becomes comparable with depth D and the angle between incident and reflected rays exceeds, say 40°–60°, the contribution of cross-covariances C_{du} and C_{ud} becomes small, of the order of $l_{\mathrm{hor}}/R \ll 1$. Under these conditions only C_{dd} and C_{uu} terms in eq. (32) remain significant:

$$C_{\mathrm{t}}(\mathbf{R}_1, \mathbf{R}_2) \cong C_{\mathrm{dd}}(\mathbf{R}_1, \mathbf{R}_2) + C_{\mathrm{uu}}(\mathbf{R}_1, \mathbf{R}_2), \quad R_{1,2} \gtrsim D, \qquad (57)$$

so that DPE completely disappears.

5.3 Statistical traveltime characteristics for oblique propagation

Let the source and the receiver be separated from each other by a distance X that is large compared with the horizontal correlation length l_{hor} (Fig. 1b). In this case, the DPE is not significant and one can neglect the cross-correlation 'du' and 'ud' terms in the expression for the covariance function, since their relative contribution does not exceed $l_{\mathrm{hor}}/X \ll 1$. Thus, for oblique incidence the covariance function $C_{\mathrm{t}}(\mathbf{R}_1, \mathbf{R}_2)$ contains only 'dd' and 'uu' terms (see eq. 57). We analyse this function in details for statistically homogeneous but anisomeric media, assuming that the normalized correlation function $K(\mathbf{r}_1, \mathbf{r}_2)$ is of the form (34). For down-going rays the difference $\mathbf{r}_1 - \mathbf{r}_2$ can be locally expressed through the longitudinal parameter ξ and the transverse vector $\boldsymbol{\delta}(s)$ by the relation

$$\mathbf{r}_1 - \mathbf{r}_2 \approx \mathbf{t}_{\mathrm{d}}\xi + \boldsymbol{\delta}(s). \qquad (58)$$

For homogeneous media and for the rays laying in the (x, z) plane $\mathbf{t}_{\mathrm{d}} = (S, 0, C)$, where $C = \cos\,\Theta$, $S = \sin\,\Theta$ (Θ is the angle of ray incidence at the interface as depicted in Fig. 10). In this notation $\boldsymbol{\delta} = (C\,\delta\,, 0, -S\,\delta)$ and the dimensionless parameter g^2 in expression

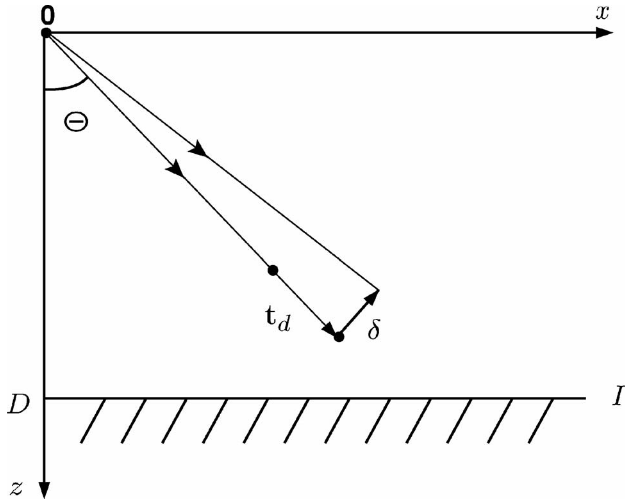

Figure 10. Two down-going rays in a homogeneous medium. The angle θ denotes the ray incidence on interface.

(34) takes the form

$$g^2 = \frac{(S\xi + C\delta)^2}{l_x^2} + \frac{(S\xi + C\delta)^2}{l_z^2} = \frac{(S^2\xi^2 + 2CS\xi\delta + C^2\delta^2)}{l_x^2} + \frac{(C^2\xi^2 - 2CS\xi\delta + S^2\delta^2)}{l_z^2}. \qquad (59)$$

In order to perform integration over variable ξ, we first isolate in eq. (59) the term containing the full square of linear form with ξ:

$$g^2 = \left[\xi\left(S^2/l_x^2 + C^2/l_z^2\right)^{1/2} + CS\delta\frac{1/l_x^2 - 1/l_z^2}{\left(S^2/l_x^2 + C^2/l_z^2\right)^{1/2}}\right]^2 + \delta^2\left[C^2/l_x^2 + S^2/l_z^2 - C^2S^2\frac{\left(1/l_x^2 - 1/l_z^2\right)^2}{S^2/l_x^2 + C^2/l_z^2}\right] = (\xi + \alpha)^2/l_\xi^2 + \delta^2/l_\delta^2, \qquad (60)$$

where $\alpha = CS\delta l_\xi^2(1/l_x^2 - 1/l_z^2)$ and

$$l_\xi = \left(S^2/l_x^2 + C^2/l_z^2\right)^{-1/2} \qquad (61)$$

is an effective characteristic length along the ray, and

$$l_\delta = \left(S^2 l_z^2 + C^2 l_x^2\right)^{1/2} \qquad (62)$$

is a characteristic scale, transverse to the ray. It is easy to verify that the product of l_ξ and l_δ is a constant equal to $l_x l_z$:

$$l_\xi l_\delta = l_x l_z. \tag{63}$$

To our knowledge this very simple relation for random media with anisomeric inhomogeneities has not been reported so far. According to eqs (61)–(63), at small angles of incidence the longitudinal scale l_ξ is close to l_z, whereas $l_\delta \approx l_x$. Conversely, at large angles $\Theta \to \pi/2$ one has $l_\xi \approx l_x$ and $l_\delta \approx l_z$. Consequently, in practically relevant situations, where $l_x > l_z$, the longitudinal scale is much larger then the transverse scale for large angles. This can also be observed in Fig. 11, where the functions $l_\xi(\Theta)$ and $l_\delta(\Theta)$ are depicted for $l_x = 5\, l_z$.

Integration of the Gaussian correlation function $K(\mathbf{r}_1, \mathbf{r}_2) = \exp(-g^2)$ in ξ gives

$$\int_{-\infty}^{\infty} K(\mathbf{r}_1, \mathbf{r}_2)\, d\xi = 2l_{\rm e}(\Theta) \exp\left(-\delta^2/l_\delta^2\right), \tag{64}$$

where $l_{\rm e} = \sqrt{\pi/2} l_\xi(\Theta)$ is an effective correlation length, which differs from $l_\xi(\Theta)$ only by a factor of $\sqrt{\pi/2}$. Using eq. (64) at $\delta = 0$, one can calculate the traveltime variance on the down-going section of the ray, and the total variance will be twice as large:

$$\sigma_{\rm t}^2 = \sigma_\mu^2 l_{\rm e}(\Theta) S(\Theta) = 4\sigma_\mu^2 l_{\rm e}(\Theta) X / \sin(\Theta), \tag{65}$$

where $S(\Theta) = \sqrt{D^2 + X^2/4} = X/\sin(\Theta)$ was used for the length of down- and up-going sections of the ray: $S_{\rm d} = S_{\rm u} = S(\Theta)$. From eqs (64) and (65) on can derive the following expression for the longitudinal traveltime covariance function:

$$C_{||}(X_1, X_2) = \frac{\sigma_{\rm t}^2(X_1)}{2S(\Theta)} \int_0^{2S(\Theta)} ds \exp\left[-\frac{s^2}{4S^2(\Theta)} \frac{\rho_x^2 \cos^2(\Theta)}{l_\delta^2(\Theta)}\right], \tag{66}$$

where $\rho_x = X_2 - X_1$ is the distance between the points of observation. Here we combined the expressions for $C_{\rm dd}$ and $C_{\rm uu}$ into a single formula, and took into account that the transverse distance δ between rays equals ρ cos (Θ) near the points of observation. In fact, formula (66) is identical to the covariance function for a spherical wave emitted from the virtual 'mirror' source (Fig. 12). The only effect that is not included in this simplified analysis is that the rays are closer to each other between the reflection points and thus implies an additional correlation. However, the resulting inaccuracy is not significant because of the smallness of the difference ρ_x relative to X. Finally, the longitudinal correlation scale $\rho_{||}$ estimated from eq. (66) is

$$\rho_{||} \approx \frac{l_\delta(\Theta)}{\cos(\Theta)} \approx \frac{\left[l_z^2 \sin^2(\Theta) + l_x^2 \cos^2(\Theta)\right]^{1/2}}{\cos(\Theta)} = \left[l_x^2 + l_z^2 \tan^2(\Theta)\right]^{1/2}. \tag{67}$$

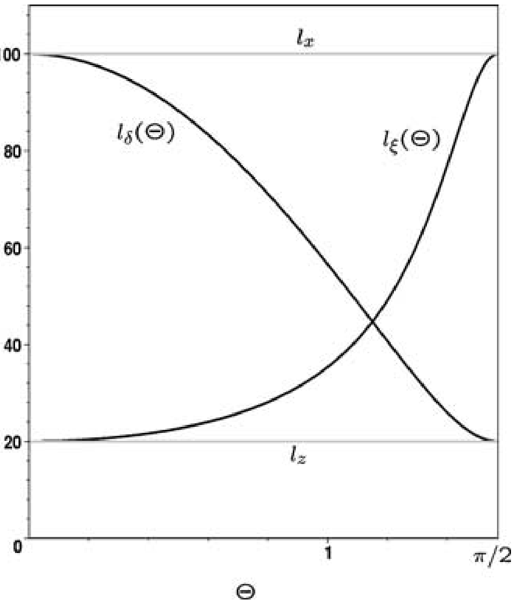

Figure 11. The effective characteristic lengths along the ray (l_ξ) and transverse to the ray (l_δ) as a function of the angle of incidence Θ for a Gaussian correlated random medium with $l_z = 20$ m and $l_x = 100$ m.

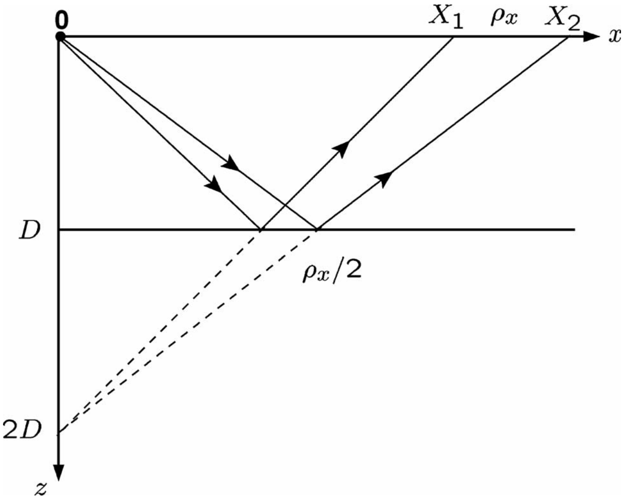

Figure 12. The longitudinal traveltime covariance function (66) is the same for the reflected rays at interface depth D and for a 'virtual' point source at the depth $2D$.

A formula similar to eq. (66) can also be written if there is a transverse displacement of the receiver no 2 relative to receiver no 1. Let $\mathbf{R}_1 = (X_1, 0)$ and $\mathbf{R}_2 = (X_1, \rho_y)$. Then, by analogy with eq. (66), the transverse traveltime covariance function is

$$C_\perp(\rho_y) = \frac{\sigma_{\rm t}^2(X_1)}{2S(\Theta)} \int_0^{2S(\Theta)} ds \exp\left[-\frac{s^2}{4S^2(\Theta)} \frac{\rho_y^2}{l_y^2}\right]. \tag{68}$$

In this case the corresponding transverse correlation scale is $\rho_\perp \approx l_y$.

Eqs (66) and (68) define the total traveltime covariance for the case of oblique propagation.

6 CONCLUSIONS

The double-passage effect, which has been known for decades in other geophysical situations (e.g. radio wave reflection from the ionosphere, phase fluctuations of light in the turbulent troposphere) is becoming a powerful instrument for measurements of the statistical properties in elastic random media (as demonstrated in the publications of Touati 1996; Iooss 1998; Gaerets *et al.* 2001). In this paper we have studied the statistical properties of seismic reflection traveltimes in order to characterize the inhomogeneities of the reflector overburden. Detailed analysis of these statistical properties on the basis of the geometrical optics approximation was performed for more complicated situations than before, namely 3-D geometry, quasi-homogeneous fluctuations of the medium parameters and curved rays.

As a result, few new features of the problem were revealed.

(1) From measurements of the traveltime variance for small offsets the geometry of the inhomogeneities cannot be uniquely determined: for horizontally anisomeric inhomogeneities and isomeric, but tilted inhomogeneities, the traveltime variance shows similar patterns.

(2) The asymptotic behaviour of the traveltime covariance can also be used in order to infer the horizontal correlation length.

(3) For offsets comparable to the reflector depth the contributions of the cross-covariances become small and the double-passage effect nearly disappears.

(4) In the case of oblique propagation in anisomeric random media the product of characteristic length-scales associated with a ray (one along the ray and another transverse to the ray) is a constant, which equals the product of horizontal and vertical correlation lengths of the inhomogeneities.

In future publications we intend to present the results of numerical experiments demonstrating opportunities for extracting the elastic medium statistical parameters from traveltime fluctuations and thus supporting the results of this study.

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